Hw1

1. (a) We have: $\vec{u}\cdot\vec{v} = A\vec{u}\cdot A\vec{v} = (A^TA\vec{u})\cdot\vec{v}$ for all $\vec{u},\vec{v}\in\mathbb{R}^3$. Since $\vec{v}$ is arbitrary, and dot product on $\mathbb{R}^3$ is non-degenerate, we have $A^TA\vec{u} = \vec{u}$ for every $\vec{u}\in\mathbb{R}^3$. Hence $A^TA = I$. For the other order, note that we now have $A^{-1} = A^T$, so that $I = AA^{-1} = AA^T$. Taking determinants of both sides and using that $\det A^T = \det A$, we have $1 = (\det A)^2$, so that $\det A = \pm 1$.

   (b) Differentiating $A(t)\vec{u}\cdot A(t)\vec{v} = \vec{u}\cdot\vec{v}$ gives $\dot{A}(t)\vec{u}\cdot A(t)\vec{v} + A(t)\vec{u}\cdot\dot{A}(t)\vec{v} = 0$, for any $\vec{u},\vec{v}\in\mathbb{R}^3$. Now evaluate at $t=0$, where $\dot{A}(0) = \Omega$ and $A(0) = I$ gives $\Omega\vec{u}\cdot\vec{v} = -\vec{u}\cdot\Omega\vec{v} = -(\Omega^T\vec{u})\cdot\vec{v}$, as this holds for all $\vec{v}\in\mathbb{R}^3$, we have $\Omega\vec{u} = -\Omega^T\vec{u}$ for all $\vec{u}$, i.e. $\Omega = -\Omega^T$ is skew-symmetric.

   (c) For any $\vec{u},\vec{v}\in\mathbb{R}^3$, we have $\Omega_{\vec{\omega}}\vec{u}\cdot\vec{v} = (\vec{\omega}\times\vec{u})\cdot\vec{v} = -(\vec{\omega}\times\vec{v})\cdot\vec{u} = -\vec{u}\cdot\Omega_{\vec{\omega}}\vec{v} = -(\Omega_{\vec{\omega}}^T\vec{u})\cdot\vec{v}$, so that $\Omega_{\vec{\omega}} = -\Omega_{\vec{\omega}}^T$ is skew-symmetric.

   This map is linear from cross product properties: $\Omega_{\vec{\omega}+\lambda\vec{\nu}}\vec{u} = (\vec{\omega}+\lambda\vec{\nu})\times\vec{u} = \vec{\omega}\times\vec{u} + \lambda(\vec{\nu}\times\vec{u}) = (\Omega_{\vec{\omega}} + \lambda\Omega_{\vec{\nu}})\vec{u}$. It is also injective, since if $\vec{\omega}\times\vec{u} = 0$ for all $\vec{u}$, then for $\vec{u}$ a unit vector in $\vec{\omega}^{\perp}$, we have $0 = |\vec{\omega}\times\vec{u}| = |\vec{\omega}|$, so that $\vec{\omega} = 0$. Since both vector spaces are 3-dimensional the linear map is an isomorphism.

   It is also worthwhile to workout a formula for this map: $\vec{\omega} = \omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k} \mapsto \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$, since for example, $\vec{\omega}\times\hat{i} = -\omega_2\hat{k} + \omega_3\hat{j}$ gives the entries of the 1st column.

   (d) Note that rotations about a common axis commute. Hence $A(t)A^{-1}(s) = A^{-1}(s)A(t) = A(t-s)$. Let $\tau = t-s$ so that $\frac{d}{dt}|_{t=s}A(t-s) = \frac{d}{d\tau}|_{\tau=0}A(\tau) = \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \Omega_{\vec{\omega}}$.

   (e) We compute:

   $(*)\qquad \dot{\vec{x}} = \dot{A}\vec{y} + A\dot{\vec{y}} = \dot{A}A^{-1}\vec{x} + A\dot{\vec{y}} = \vec{\omega}\times\vec{x} + A\dot{\vec{y}}$

   where we use product rule and part (d) for the last equality.

   Taking A^{-1} of the first equality in $(*)$ gives: $A^{-1}\dot{\vec{x}} = A^{-1}\dot{A}\vec{y} + \dot{\vec{y}}$, or $\dot{\vec{y}} = -\vec{\omega}\times\vec{y} + A^{-1}\dot{\vec{x}}$.

   Since $\vec{\omega}$ is fixed, we have:

   $(**)\qquad \ddot{\vec{x}} = \vec{\omega}\times\dot{\vec{x}} + \dot{A}\dot{\vec{y}} + A\ddot{\vec{y}} = \vec{\omega}\times\dot{\vec{x}} + \dot{A}A^{-1}(A\dot{\vec{y}}) + A\ddot{\vec{y}} =$

   $= \vec{\omega}\times\dot{\vec{x}} + \vec{\omega}\times(\dot{\vec{x}} - \vec{\omega}\times\vec{x}) + A\ddot{\vec{y}} = 2\vec{\omega}\times\dot{\vec{x}} - \vec{\omega}\times(\vec{\omega}\times\vec{x}) + A\ddot{\vec{y}}$.

   Taking A^{-1} of the first equality in $(**)$ and using that $A(\vec{u}\times\vec{v}) = A\vec{u}\times A\vec{v}$ for rotations, we get:

   $A^{-1}\ddot{\vec{x}} = \vec{\omega}\times A^{-1}\dot{\vec{x}} + \vec{\omega}\times\dot{\vec{y}} + \ddot{\vec{y}} = \vec{\omega}\times(\dot{\vec{y}} + \vec{\omega}\times\vec{y}) + \vec{\omega}\times\dot{\vec{y}} + \ddot{\vec{y}}$, or

   $\ddot{\vec{y}} = -2\vec{\omega}\times\dot{\vec{y}} - \vec{\omega}\times(\vec{\omega}\times\vec{y}) + A^{-1}\ddot{\vec{x}}$.

2. (a) Set $e_\rho := (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$, $e_\varphi := (\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi)$, $e_\theta := (-\sin\theta, \cos\theta, 0)$. Thes vectors are orthonormal. We compute:

   $\dot{e}_\rho = \dot{\varphi}e_\varphi + \sin\varphi\dot{\theta}e_\theta$,

   $\dot{e}_\varphi = -\dot{\varphi}e_\rho + \cos\varphi\dot{\theta}e_\theta$,

   $\dot{e}_\theta = \dot{\theta}(-\cos\theta, -\sin\theta, 0) = (\dot{e}_\theta\cdot e_\rho)e_\rho + (\dot{e}_\theta\cdot e_\varphi)e_\varphi = -\sin\varphi\dot{\theta}e_\rho - \cos\varphi\dot{\theta}e_\varphi$.

   Now, with $q := (x,y,z) = \rho e_\rho$, we find:

   $\dot{q} = \dot{\rho}e_\rho + \rho\dot{e}_\rho = \dot{\rho}e_\rho + \rho\dot{\varphi}e_\varphi + \rho\sin\varphi\dot{\theta}e_\theta$,

   $\ddot{q} = \ddot{\rho}e_\rho + \dot{\rho}\dot{e}_\rho + \frac{d}{dt}(\rho\dot{\varphi})e_\varphi + \rho\dot{\varphi}\dot{e}_\varphi + \frac{d}{dt}(\rho\sin\varphi\dot{\theta})e_\theta + \rho\sin\varphi\dot{\theta}\dot{e}_\theta$

   $= (\ddot{\rho} - \rho\dot{\varphi}^2 - \rho\sin^2\varphi\dot{\theta}^2)e_\rho + (\dot{\rho}\dot{\varphi} + \frac{d}{dt}(\rho\dot{\varphi}) - \rho\sin\varphi\cos\varphi\dot{\theta}^2)e_\varphi + (\dot{\rho}\sin\varphi\dot{\theta} + \rho\dot{\varphi}\cos\varphi\dot{\theta} + \frac{d}{dt}(\rho\sin\varphi\dot{\theta}))e_\theta$.

Note that, with $C := \rho^2 \sin^2\varphi\dot\theta$, we have:

$\ddot q = (\ddot\rho - \rho\dot\varphi^2 - \frac{C^2}{\rho^3\sin^2\varphi})e_\rho + (2\dot\rho\dot\varphi + \rho\ddot\varphi - \rho\sin\varphi\cos\varphi\dot\theta^2)e_\varphi + \frac{\dot C}{\rho\sin\varphi}e_\theta$.

(b) From (a) and $e_\rho, e_\varphi, e_\theta$ being orthonormal, we have: $|\dot q|^2 = \dot\rho^2 + \rho^2\dot\varphi^2 + \rho^2\sin^2\varphi\dot\theta^2$.

3. (a) Differentiating $|q|^2 = R^2 = cst.$, we find: $q\cdot\dot q = 0$ and again gives:

   $(*)\qquad |\dot q|^2 + q\cdot\ddot q = 0.$

   Now write $\ddot q = \ddot q_{tan} + \ddot q_{nor} = \ddot q_{tan} + \lambda q$. Note that $\ddot q_{tan}\cdot q = 0$, since q is normal to the sphere at q. Plugging in $\ddot q$ to $(*)$ gives: $|\dot q|^2 + \lambda R^2 = 0$, or $\ddot q_{nor} = -\frac{|\dot q|^2}{R^2}q$.

   (b) For free motion, we have $\ddot q = \ddot q_{nor}$ is normal to the sphere at q. First note that $\frac{d}{dt}|\dot q|^2 = 2\dot q\cdot\ddot q = 0$, since $\dot q$ is tangent to the sphere and $\ddot q$ is normal to the sphere.

   Let us assume that $\dot q \neq 0$ (so that the particle is actually moving). Then, by (a), q satisfies:

   $\ddot q = -k^2 q$, where $k^2 = \frac{|\dot q|^2}{R^2} > 0$ is a constant.

   The solutions to this second order ODE are $q(t) = q_o\cos kt + \frac{\dot q_o}{k}\sin kt$, which parametrize great circles.

   (c) Consider a latitude, $\varphi = \varphi_o = cst$. We parametrize this latitude with unit speed by:

   $R(\sin\varphi_o\cos\omega t, \sin\varphi_o\sin\omega t, \cos\varphi_o)$, where $\omega = \frac{1}{R\sin\varphi_o} = \dot\theta = cst.$ (consider the formula from 2(b)). By our acceleration expression in 2(a) (with $\rho = R = cst., \varphi = \varphi_o = cst., \dot\theta = \omega = cst.$) we have $\ddot q_{tan} = -R\sin\varphi_o\cos\varphi_o\omega^2 e_\varphi$, which has norm: $\frac{\cot\varphi_o}{R} = \kappa_{esf}(\varphi = \varphi_o)$.

   (note that for $\varphi_o = \frac{\pi}{2}$, when the latitude is the equator (a great circle), the curvature is zero).

4. Set $v = \dot h$. We then have a 1st order ode $\dot v = -g + \frac{\gamma}{m}v^2$. The equilibrium solutions are $v_\pm = \pm\sqrt{\frac{gm}{\gamma}}$. Sketching the slope field $\dot v(v)$ in the (t, v)-plane, one sees the slope is negative for $|v| < \sqrt{\frac{gm}{\gamma}}$ and positive for $v < -\sqrt{\frac{gm}{\gamma}}$. In particular, a solution with $v(0) \le 0$ tends as $t\to\infty$ to the equilibrium solution v_-.

5. Rewrite this 2nd order ODE as a linear 1st order system: $\dot X = \begin{pmatrix} 0 & 1 \\ -1 & -\gamma \end{pmatrix} X = AX$, where $X = \begin{pmatrix} x \\ \dot x \end{pmatrix}$.

   One may sketch solutions by diagonalizing A (see figures at end –taken from Fernando's hw). The eigenvalues, λ, of A are the roots of: $\lambda^2 + \gamma\lambda + 1 = 0$, that is $\lambda_\pm = \frac{-\gamma\pm\sqrt{\gamma^2-4}}{2}$.

   (a) The eigenvalues are imaginary, with a negative real part: solutions spiral towards the origin.

   (b) There is one repeated (negative) eigenvalue: solutions are forwards asymptotic to the origin (approaching an invariant line)

   (c) There are two negative real eigenvalues: solutions are forwards asymptotic to the origin (two invariant lines).

6. (a) The velocities on the particles due to $\vec\omega$ are $\vec v_j = \vec\omega\times\vec q_j$. By definition of angular momentum: $\vec C = \sum m_j\vec q_j\times(\vec\omega\times\vec q_j)$.

   (b) For $\vec\omega_1, \vec\omega_2 \in \mathbb{R}^3$ and $\lambda\in\mathbb{R}$, we have $\mathbb{I}(\vec\omega_1 + \lambda\vec\omega_2) = \sum m_j\vec q_j\times(\vec\omega_1\times\vec q_j + \lambda\vec\omega_2\times\vec q_j) = \sum m_j\vec q_j\times(\vec\omega_1\times\vec q_j) + \lambda\sum m_j\vec q_j\times(\vec\omega_2\times\vec q_j) = \mathbb{I}(\vec\omega_1) + \lambda\mathbb{I}(\vec\omega_2)$, so that $\mathbb{I}$ is linear.

   Also, $\mathbb{I}\vec\omega_1\cdot\vec\omega_2 = \sum m_j[\vec q_j\times(\vec\omega_1\times\vec q_j)]\cdot\vec\omega_2 = -\sum m_j(\vec q_j\times\vec\omega_2)\cdot(\vec\omega_1\times\vec q_j) = \sum m_j(\vec\omega_2\times\vec q_j)\cdot(\vec\omega_1\times\vec q_j) = \mathbb{I}\vec\omega_2\cdot\vec\omega_1$, so that $\mathbb{I}$ is symmetric.

   (c) From (b), we have $\mathbb{I}\vec\omega\cdot\vec\omega = \sum m_j(\vec\omega\times\vec q_j)\cdot(\vec\omega\times\vec q_j) = \sum m_j|\vec\omega\times\vec q_j|^2$.

   (d) Let $\hat\imath, \hat\jmath, \hat k$ be an (oriented) orthonormal basis, and $\vec q_j = x_j\hat\imath + y_j\hat\jmath + z_j\hat k$ in this basis.

   Note that $\hat\imath\times\vec q_j = -z_j\hat\jmath + y_j\hat k,\quad \hat\jmath\times\vec q_j = z_j\hat\imath - x_j\hat k,\quad \hat k\times\vec q_j = -y_j\hat\imath + x_j\hat\jmath.$

   Now: $\mathbb{I}\hat\imath\cdot\hat\imath = \sum m_j|\hat\imath\times\vec q_j|^2 = \sum m_j(y_j^2 + z_j^2)$ and $\mathbb{I}\hat\imath\cdot\hat\jmath = \sum m_j(\hat\imath\times\vec q_j)\cdot(\hat\jmath\times\vec q_j) = -\sum m_j x_j y_j$.

Likewise: $\mathbb{I}\hat{j}\cdot\hat{j} = \sum m_j(x_j^2+z_j^2)$, $\mathbb{I}\hat{k}\cdot\hat{k} = \sum m_j(x_j^2+y_j^2)$, $\mathbb{I}\hat{i}\cdot\hat{k} = -\sum m_j x_j z_j$, $\mathbb{I}\hat{j}\cdot\hat{k} = -\sum m_j y_j z_j$.

So that, in this basis, $\mathbb{I} = \sum \begin{pmatrix} m_j(y_j^2+z_j^2) & -m_jx_jy_j & -m_jx_jz_j \\ -m_jx_jy_j & m_j(x_j^2+z_j^2) & -m_jy_jz_j \\ -m_jx_jz_j & -m_jy_jz_j & m_j(x_j^2+y_j^2) \end{pmatrix}$.

7. (a) By definition, $Q_j = \frac{\sum_{j\in I_j} q_j}{M_j}$, or $M_jQ_j = \sum_{j\in I_j} q_j$. Since I_j is a partition, we have $M_1+...+M_k = m_1+...+m_N = M$ and $M_1Q_1+...+M_kQ_k = q_1+...+q_N$. Hence $Q_{cm} = \frac{q_1+...+q_N}{M} = q_{cm}$.

   (b) **to get the general idea, you may want to first try relating the theorem that the medians of a triangle are concurrent with part (a) when equal masses are placed at the vertices of the triangle**

   ($\Leftarrow$) Suppose the ratios of Ceva's theorem hold for given A, A', B, B', C, C'. We will choose masses m_A, m_B, m_C on the vertices so that the lines are concurrent at the center of mass of the triangle.

   First, fix $m_A > 0$. We then choose m_B so that C' is the center of mass of the A, B system: $m_A|C'A| = m_B|BC'|$, or $m_B = m_A\frac{|C'A|}{|BC'|}$. Having determined m_B, we choose $m_C = m_B\frac{|A'B|}{|CA'|}$, so that A' is the center of mass of the C, B system.

   By substitution, $m_C = m_A\frac{|C'A|}{|BC'|}\frac{|A'B|}{|CA'|} = m_A\frac{|AB'|}{|B'C|}$ (use the ratios in Ceva's theorem for the last equality), so that B' is the center of mass of the A, C system.

   Now, let P be the center of mass of the triangle with masses m_A, m_B, m_C at the vertices. Consider the partition of masses into $\{m_A, m_B\}$ and $\{m_C\}$. By construction C' is the center of mass of the A, B system. By part (a), the center of mass of C' and C is P. In particular, P lies on the line CC'. Likewise P lies on AA' and BB', so that the lines AA', BB', CC' are concurrent (at P).

   ($\Rightarrow$) Suppose the lines of Ceva's theorem are concurrent at P. The interior of the triangle is parametrized by $\{m_AA+m_BB+m_CC : m_A+m_B+m_C = 1, m_A, m_B, m_C > 0\}$ (every interior point is a center of mass for some choice of masses). In particular, there exist $m_A, m_B, m_C > 0$ with $m_A+m_B+m_C = 1$ for which the center of mass of the triangle with these masses at the vertices is P.

   Partition the masses into $\{m_A, m_B\}$ and $\{m_C\}$. Let $\tilde{C}$ on the segment AB be the center of mass of the A, B system. By (a), P lies on the line $C\tilde{C}$. The line CP intersects the segment AB in one point, namely C'. Hence $\tilde{C} = C'$, so C' is the center of mass of the A, B system. Likewise, B' is the center of mass of the A, C system and A' is the center of mass of the B, C system.

   By definition of center of mass: $m_A|C'A| = m_B|BC'|, m_B|A'B| = m_C|CA'|, m_C|B'C| = m_A|AB'|$. Hence: $\frac{|AB'|}{|B'C|}\frac{|CA'|}{|A'B|}\frac{|BC'|}{|C'A|} = \frac{m_C}{m_A}\frac{m_B}{m_C}\frac{m_A}{m_B} = 1$.

8. (a) Let the curve be parametrized by arc-length $c(s)$, with $c(0)$ the point at which the string is attached. If the length of the string is ℓ, then the involute (evolvente), may be parametrized by: $i(s) = c(s) + (\ell - s)c'(s)$, $s \in [0, \ell]$.

   For each s, the tangent to the involute is: $i'(s) = c'(s) - c'(s) + (\ell - s)c''(s) = (\ell - s)c''(s)$, while the string has direction $c'(s)$.

   Since s is the arc-length parameter, we have $|c'(s)| = 1$, and in particular $c'(s)\cdot c''(s) = 0$, so that indeed, the involute is perpendicular to the string at each instant.

   (b) For a circle of radius R, an arc may be parametrized as: $c(\theta) = R(\theta - \sin\theta, 1-\cos\theta), \theta \in [0, 2\pi]$.

   Then $\frac{dc}{d\theta}(\theta) = R(1-\cos\theta, \sin\theta)$, and the length is: $R\int_0^{2\pi}\sqrt{2-2\cos\theta}\ d\theta = R\int_0^{2\pi} 2\sin\frac{\theta}{2}\ d\theta = 8R$.

   Note that the arc-length, $s(\theta) = 4R(1-\cos\frac{\theta}{2})$, and $ds = 2R\sin\frac{\theta}{2}d\theta$.

   The involute may be parametrized by $i(\theta) = c(\theta) + (4R - s(\theta))\frac{dc}{d\theta}\frac{d\theta}{ds}$.

   We have: $\frac{dc}{ds} = \frac{dc}{d\theta}\frac{d\theta}{ds} = \frac{(1-\cos\theta, \sin\theta)}{2\sin\frac{\theta}{2}} = \frac{(\sin^2\frac{\theta}{2}, \sin\frac{\theta}{2}\cos\frac{\theta}{2})}{\sin\frac{\theta}{2}} = (\sin\frac{\theta}{2}, \cos\frac{\theta}{2})$ and $4R - s = 4R\cos\frac{\theta}{2}$. So: $(4R-s)\frac{dc}{ds} = 2R(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}, 2\cos^2\frac{\theta}{2}) = 2R(\sin\theta, 1+\cos\theta)$.

   Hence, $i(\theta) = R(\theta + \sin\theta, 3 + \cos\theta)$, which parametrizes a cycloid obtained by rolling a circle of radius R along the line $y = 2R$.

9. (a) The force due to each spring is $F_j = k_j L_j$. Since $L_1 = L_2 = L$ is the displacement of both springs, the total force on the endpoint is $F_1 + F_2 = (k_1 + k_2)L$, so $\ddot{L} = -(k_1 + k_2)L$.

(b) We assume that the force on the endpoint when it is displaced a distance L depends only on L. Stretch the spring a distance L, and hold it there. Then, the connection between the springs is in equilibrium: $k_1 L_1 = F_1 = F_2 = k_2 L_2$ and $L = L_1 + L_2$. The force on the endpoint is $F = F_1 = F_2$, so that $L = (\frac{1}{k_1} + \frac{1}{k_2})F$ or $F = \frac{k_1 k_2}{k_1 + k_2} L$ and $\ddot{L} = -\frac{k_1 k_2}{k_1 + k_2} L$.

(c) The displacement L of the spring is $\sqrt{x^2 + D^2}$. By similar triangles, the component of the force due to the spring along the x-axis has norm: $F_x = |\frac{L_0 - L}{L} x|$.

We first consider when $L_0 \leq D$. Intuitively, in this case there is only one equilibrium point, $x = 0$ which is stable. Indeed here $L > L_0$ for $x \neq 0$ so that F_x is only zero when $x = 0$. Moreover, the signed force: $m\ddot{x} = \frac{L_0 - L}{L} x$ is always directed towards the origin.

Now, we consider $L_0 > D$. One should expect 3 equilibrium points and that $x = 0$ should be unstable. Indeed, for x sufficiently small, we have $L < L_0$ so that the signed force $m\ddot{x} = \frac{L_0 - L}{L} x$ is directed away from the origin. The equilibrium points with $x \neq 0$ occur when $L = L_0$, that is when $x_\pm = \pm\sqrt{L_0^2 - D^2}$. These equilibrium points are stable. The directions of the forces around $x_\pm$ can be found by considering the graph of $y = L_0 - L(x)$, which is a downward turning hyperbola with peak at $x = 0, y = L_0 - D > 0$.

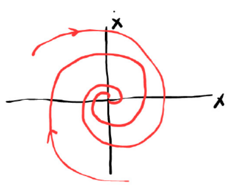

Figura 2: Spirals towards the origin $0 < \gamma < 2$

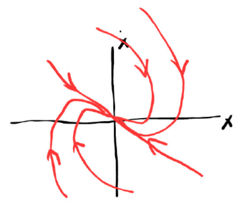

Figura 3: Improper node towards the origin $\gamma = 2$

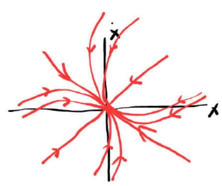

Figura 4: Node towards the origin $\gamma > 2$