Hw3

1. (a) We use product rule and the equations of motion:
   $\dot{C} = -S\dot{\Omega} - \dot{\Omega}S = \Omega^2 S - S\Omega^2 = \Omega(\Omega S + S\Omega) - (S\Omega + \Omega S)\Omega = C\Omega - \Omega C.$
   From $gC = cg$ we get $g\dot{C} + \dot{g}C = \dot{c}g + c\dot{g}$. Since $g^{-1}\dot{g} = \Omega$, we have $c\dot{g} = gCg^{-1}\dot{g} = gC\Omega$, hence:
   $g^{-1}\dot{c}g + C\Omega = \dot{C} + \Omega C$, or $g^{-1}\dot{c}g = \dot{C} + [\Omega, C] = 0 \Rightarrow \dot{c} = 0$.
   (b) Using $tr(AB) = tr(BA)$, and $\mathbb{I}_B\dot{\Omega} = \Omega^2 S - S\Omega^2$, we get:
   $\dot{K} = tr(\Omega\mathbb{I}_B\dot{\Omega}) = tr(\Omega(\Omega^2 S - S\Omega^2)) = tr(\Omega^3 S) - tr(\Omega S\Omega^2) = tr(\Omega^3 S) - tr(\Omega^3 S) = 0.$
   (c) Let $\vec{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$ in an orthonormal diagonalizing basis of $\mathbb{I}_B$. Take $x = I_1\Omega_1, y = I_2\Omega_2, z = I_3\Omega_3$. Then:
   $K = \sum I_j\Omega_j^2 = \frac{x^2}{I_1} + \frac{y^2}{I_2} + \frac{z^2}{I_3}$, so that $K = 1$ is an ellipsoid in x, y, z coordinates and
   the level sets of $|\vec{C}|^2 = \sum I_j^2\Omega_j^2 = x^2 + y^2 + z^2$, are spheres.
   (see end of lecture 9, for a sketch of the intersections of the $K = 1$ ellipsoid with some of these angular momentum spheres).

2. (a) One may consider free motion –geodesics– on the sphere. These are extremals of the kinetic energy Lagrangian and parametrize great circles. Antipodal points are connected by many extremals.
   (b) One may consider straight lines in the punctured plane, $\mathbb{R}^2\backslash(0,0)$, which are extremals of the length functional. The points $\pm q$ for $q \neq (0,0)$ are not connected by any extremal.

3. First recall a general formula. Let $\gamma_s \in \Gamma$ with $\gamma_0 = \gamma$ be a variation of γ. Then:

$$(*) \quad \frac{d}{ds}\Big|_{s=0} \int_0^1 L(\gamma_s, \dot{\gamma}_s)\, dt = \int_0^1 \left(\partial_q L(\gamma, \dot{\gamma}) - \frac{d}{dt}\partial_v L(\gamma, \dot{\gamma})\right) \cdot \delta\gamma\, dt + \partial_v L(\gamma, \dot{\gamma}) \cdot \delta\gamma\Big|_{t=0}^{t=1}$$

   where $\delta\gamma(t) := \frac{d}{ds}|_{s=0}\gamma_s(t)$.
   (a) Suppose γ is an extremal over this class of 1-periodic loops: the first variation, $(*)$, vanishes for any $\delta\gamma$'s arising from variations of γ by such loops.

   In particular, we may take any variation γ_s with fixed endpoints, $\gamma_s(0) = \gamma(0), \gamma_s(1) = \gamma(1)$. For such choice of variation, the boundary terms in $(*)$ vanish and so, as we argued in class, γ must satisfy the Euler-Lagrange equations for L.

   So the integral term in $(*)$ is always zero. Now consider a general variation γ_s of γ by such loops. Then $\gamma_s(0) = \gamma_s(1)$ and by differentiating at $s = 0$ we have: $\delta\gamma(0) = \delta\gamma(1)$. For γ to be an extremal we then have the boundary condition: $(\partial_v L(\gamma(1), \dot{\gamma}(1)) - \partial_v L(\gamma(0), \dot{\gamma}(0))) \cdot \delta\gamma(0) = 0$ for any $\delta\gamma(0) \in \mathbb{R}^2$. Hence the extremal γ satisfies in addition to the Euler-Lagrange equations that:

$$\partial_v L(\gamma(1), \dot{\gamma}(1)) = \partial_v L(\gamma(0), \dot{\gamma}(0)).$$

   **Note in the 'mechanical case' $L = \frac{|v|^2}{2} + U(q)$, we have $\partial_v L = v$, so that this boundary condition means that γ be a smooth loop: $\gamma(0) = \gamma(1)$ *and* $\dot{\gamma}(0) = \dot{\gamma}(1)$.

   (b) Suppose γ is an extremal in this class of curves connecting ℓ_0 to ℓ_1. As in part (a), we may still take any fixed endpoint variation, γ_s, of γ to conclude that the integral term in $(*)$ vanishes, that is γ satisfies the Euler-Lagrange equations.

   We consider now the boundary terms arising from a general variation γ_s of γ by curves connecting the given lines. Then $\gamma_s(0) \in \ell_0, \gamma_s(1) \in \ell_1$ and differentiating at $s = 0$ gives $\delta\gamma(0)$ is tangent to ℓ_0 and $\delta\gamma(1)$ is tangent to ℓ_1.

   From a variation of the endpoint in ℓ_0 and that γ be an extremal we obtain: $\partial_v L(\gamma(0), \dot{\gamma}(0)) \cdot \delta\gamma(0) = 0$ for any $\delta\gamma(0)$ tangent to ℓ_0. So the extremal satisfies –in addition to the E-L eqs– that:

$$\partial_v L(\gamma(0), \dot{\gamma}(0)) \text{ is perpendicular to } \ell_0,$$

and likewise, $\partial_v L(\gamma(1), \dot\gamma(1))$ is perpendicular to ℓ_1.

**Note in the 'mechanical case' $L = \frac{|v|^2}{2} + U(q)$, we have $\partial_v L = v$, so this boundary condition means that the extremals are perpendicular to the lines at their endpoints: $\dot\gamma(0) \perp \ell_0, \dot\gamma(1) \perp \ell_1$.

4. The Lagrangian in spherical coordinates is:

$$L = m\ell^2 \frac{\dot\varphi^2 + \sin^2\varphi\ \dot\theta^2}{2} + mg\ell(\cos\varphi - 1)$$

we take units with $m = \ell = g = 1$. There are two constants of motion, angular momentum along the 'z-axis' and energy:

$$C = \sin^2\varphi\ \dot\theta, \quad E = \frac{\dot\varphi^2}{2} + \frac{C^2}{2\sin^2\varphi} + 1 - \cos\varphi.$$

Consider first the case $C = 0$. Then either $\dot\theta = 0$ or $\varphi = 0, \pi$. If $\dot\theta = 0$, then the motion of φ is that of the planar pendulum, $\ddot\varphi = -\sin\varphi$ along the great circle $\theta = cst.$. If $\varphi = 0, \pi$ then we are at the lower stable equilibrium or upper unstable equilibrium resp.

When $C \neq 0$, consider the effective potential $V_{eff}(\varphi) = \frac{C^2}{2\sin^2\varphi} + 1 - \cos\varphi$. A motion with energy E has $E \geq V_{eff}(\varphi)$ with equality if and only if $\dot\varphi = 0$.

See figures for a graph of V_{eff}. Note the minimum of the effective potential occurs at φ_c satisfying $C^2 \cos\varphi_c = \sin^4\varphi_c$. In particular $\cos\varphi_c > 0$, so that $\varphi_c \in (0, \frac{\pi}{2})$.

So, for fixed $C > 0$, as the energy E is raised the orbits of the spherical pendulum are (see figures):

1. $\varphi(t) = \varphi_c$, and $\theta(t)$ increasing. The pendulum spins around on a latitude,

2. $\varphi(t)$ oscillates between φ_m and φ_M, and $\theta(t)$ is increasing. The pendulum bobs up and down in a 'band'. We always have $\varphi_m < \varphi_c < \frac{\pi}{2}$, while for high enough energy we have $\varphi_M > \frac{\pi}{2}$.

5. (a) Let P be a point on the interface of the two media (the x-axis), A a point in the upper half plane and B a point in the lower half plane. The time along a ray from A to B which crosses the x-axis at P is:
$\frac{|AP|}{v_1} + \frac{|PB|}{v_2} = T(P) = T(x)$.

We compute that $\frac{d}{dx}|AP|^2 = 2(P - A) \cdot (1, 0) = 2|AP|\frac{d}{dx}|AP|$, so that $\frac{d}{dx}|AP| = \frac{(P-A)}{|PA|} \cdot (1, 0)$.

Likewise $\frac{d}{dx}|PB| = \frac{(P-B)}{|PB|} \cdot (1, 0)$.

Note $\frac{(P-A)}{|PA|} \cdot (1, 0) = \cos\theta_1' = \sin\theta_1$ where $\theta_1' + \theta_1 = \frac{\pi}{2}$ for θ_1 the angle from the vertical and likewise, $\frac{(P-B)}{|PB|} \cdot (1, 0) = -\sin\theta_2$.

So the critical points of $T(x)$ are at: $0 = \frac{d}{dx}T(x) = \frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2}$, giving Snell's law.

(b) Note that the normal to the graph has direction $N = (\frac{dy}{dx}, -1)$ and the angle from the vertical, $\theta(x)$, has $\sin\theta(x) = \frac{N}{|N|} \cdot (0, -1) = \frac{1}{\sqrt{1+(y')^2}}$.

Hence, the condition $\frac{v}{\sin\theta} = cst.$, with $v = \sqrt{y}$ may be written as the ode, $y(1 + (y')^2) = k^2$ or $(y')^2 = \frac{k^2 - y}{y}$, the same ode we solved to get the cycloid in lecture 11.

6. Using the Lagrange multiplier method for parametrized curves giving the chain $(x(s), y(s))$ with the constraint $\dot x^2 + \dot y^2 = 1$ parametrized from 0 to ℓ by arc-length, we seek extremals of:

$$\int_0^\ell \rho g y + \lambda(\dot x^2 + \dot y^2)\ ds.$$

The 'loaded chain condition' means that the weight on the chain is proportional to the distance x, $\int_0^s \rho(s)\, ds = \sigma x(s)$, i.e. $\rho = \sigma \dot{x}$, where σ is some constant.

The Euler-Lagrange equations give: $\lambda \dot{x} = k = cst.$ and $2\frac{d}{ds}\lambda \dot{y} = \rho g = g\sigma \dot{x}$. Substituting $\lambda = \frac{k}{\dot{x}}$ and using $y' = \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$, we get:

$\sigma g \dot{x} = 2k\frac{d}{ds}\frac{dy}{dx} = 2k\frac{dx}{ds}\frac{d^2y}{dx^2} = 2k\dot{x}y''$

so that $y'' = cst.$, and the loaded chain has the shape of a parabola.

7. The area of the surface of revolution is given by $A(f) = 2\pi \int_{z_0}^{z_1} f\sqrt{1+(f')^2}\, dz$. Since the Lagrangian is 'autonomous' depending only on f, f', we have that over extremals:

$$cst. = f'\partial_{f'}L - L = -\frac{f}{\sqrt{1+(f')^2}}$$

or $(f')^2 + 1 = \frac{f^2}{k^2}$ for some constant k, which is the same ode as when we solved for the catenary in lecture 11. So the profile of these extremal area surfaces of revolution are catenary curves.

**there was some careless phrasing in this question, with the condition that f be a diffeomorphism (the way I was imagining the question should have this condition omitted). One would need to determine which catenary curves from z_0, z_1 to y_0, y_1 are diffeomorphisms, which only gives lines, and so cones as the extremal surfaces of revolution.

8. See figures. To make the sketches, we use the 'Clairaut integral' (derived from the angular momentum along the axis of revolution, also see Arnold pg. 86).

Consider a (local) parametrization of a general surface of revolution as $(r\cos\theta, r\sin\theta, z(r))$, where r is the distance from the axis of revolution. The angular momentum along this axis is $C = r^2\dot{\theta} = cst.$ as may also be seen by considering the Euler-Lagrange equations of $L = \frac{(1+z'(r)^2)\dot{r}^2 + r^2\dot{\theta}^2}{2}$.

Now consider a free motion along this surface, $q(t)$ having say unit speed, $|\dot{q}| = 1$. Then $\dot{q} = \dot{r}\partial_r + \dot{\theta}\partial_\theta$, where ∂_r and ∂_θ are orthogonal and $\partial_\theta = (-r\sin\theta, r\cos\theta, 0)$. The Clairaut integral comes from writing:

$$C = \dot{q}\cdot\partial_\theta = |q||\partial_\theta|\cos\beta = r\cos\beta$$

where β is the angle between the velocity $\dot{q}$ and the 'horizontal' or longitudinal direction ∂_θ. It is often written in the equivalent form

$$C = r\sin\alpha$$

where α is the angle between the velocity $\dot{q}$ and the 'vertical' or latitude direction ∂_z.

This Clairaut form of writing the angular momentum is useful for sketching the free motions (geodesics). It says that for an orbit with momentum C fixed, as a geodesic moves further from the axis of rotation (r increases), then the geodesic gets 'more vertical' (α gets nearer 0 or π).

Moreover, for certain values of C, the condition $|C| = r|\sin\alpha| \leq r$ may constrain the motion to 'bands' $r \geq C$. Note that equality occurs only when $\alpha = \frac{\pi}{2}$, that is the velocity is horizontal.

Returning to the torus, one may proceed as follows. Fix $E = 1$ –unit speed motions– as other energy values just correspond to reparametrizations. We will vary C and describe possible motions. Note for $C > 0$ that θ is always increasing, i.e. the geodesics are always 'winding' around the torus. Let r_m be the distance from the interior ring of the torus to the axis and r_M the outer rings distance to the axis.

1. when $C = 0$, then since r is never zero, we have $\alpha = 0, \pi$, i.e. the motions are along the latitudes.

2. with $C \in (0, r_m)$, we obtain geodesics winding around the whole torus–they may be periodic or not depending on the value C.

3. for $C = r_m$, we have the inner ring of the torus as a geodesic and those asymptotic towards this interior ring.

4. for $C \in (r_m, r_M)$, the geodesics are constrained to 'bands'.

5. for $C = r_M$, we obtain just the outer ring of the torus as a geodesic. It is not possible to have unit speed motions with $C > r_M$.

9. Let $T(E) = T$ be the period of this elliptic orbit with energy E. Then its action is:

$$\int_0^T \frac{|\dot{q}|^2}{2} + \frac{1}{|q|}\ dt = \int_0^T E + \frac{2}{|q|}\ dt = ET + 2\int_0^T \frac{dt}{|q|}.$$

To evaluate the integral term, we parametrize by 'eccentric anomaly' $u \in [0, 2\pi]$ (end of lecture 6). Recall that $|q| = a(1 - e\cos u)$ and $a^{-3/2}t = u - e\sin u$ so that

$$dt = a^{3/2}(1 - e\cos u)\ du = a^{1/2}|q|\ du,$$

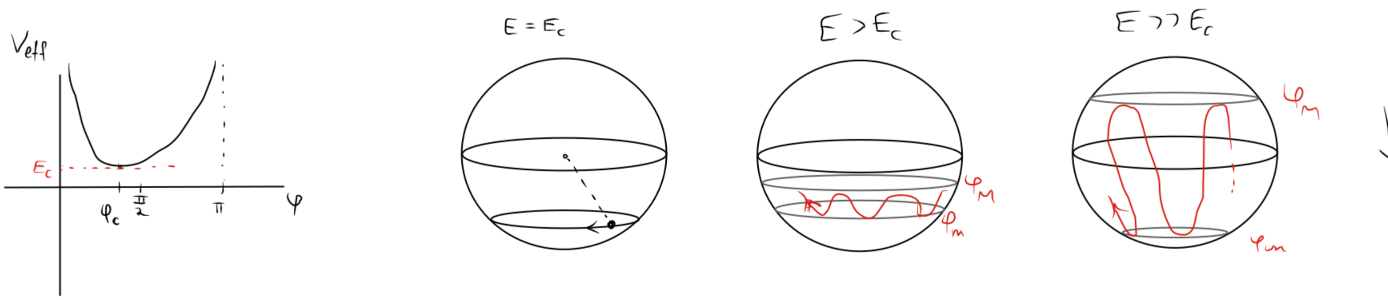
where a is the major axis of the orbit. Hence the action is:

$$(*) \quad ET + 2\int_0^{2\pi} \frac{a^{1/2}|q|}{|q|}\ du = ET + 4\pi a^{1/2}.$$

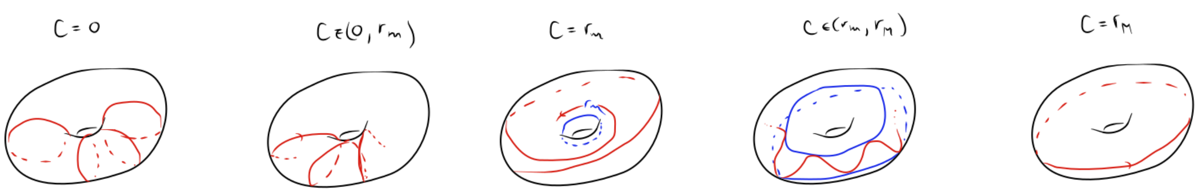
We may rewrite $(*)$ to depend only on the energy using $E = -\frac{1}{2a}$ and $T^2 = 4\pi^2 a^3$ (K3), to get:

$$\frac{3\pi}{\sqrt{-2E}} = 3\pi\sqrt{a} = \frac{3\pi}{(2\pi)^{1/3}} T^{1/3}$$

for the action over an elliptic orbit with energy $E < 0$, major axis a and period T.



#4 : *Effective potential for the spherical pendulum and some rough sketches of orbits.*



#8 : *some geodesics on a torus of revolution.*