Hw2

1. (a) We verify this by differentiation along a solution: $\frac{d}{dt}(u_1/u_2) = \frac{\dot{u}_1 u_2 - u_1 \dot{u}_2}{u_2^2}$. Note that $\dot{u}_1 = v - iq = -iu_1$ and $\dot{u}_2 = \bar{v} - i\bar{q} = -iu_2$, so that $\dot{u}_1 u_2 - u_1 \dot{u}_2 = -i(u_1 u_2 - u_1 u_2) = 0$. Hence $\frac{d}{dt}(u_1/u_2) = 0$, or $u_1/u_2 = cst.$ over solutions.

   (b) Let $(u_1^o, u_2^o) \in \mathbb{C}^2 \backslash (0,0)$ be a representative of ℓ (so $\ell = (u_1^o : u_2^o)$). Then, the set of points, $(u_1, u_2) \in \mathbb{C}^2$ with $(u_1 : u_2) = \ell$ is the complex line: $(u_1, u_2) = (\lambda u_1^o, \lambda u_2^o)$, for $\lambda \in \mathbb{C}$. The additional condition $|u_1|^2 + |u_2|^2 = 1$ implies that $|\lambda|^2 = \frac{1}{|u_1^o|^2 + |u_2^o|^2} = cst. > 0$, so that $\lambda = ke^{i\theta}$ for $k > 0$ some constant. Hence, we parametrize this set by $\{e^{i\theta}(ku_1^o, ku_2^o)\} \subset \mathbb{C}^2$ as $e^{i\theta}$ runs over the unit circle.

   (c) In terms of $(u_1, u_2) \in \mathbb{C}^2 \backslash (0,0)$, and $u_2 \neq 0$, we have $H(u_1, u_2) = \left(\frac{2\frac{u_1}{u_2}}{\frac{|u_1|^2}{|u_2|^2}+1}, \frac{\frac{|u_1|^2}{|u_2|^2}-1}{\frac{|u_1|^2}{|u_2|^2}+1}\right)$, multiplying by $\frac{|u_2|^2}{|u_2|^2}$ gives: $H(u_1, u_2) = \left(\frac{2u_1\bar{u}_2}{|u_1|^2+|u_2|^2}, \frac{|u_1|^2-|u_2|^2}{|u_1|^2+|u_2|^2}\right)$, which is defined for any $(u_1, u_2) \in \mathbb{C}^2 \backslash (0,0)$.

   **observe that the three components of $H(u_1, u_2)$ are constants of motion for the planar Hooke problem. They relate to those we found in lecture 5 (E, C, F, I) by:

   $|u_1|^2 + |u_2|^2 = 4E$, $|u_1|^2 - |u_2|^2 = -4C$, $u_1\bar{u}_2 = q^2 + v^2 = F + 2iI$, so that $H(q, v) = (\frac{F}{2E}, \frac{I}{E}, -\frac{C}{E})$ is in the unit sphere, and we have the relation: $F^2 + 4I^2 + 4C^2 = 4E^2$ between these integrals.

2. These central forces are derived from a force function $U(r) = \frac{1}{\alpha r^\alpha}$. Being central, the angular momentum C is a constant of motion, as is the energy, E, which we write as: $E = \frac{\dot{r}^2}{2} + \frac{C^2}{2r^2} - \frac{1}{\alpha r^\alpha}$.

   We may use the effective potential method to determine when bounded motions ($r \leq cst.$) are possible.

   There are 3 main cases to consider (see figures at end):

   1. $0 < \alpha < 2$. The effective potential takes the same form as in the Kepler problem. In these cases, when $E \geq 0$, the motions are always unbounded. When $E < 0$ and $C = 0$ the motions are bounded. Also, when $C \neq 0$ we have bounded motions for $E \in [E_{cir}, 0)$. The energy value $E_{cir} = \frac{\alpha-2}{2\alpha} C^{\frac{2\alpha}{\alpha-2}} < 0$ gives circular orbits with momentum C. In summary, for every negative energy, $E < 0$, bounded motions are possible.

   2. $\alpha = 2$. Here the graph of the effective potential depends strongly on C. For $|C| < 1$, bounded (collision) orbits occur for any $E < 0$, while for $E > 0$ we get unbounded collision orbits. For $C^2 = 1$, we get circular orbits with $E = 0$, and unbounded orbits when $E > 0$ (there are no orbits with $E < 0$ and $C = \pm 1$). Finally, when $|C| > 1$, it is only possible to have $E > 0$ which yield unbounded orbits. In summary, bounded orbits are possible for any non-positive energy, $E \leq 0$.

   3. $\alpha > 2$. The effective potential has the form of a reflected Kepler problem type potential. First, for $C = 0$, we find every negative energy gives bounded collision orbits, and any $E \geq 0$ gives unbounded collision orbits. Next, when $C \neq 0$, we have the possibility of bounded motions for $E \in (-\infty, E_{cir}]$, where $E_{cir} = \frac{\alpha-2}{2\alpha} C^{\frac{2\alpha}{\alpha-2}} > 0$ gives circular orbits with momentum C. In summary, bounded motions are possible for any energy value.

3. (a) See for example wiki.

   (b) Let us consider such Cartesian coordinates, based at f_1 and with X-axis along the $f_1 f_2$ line. Then $R = d(p, f_1)$ and $\rho^2 = d^2(p, f_2) = (X + 2ae)^2 + Y^2 = R^2 + (2ae)^2 + 4aeX$, where $2ae := d(f_1, f_2)$. The defining condition $R + \rho = 2a$, may be written $\rho^2 = R^2 + (2a)^2 - 4aR$ or $R = a(1 - e^2) - eX$, of the desired form. Note that if we rotate the coordinates so the X-axis is not aligned with the $f_1 f_2$ line, we get an equation of the form $R = AX + BY + C$.

   (c) Let the line be given by $ax + by + c = 0$, for Cartesian coordinates centered at f. Then, if p has coordinates (x, y), we have: $d(p, f) = r = \sqrt{x^2 + y^2}$ and $d(p, \ell) = (p - p_o) \cdot n$, where $p_o = (0, -c/b) \in \ell$ and $n = \frac{(a,b)}{\sqrt{a^2+b^2}}$ is the unit normal to ℓ. Hence $d(p, \ell) = \frac{ax+by+c}{\sqrt{a^2+b^2}} = a'x + b'y + c'$, so $d(p, f)/d(p, \ell) = cst.$ is equivalent to $r = Ax + By + C$, for some constants A, B, C.

   (d) Write the plane π as $z = Ax + By + C$. The intersection is defined by the equations: $\{z = \sqrt{x^2 + y^2}\} \cap \{z = Ax + By + C\}$ or $z = r = ax + by + c$ which is a conic section in the xy-plane.

4. The lift is given by $Q(t) = (q(t), r(t))$, where $r = |q|$. Hence $\ddot{Q} = (\ddot{q}, \ddot{r}) = \frac{(-q, C^2 - r)}{r^3} = \frac{-Q + C^2}{r^3}$. So, for $X = Q - (0, 0, C^2)$, we have $\ddot{X} = \ddot{Q} = -\frac{X}{r^3}$.

   In particular, $N = X \times \dot{X}$ is constant, which is to say that $X(t)$ lies in the fixed plane $N^\perp$. Hence $q(t)$ is the projection to the xy-plane of the intersection of the cone with a plane – a conic section.

5. (a) Differentiating $c \cdot c = |c|^2 = 1$, we have $c \cdot c' = 0$. By assumption $|c'| = 1$, so that the vectors $c, c', n = c \times c'$ are an (oriented) orthonormal basis at each s.

   Hence, we may write $c'' = a_1 c + a_2 c' + a_3 n$. Now differentiating $c \cdot c' = 0$ gives $c'' \cdot c = -1$, while differentiating $1 = c' \cdot c'$ gives $c'' \cdot c' = 0$. So, for each s, we have $c''(s) = -c(s) + k(s)n(s)$ for some $k(s) \in \mathbb{R}$.

   (b) We make a computation, using $\dot{c} = vc'$ (chain rule). Start from $\vec{C} = Cc$, so that $\dot{\vec{C}} = \dot{C}c + Cvc'$. We aim to find $\ddot{\vec{C}} \cdot n$, so we will not worry about terms which are mutliples of c or $\dot{c}$ (since $c, c' = \frac{\dot{c}}{v}, n$ are orthonormal). Now $\ddot{\vec{C}} = (*)c + (*)c' + Cv^2 c''$, so that $\ddot{\vec{C}} \cdot n = Cv^2 c'' \cdot n = Ckv^2$.

   (c) By assumption $\vec{C} = \vec{R} - \vec{r}$, where the radial vectors $\vec{R}, \vec{r}$ solve the Kepler problem. Note that n is perpendicular to $\vec{C}$, so that $0 = \vec{C} \cdot n \Rightarrow \vec{R} \cdot n = \vec{r} \cdot n$. Hence $\ddot{\vec{C}} \cdot n = (\ddot{\vec{R}} - \ddot{\vec{r}}) \cdot n = -\frac{\vec{R} \cdot n}{R^3} + \frac{\vec{r} \cdot n}{r^3} = \vec{r} \cdot n (\frac{1}{r^3} - \frac{1}{R^3})$.

6. (a) Let p be the interior point to the sphere, S^2, centered at C.

   Any line through p intersects the sphere in two points x_1, x_2. The idea is that the force from the sphere due to these two 'sides' cancels out. One may write this in symbols using solid angle.

   First, $F_{tot} = \int \sigma \frac{p - x_1}{|px_1|^3}\, dA = \int \sigma \frac{p - x_1}{|px_1|} \frac{d\Omega_p}{\cos \alpha_1}$, where $\sigma = cst.$ is the density, $d\Omega_p$ is solid angle from p, and α_1 is the angle between Cx_1 and px_1.

   On the other hand, $F_{tot} = \int \sigma \frac{p - x_2}{|px_2|} \frac{d\Omega_p}{\cos \alpha_2}$, where α_2 is the angle between Cx_2 and px_2.

   As x_1, x_2 lie on a chord through p, we have $\alpha_1 = \alpha_2$ and $\frac{p - x_1}{|px_1|} = -\frac{p - x_2}{|px_2|}$, so $F_{tot} = -F_{tot} \Rightarrow F_{tot} = 0$.

   (b) Consider a point p in the interior of the solid ball, say of radius R. Let ρ be the constant density of the ball and r the distance from p to the center of the ball. From part (a), the forces on p due to the spherical shells of radius between r and R is zero. From class, we know that the force due to the spherical shells of radius less than r is directed towards the center of the ball with strength proportional to the mass over distance to the center squared. In symbols, the magnitude due to these interior shells is: $\rho \frac{4\pi r^3}{3r^2} = cst.r$, so indeed the force is directed towards the center of the sphere with a strength proportional to the distance to the center.

7. We measure angular momentum from a point located on the rotation axis. By d'Alembert's principle, the acceleration of a free particle is always orthogonal to the surface. In particular, $\ddot{\vec{q}}$ lies in the plane containing $\vec{q}$ and the axis of rotation, so that $\vec{q} \times \ddot{\vec{q}}$ is perpendicular to the axis, $\hat{k}$ of rotation.

   So, we have $\frac{d}{dt}(\vec{q} \times \dot{\vec{q}}) \cdot \hat{k} = (\vec{q} \times \ddot{\vec{q}}) \cdot \hat{k} = 0$.

   If there is a uniform gravitational force parallel to the axis $\hat{k}$, then again by d'Alembert, we have that $\ddot{\vec{q}} - g\hat{k}$ is perpendicular to the surface, and still have $\ddot{\vec{q}}$ contained in the $\vec{q}, \hat{k}$ plane so that the $\hat{k}$-component of angular momentum is still conserved.

8. We have shown in hw1, that the involute (evolvente) of the cycloid is another cycloid. Let's for simplicity consider a unit cycloid, parametrized as $(\theta + \sin\theta, -1 - \cos\theta)$ (see figures).

   We have the relations $ds = 2\cos\frac{\theta}{2}\, d\theta$ and $s = 4 \sin \frac{\theta}{2}$ (note we are using signed arc-length).

   The particle is subject to acceleration $(0, -g)$ of which only the tangential component to the cycloid effects the motion. To find this tangential component, we compute the projection: $\frac{dc}{ds} \cdot (0, -g) = \frac{d\theta}{ds}(1 + \cos\theta, \sin\theta) \cdot (0, -g) = -\frac{g \sin\theta}{2\cos\frac{\theta}{2}} = -g \sin\frac{\theta}{2} = -\frac{g}{4}s$, so our equation of motion is:

   $\ddot{s} = -\frac{g}{4}s$.

9. Let $\gamma_E = \partial D_E$ be the closed curve of energy E enclosing the region D_E. Consider the vector field $v = \frac{\nabla E}{|\nabla E|^2}$, with flow ϕ_v^s. Note that $\frac{d}{ds}E(\phi_v^s(p)) = 1$ for any point p in the plane, so that the flow of v increases the energy of a point at constant rate: $E(\phi_v^s(p)) = E(p) + s$.

   As $A(E) = \int_{D_E} dA$, we have $\frac{dA}{dE} = \frac{d}{ds}|_{s=0} \int_{\phi_v^s(D_E)} dA = \int_{D_E} div(v)\ dA$. By the planar divergence theorem, $\frac{dA}{dE} = \int_{\gamma_E} v \cdot n\ ds$. Note that v is normal to the curve, so that $v \cdot n = |v| = \frac{1}{|\nabla E|}$. Also for $\gamma_E(t)$ parametrized with $t \in [0, T]$, we have $\dot\gamma_E = (v, U'(x))$ so that $ds = |\dot\gamma_E| dt = |\nabla E| dt$.

   Now, $\frac{dA}{dE} = \int_0^T |v|\ ds = \int_0^T dt = T$.

10. (a) We just translate the condition that the particles are ordered $0 \le x \le y$ to the new coordinates by multiplying by $\sqrt{M} > 0$, to get: $0 \le \sqrt{M}x \le Y$ or $0 \le \sqrt{\frac{M}{m}}X \le Y$.

    (b) We have $U = \sqrt{m}u, V = \sqrt{M}v$.

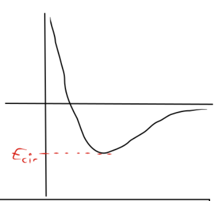
    A collision with the wall replaces $u \to -u$ and so as well $U \to -U$. This is equivalent to reflecting $\vec{v}$ over the Y-axis.

    At a collision between the particles, observe that $|\vec{v}|^2 = mu^2 + Mv^2$ is the Kinetic energy. So the new velocity, $\vec{v}'$ has the same norm as the previous $\vec{v}$, that is $|\vec{v}| = |\vec{v}'|$ so they lie on a common circle. Next, note that the direction of the collision line, $Y = \sqrt{\frac{M}{m}}X$, is $(\sqrt{m}, \sqrt{M}) = \vec{m}$ and that $\vec{m} \cdot \vec{v} = mu + Mv$ is the linear momentum as well as being a constant multiple of the component of $\vec{v}$ along the collision line. So conservation of linear momentum $\vec{m} \cdot \vec{v} = \vec{m} \cdot \vec{v}'$ means that the components of $\vec{v}$ and $\vec{v}'$ have the same projections onto the collision line. Hence $\vec{v}', \vec{v}$ are related by reflection over the collision line.

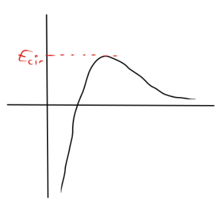
    (c) It suffices to count the number of intersections of the straight line $(X_o, Y_o) + t\vec{v}$ with the sides of the sectors $0 \le \sqrt{\frac{M}{m}}X \le Y$ reflected over the half plane $X \ge 0$.

    The interior angle of these sectors, α, satisfies: $\tan\alpha = \sqrt{\frac{m}{M}}$. Let $n \in N$ be such that $n\alpha < \pi \le (n+1)\alpha$. Then there are $n = \lfloor \frac{\pi}{\alpha} \rfloor$ collisions.

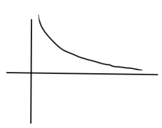
#2 :



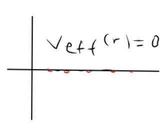
effective potential with $0 < \alpha < 2$ and $C \neq 0$.



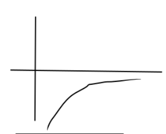
effective potential with $2 < \alpha$ and $C \neq 0$.



effective potential when $\alpha = 2$ and $|C| > 1$

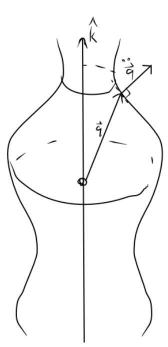


$\alpha = 2$ and $C = \pm 1$, the effective potential graph is the 'x − axis'



$\alpha = 2$ and $|C| < 1$.

#7 :



#8 :