# Hw4

1. (a) We can make the same computation using integration by parts as we did for the autonomous case. Let $\gamma$ be an extremal and $\gamma_s$ a fixed endpoint variation, with $\delta\gamma := \frac{d}{ds}|_{s=0}\gamma_s$. Then:

$$0 = \frac{d}{ds}|_{s=0} A(\gamma_s) = \int_0^T \partial_q L \cdot \delta\gamma + \partial_v L \cdot \delta\dot{\gamma}\ dt = \int_0^T (\partial_q L - \frac{d}{dt}\partial_v L) \cdot \delta\gamma\ dt + \partial_v L \cdot \delta\gamma|_0^T.$$

As before, the boundary terms are zero since the endpoints are fixed. This equality holds for any vector field $\delta\gamma$ along the curve vanishing at the endpoints, implying that the Euler-Lagrange equations hold over the extremal.

Now, over an extremal $\gamma$, we have:

$$\frac{d}{dt}E = \frac{d}{dt}\partial_v L \cdot \dot{\gamma} + \partial_v L \cdot \ddot{\gamma} - \partial_q L \cdot \dot{\gamma} - \partial_v L \cdot \ddot{\gamma} - \partial_t L = -\partial_t L,$$

since $\frac{d}{dt}\partial_v L = \partial_q L$.

**note there was a sign error here in the problem set**

(b) We have $v = \dot{q} = e^{i\omega t}(\dot{Q} + i\omega Q) = e^{i\omega t}(V + i\omega Q)$, hence

$L = \frac{|V+i\omega Q|^2}{2} + U(|Q|) = \frac{|V|^2+\omega^2|Q|^2}{2} + \omega V \cdot iQ + U(|Q|)$

$E = (V + i\omega Q) \cdot V - L = \frac{|V|^2-\omega^2|Q|^2}{2} - U(|Q|)$.

Note that the Euler-Lagrange equations for $L(Q, V)$ are the equations of motion in a rotating frame: $\ddot{Q} = -2i\omega Q + \omega^2 Q + \partial_Q U$ (as they should be). Moreover since $L(Q, V)$ has no time dependence the energy $E$ is still a constant of motion in these rotating coordinates (due to the fact that the potential is rotation invariant).

2. (a) We have, $S(s_0, s_1)^2 = |\gamma(s_0) - \gamma(s_1)|^2$, so that $\partial_{s_0} S^2 = 2S\partial_{s_0} S = 2(\gamma(s_0) - \gamma(s_1)) \cdot \gamma'(s_0)$. Hence

$$\partial_{s_0} S = \frac{\gamma(s_0) - \gamma(s_1)}{|\gamma(s_0) - \gamma(s_1)|} \cdot \gamma'(s_0) = \cos\theta_0$$

where $\theta_0$ is the angle between $\gamma'(s_0)$ and $\gamma(s_0) - \gamma(s_1)$. Comparing with the figure, we see that $\theta_0 = \pi - \varphi_0$ so that indeed. $\partial_{s_0} S = -\cos\varphi_0$. Likewise, we get $\partial_{s_1} S = \cos\varphi_1$.

(b) Observe that at a maximum of $A$, no two points coincide since the distance may always be increased from zero. In particular at a maximum, $A$ is differentiable and this is a critical point. At each vertex, $\gamma(s_j)$ of this critical point, let $\varphi_j, \varphi_j'$ be the 'ingoing' and 'outgoing' angles of incidence. Then setting the partials of $A$ to zero gives the conditions:

$$\partial_{s_j} S = -\cos\varphi_j + \cos\varphi_j' \Rightarrow \varphi_j = \varphi_j'$$

meaning that the critical point of $A$ satisfies the reflection rules for a billiard trajectory.

3. (a) Using $v(t) \le c(t) + R(t)$, we have $\frac{dR}{dt} = u(t)v(t) \le u(t)c(t) + u(t)R(t)$ or $\frac{dR}{dt} - u(t)R(t) \le u(t)c(t)$.

**note! here we assume $u$ is a positive function, so that the inequality does not 'flip' when multiplying by $u$ – this was not properly stated in the problem set (thanks Fernando for catching this)**

(b) Let $\mu > 0$ satisfy $\mu' = -\mu u$. Then multiplying the inequality of (a) by $\mu$ gives: $\frac{d}{dt}(\mu R) \le \mu uc$.

(c) For given $t \in [0, T]$, integrating both sides of (b) from 0 to $t$ gives:

$\mu(t)R(t) - \mu(0)R(0) \le -\int_0^t \mu'(s)c(s)\ ds = -c(s)\mu(s)|_0^t + \int_0^t c'(s)\mu(s)\ ds$.

For our integrating factor, $\mu$, we take the solution of $\mu' = -\mu u$ with $\mu(t) = 1$, that is:

$\mu(s) = \exp\left(-\int_t^s u(\tau)\ d\tau\right) = \exp\left(\int_s^t u(\tau)\ d\tau\right)$. Then since $R(0) = 0$, we have:

$R(t) \le c(0)\mu(0) - c(t) + \int_0^t c'(s)\mu(s)\ ds \Rightarrow R(t) + c(t) \le c(0)\exp\left(\int_0^t u(s)\ ds\right) + \int_0^t c'(s)\exp\left(\int_s^t u(\tau)\ d\tau\right)\ ds.$

Finally, we use that $v(t) \le c(t) + R(t)$ to get the Gronwall lemma we used in class.

4. First recall that given an inner product space $V, \langle\cdot,\cdot\rangle$, and a symmetric linear map $L : V \to V$ then there exists an orthonormal basis of $V$ of eigenvectors of $L$ with real eigenvalues.

   Taking this for granted, we proceed to the setting of the problem. Let $V$ be a vector space with dual $V^*$. We write $(\nu, v) := \nu(v) \in \mathbb{R}$ for the natural pairing of $\nu \in V^*$ with $v \in V$.

   A linear map $L : V \to V^*$ is symmetric when $(Lu, v) = (Lv, u), \quad \forall u, v \in V$ (meaning that $L = L^*$ upon identifying $V^{**} = V$). Such a symmetric map is positive definite when $(Lv, v) > 0, \quad \forall v \neq 0$.

   Now, given two symmetric maps $A, B : V \to V^*$, with $A$ positive definite, we may define an inner product on $V$ by $\langle u, v\rangle := (Au, v)$.

   With respect to this inner product on $V$, the operator $A^{-1}B$ is symmetric: $\langle A^{-1}Bu, v\rangle = (Bu, v) = (Bv, u) = \langle u, A^{-1}Bv\rangle$.

   By the theorem we recalled at the start, there exists a $\langle\cdot,\cdot\rangle$ orthonormal basis $e_1, ..., e_n$ of $V$ consisting of eigenvectors of $A^{-1}B$ with real eigenvalues, $\lambda_j \in \mathbb{R}$.

   That this basis of $V$ is $\langle\cdot,\cdot\rangle$ orthonormal, means $\delta_{jk} = (Ae_j, e_k)$, i.e. $Ae_j = e_j^*$, where $e_j^*$ is the dual basis of $V^*$ corresponding to $e_j$.

   Hence, from $A^{-1}Be_j = \lambda_j e_j$, we have $Be_j = \lambda_j e_j^*$.

   Now, let $P : \mathbb{R}^n \to V$ be coordinates from the basis $e_j$ of $V$, $P(v^1, ..., v^n) := v^1 e_1 + ... + v^n e_n$. Then the dual $P^* : V^* \to \mathbb{R}^n$ are the coordinates corresponding to the dual basis $e_j^*$ of $V^*$, $P^*(\nu^1 e_1^* + ... + \nu^n e_n^*) = (\nu^1, ..., \nu^n)$. In these bases, the linear maps $A, B$ are represented by:

   $P^*AP = id, \quad P^*BP = D$ where $D$ has diagonal entries $\lambda_1, ..., \lambda_n$.

   Note that upon giving $V$ a general basis (so that $V^*$ becomes equipped with the corresponding dual), then $P, P^*$'s matrix representations are given as $P, P^T$.

   Finally, the $\lambda_j$ are the roots of $0 = \det(D - \lambda I)$, since $\det P^{-1} = \det(P^*)^{-1} \neq 0$, multiplying on the left and right preserves these roots, so $\lambda_j$ are the roots of $0 = \det\big((P^*)^{-1}DP^{-1} - \lambda(P^*)^{-1}P^{-1}\big) = \det(B - \lambda A)$.

5. We set up our coordinates with a vertical $x$-axis, so that $q_1 = e^{i\theta_1}, q_2 = e^{i\theta_1} + e^{i\theta_2}$.

   The Kinetic energy is: $2K = |\dot{q}_1|^2 + |\dot{q}_2|^2 = |i\dot{\theta}_1 e^{i\theta_1}|^2 + |i\dot{\theta}_1 e^{i\theta_1} + i\dot{\theta}_2 e^{i\theta_2}|^2 = 2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 + \theta_2)$

   Upto a constant, the potential is $V = -2\cos\theta_1 - \cos\theta_2$.

   The Lagrangian in these coordinates is then: $L = \dot{\theta}_1^2 + \frac{\dot{\theta}_2^2}{2} + \dot{\theta}_1\dot{\theta}_2\cos(\theta_1 + \theta_2) + 2\cos\theta_1 + \cos\theta_2$

   The equilibrium points are critical points of $V$, when $\theta_j = 0, \pi$. There are four combinations, the linearized Lagrangian for each choice is:

   1) $\theta_j = 0$, $L_1 = A_1\dot{\theta}\cdot\dot{\theta} + B_1\theta\cdot\theta$, where $A_1 = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}, B_1 = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$. The behaviour of small oscillations around this equilibrium point is governed by the eigenvalues

   $0 = \det(B_1 - \lambda A_1) = \begin{vmatrix} -2-\lambda & -1/2 \\ -1/2 & -1-\lambda/2 \end{vmatrix}$

   which are $\lambda = -2 \pm \sqrt{1/2}$, in particular both negative. The small oscillations around this equilibrium point are stable.

   2) $\theta_1 = 0, \theta_2 = \pi$, $L_2 = A_2\dot{\theta}\cdot\dot{\theta} + B_2\theta\cdot\theta$, where $A_2 = \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}, B_2 = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$. The behaviour of small oscillations around this equilibrium point is governed by the eigenvalues

   $0 = \det(B_2 - \lambda A_2) = \begin{vmatrix} -2-\lambda & 1/2 \\ 1/2 & 1-\lambda/2 \end{vmatrix}$

   which are $\lambda = \pm\sqrt{9/2}$, one positive one negative. In the negative eigenspace the small oscilations are unstable.

3) $\theta_1 = \pi, \theta_2 = 0$, $L_3 = A_3\dot\theta \cdot \dot\theta + B_3\theta \cdot \theta$, where $A_3 = \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}, B_3 = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$. The behaviour of small oscillations around this equilibrium point is governed by the eigenvalues

$0 = \det(B_3 - \lambda A_3) = \begin{vmatrix} 2-\lambda & 1/2 \\ 1/2 & -1-\lambda/2 \end{vmatrix}$

which are $\lambda = \pm\sqrt{9/2}$, one positive one negative. In the negative eigenspace the small oscilations are unstable.

4) $\theta_j = \pi$, $L_4 = A_4\dot\theta \cdot \dot\theta + B_4\theta \cdot \theta$, where $A_4 = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}, B_4 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. The behaviour of small oscillations around this equilibrium point is governed by the eigenvalues

$0 = \det(B_4 - \lambda A_4) = \begin{vmatrix} 2-\lambda & -1/2 \\ -1/2 & 1-\lambda/2 \end{vmatrix}$

which are $\lambda = 2 \pm \sqrt{1/2}$, both positive. The small oscilations are unstable.

6. We seek transformations of the plane $q \mapsto f(q)$, whose lifts, $v \mapsto df_q v$, preserve the Lagrangian $L = \frac{|v|^2-|q|^2}{2}$. Hence, $f$ must preserve the norm $|q| = |f(q)|$, and the velocity term must remain independent of $q$, i.e. $f$ must be a norm preserving linear map, that is, $f$ is a rotation. The conserved quantity associated to the rotational symmetry is the angular momentum, its symmetry vector field is $X = (y, -x) = iq$ and by Noether's theorem:

   $\partial_v L \cdot X = v \cdot iq$ (the angular momentum) is conserved.

7. (a) In complex notation, the Lagrangian is: $L = \frac{|\dot z|^2}{Im(z)^2}$. Let $Z = A \cdot z = \frac{az+b}{cz+d}$. We compute:

   $\dot Z = \frac{(ad-bc)\dot z}{(cz+d)^2} = \frac{\dot z}{(cz+d)^2}$

   $Im(Z) = \frac{Im(az+b)(c\bar z+d)}{|cz+d|^2} = \frac{(ad-bc)Im(z)}{|cz+d|^2} = \frac{Im(z)}{|cz+d|^2}$,

   so indeed $L(z, \dot z) = \frac{|\dot z|^2}{Im(z)^2} = \frac{|\dot Z|^2}{Im(Z)^2} = L(Z, \dot Z)$, and these are symmetries.

   (b) Acting by $A = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{R})$ represents the symmetry by translations along the $x$-axis. It is generated by the vector field $X = (1, 0)$. By Noether, the corresponding first integral is: $\partial_v L \cdot X = \frac{\dot x}{y^2}$.

   Consider a trajectory with $\dot x = cy^2$ and $1 = \frac{\dot x^2 + \dot y^2}{y^2}$. Then by chain rule $\dot y = \frac{dy}{dx}\dot x = cy^2 y'$. Substituting for $\dot x, \dot y$ into $1 = \frac{\dot x^2+\dot y^2}{y^2}$ yields:

   $\frac{cydy}{\sqrt{1-c^2y^2}} = dx$, or $\sqrt{1-c^2y^2} = c(x - x_o)$, or $(x-x_o)^2 + y^2 = \frac{1}{c^2}$.

   The orbits are thus the upper half ($y > 0$) of circles with centers lying on the $x$-axis, or (when $c = 0$) vertical lines $x = cst$.

   **an alternate method is to find one solution of the Euler-Lagrange solutions (for example a vertical line), and then determine its image under the symmetries of (a) to obtain all solutions.**

8. The idea is to find a sequence of functions whose slopes are getting large, so that the integrand $e^{-u'(x)^2}$ tends to zero. to satisfy the boundary conditions, we take a sequence of parabolas with increasing heights: $u_n(x) = nx(1-x)$. Then:

   $A(u_n) = \int_0^1 e^{-n^2(2x-1)^2}\, dx$

   with $y = n(2x-1)$, we have $A(u_n) = \frac{1}{2n}\int_{-n}^{n} e^{-y^2}\, dy$. As $n \to \infty$, the integral term converges to the Gaussian integral: $\int_{-\infty}^{\infty} e^{-y^2}\, dy = \sqrt{\pi}$. In particular it is bounded as $n \to \infty$, so that $A(u_n) \to 0$ as $n \to \infty$.

9. (a) By the 'Eikonal' equation for distance functions, we have $|\nabla u_n|^2 = n^2$ (you may also check this directly from $u_n(q) = n(1-|q|)$).

   Hence $R(u_n) = \int_D \frac{dA}{1+n^2} = \frac{\pi}{1+n^2} \to 0$ as $n \to \infty$.

   (b) We have $|\nabla u_n|^2 = |2\pi n \sin(\pi n|q|)\cos(\pi n|q|)\nabla|q||^2 = (\pi n \sin(2\pi n|q|))^2$. Since $|\nabla\sqrt{x^2+y^2}| = 1$.

   Hence $R(u_n) = \int_D \frac{dA}{1+n^2\pi^2\sin^2(2\pi n|q|)}$. Since the integrands, $\frac{1}{1+n^2\pi^2\sin^2(2\pi n|q|)} = f_n$ are bounded, we have $\lim_{n\to\infty} R(u_n) = \lim_{n\to\infty}\int_D f_n\ dA = \int_D(\lim_{n\to\infty} f_n)\ dA = 0$, since $f_n \to 0$.

10. We make use of the computation in hw3 problem 9: the Keplerian action over one circuit of an elliptic orbit with major axis $a$ is $A = 3\pi\sqrt{a}$. Going $n$-times around such a Keplerian ellipse gives an action of $3n\pi\sqrt{a}$. As $a \to 0$, the action of going around the Keplerian ellipse $n$-times goes to zero. Hence the minimum of the Keplerian action on $\Gamma_n$ is zero (the Keplerian action is always non-negative).

    Since the action going over any Keplerian ellipse $n$-times is positive, such trajectories are not minimizers over $\Gamma_n$.

11. (a) First, we expand $\ell - 1 = \int_0^1 \sqrt{1+u_x^2}\ dx$ in $u_x$ to get:

    $\ell - 1 = \int_0^1 1 + \frac{u_x^2}{2} + O_4(u_x)\ dx - 1 = \int_0^1 \frac{u_x^2}{2} + O_4(u_x)\ dx.$

    Next, we expand $f(\ell)$ around $\ell = 1$ to get:

    $f(\ell) = f'(1)(\ell-1) + O_2(\ell-1)$, since $f(1) = 0$. Setting $k = f'(1)$, we have:

    $V = \frac{k}{2}\int_0^1 u_x^2 + O_4(u_x)\ dx$

    (this implies the statement in the problem since $O_4(u_x) \subset O_3(u_x)$).

    (b) We carry out a multivariable analogue of the argument that led us to the Euler-Lagrange equations.

    Let $u_\varepsilon(x,t)$ be a variation of $u_0(x,t) = u(x,t)$ over $t \in [t_0, t_1]$.

    As $u_\varepsilon(x,t)$ are positions of the string with fixed endpoints, we have: $u_\varepsilon(0,t) = u_\varepsilon(1,t) = 0$.

    We assume that the variation also satisfies $u_\varepsilon(x,t_0) = u(x,t_0), u_\varepsilon(x,t_1) = u(x,t_1)$, analogous to having fixed endpoints of the curve when we derived the '1-d' Euler-Lagrange equations.

    Set $\delta u(x,t) = \frac{d}{d\varepsilon}|_{\varepsilon=0} u_\varepsilon(x,t)$. By our fixed endpoint conditions, we have:

    $\delta u(0,t) = \delta u(1,t) = \delta u(x,t_0) = \delta u(x,t_1) = 0.$

    We find the analogue of the Euler-Lagrange equations for $u$ by differentiating $A(u_\varepsilon) = \int_{t_0}^{t_1} L(u_\varepsilon, u_{\varepsilon,t})\ dt$ at $\varepsilon = 0$ and requiring the expression to be zero for every $\delta u$. That is, $u$ should satisfy:

    $0 = \int_{t_0}^{t_1}\int_0^1 u_t\delta u_t - ku_x\delta u_x\ dxdt.$

    Performing two integration by parts with the orders of integration changed, we arrive at:

    $0 = \int_{t_0}^{t_1}\int_0^1 (ku_{xx} - u_{tt})\delta u\ dxdt.$

    As this holds for any $\delta u$ vanishing on the boundary of the reactangle $[0,1]\times[t_0,t_1]$, we find that such a critical point $u$ must satisfy $u_{tt} = ku_{xx}$ (the 1-d wave equation).

    (c) We seek a solution of this wave equation having the special form $u(x,t) = T(t)X(x)$. Substitution gives us the conditions:

    $XT'' = kTX'' \Rightarrow \frac{T''}{kT} = \frac{X''}{X}$. Since the left side depends only on $t$ and the right only on $x$, both sides are constant, say $-\lambda$ so that:

    $T'' = -k\lambda T, \quad X'' = -\lambda X.$

    If $\lambda \leq 0$ then – apart from the constant solution $u(x,t) = 0$ – the solutions cannot satisfy the boundary conditions of $X(0) = X(1) = 0$. So we may assume $\lambda = \omega^2 > 0$ and our solutions are given in terms of trigonometric functions (assuming $k = \kappa^2 > 0$):

    $T(t) = a\cos(\kappa\omega t) + b\sin(\kappa\omega t), \quad X(x) = A\cos(\omega t) + B\sin(\omega t)$

    where the boundar conditions $X(0) = X(1) = 0$ imply $A = 0$ and $\omega = n\pi, \quad n \in \mathbb{Z}$. Hence we have solutions:

$$u(x,t) = (a\cos(n\kappa\pi t) + b\sin(n\kappa\pi t))\sin(n\pi x)$$

for each $n \in \mathbb{Z}$ and constants $a, b \in \mathbb{R}$ (these solutions are called the normal modes for the strings vibrations, similar to our study of harmonic functions, all solutions can be expressed as linear combinations of such normal modes).