Hw5

1. Let $\Omega_{\vec{\xi}} = \frac{d}{dt}|_{t=0} R_{\vec{\xi}}(t) \in \mathfrak{so}_3$ be the infinitesimal rotation around the $\vec{\xi} \in \mathbb{R}^3$ axis, i.e. $\Omega_{\vec{\xi}}(\vec{v}) = \vec{\xi} \times \vec{v}$.

   For $g \in \mathrm{SO}_3$, by definition, $Ad_g \Omega_{\vec{\xi}} = \frac{d}{dt}|_{t=0} g R_{\vec{\xi}}(t) g^{-1} = g \Omega_{\vec{\xi}} g^{-1}$.

   Since the axis of an infinitesimal rotation is the kernel of the skew symmetric operator, we have $Ad_g \Omega_{\vec{\xi}}(g\vec{\xi}) = g\Omega_{\vec{\xi}}\vec{\xi} = 0$, so that $Ad_g \Omega_{\vec{\xi}}$ is an infinitesimal rotation about the axis $g\vec{\xi}$.

   Hence, under the identification $\mathfrak{so}_3$ with $\mathbb{R}^3$, the action of Ad_g on $\mathbb{R}^3$ is by rotation: $\vec{\xi} \mapsto g\vec{\xi}$.

2. We use that as a function of the columns say, the determinant is multilinear, so that in particular we may use product rule when differentiating.

   Let $\hat{e}_j$ be the standard basis of $\mathbb{R}^n$ and $\vec{a}_j$ be the columns of the matrix A. Then viewed as a function of the columns we have: $\det(I + tA) = \det(\hat{e}_1 + t\vec{a}_1, ..., \hat{e}_n + t\vec{a}_n)$. Product rule gives:

$$\frac{d}{dt}|_{t=0} \det(I + tA) = \det(\vec{a}_1, \hat{e}_2, ..., \hat{e}_n) + \det(\hat{e}_1, \vec{a}_2, \hat{e}_3, ..., \hat{e}_n) + ... + \det(\hat{e}_1, ..., \hat{e}_{n-1}, \vec{a}_n)$$
$$= a_{11} + ... + a_{nn} = tr(A).$$

3. (a) First, we compute in Cartesian coordinates:

   $\omega_u^1 \wedge \omega_v^2 = (u_1 dx^1 + u_2 dx^2 + u_3 dx^3) \wedge (v_1 dx^2 \wedge dx^3 + v_2 dx^3 \wedge dx^2 + v_3 dx^2 \wedge dx^3)$

   $= (u_1 v_1 + u_2 v_2 + u_3 v_3) dx^1 \wedge dx^2 \wedge dx^3 = (u \cdot v)\omega_{vol} = \omega_v^1 \wedge \omega_u^2$.

   Next, by Cartan's formula we have:

   $d(i_u i_v \omega_{vol}) = \mathscr{L}_u(i_v\omega_{vol}) - i_u(d i_v \omega_{vol}) = \mathscr{L}_u(i_v \omega_{vol})$,

   since $d(i_v\omega_{vol}) = d\omega_v^2 = div(v)\omega_{vol} = 0$, as v has divergence zero.

   Using the general formula from the differential forms notes for the Lie derivative of an interior derivative:

   $d(i_u i_v \omega_{vol}) = \mathscr{L}_u(i_v\omega_{vol}) = i_{[u,v]}\omega_{vol} + i_v \mathscr{L}_v \omega_{vol} = i_{[u,v]}\omega_{vol}$,

   since again by Cartan's formula, $\mathscr{L}_u \omega_{vol} = d i_u \omega_{vol} + i_u d\omega_{vol} = 0$ since $div(u) = 0$ and $d\omega_{vol} = 0$.

   (b) Using part a, we have:

   $\langle [u, v], w\rangle = \int_D \omega_w^1 \wedge \omega_{[u,v]}^2 = \int_D \omega_w^1 \wedge d(i_u i_v \omega_{vol})$.

   Now, by product rule for exterior derivative of wedge products:

   $\omega_w^1 \wedge d(i_u i_v \omega_{vol}) = d\omega_w^1 \wedge (i_u i_v \omega_{vol}) - d(\omega_w^1 \wedge (i_u i_v \omega_{vol}))$.

   By Stoke's, $\int_D d(\omega_w^1 \wedge (i_u i_v \omega_{vol})) = \int_{\partial D} \omega_w^1 \wedge (i_u i_v \omega_{vol}) = 0$, since u, v are tangent to ∂D, so in particular $i_u i_v \omega_{vol}|_{\partial D} = (v \times u) \cdot (*) = 0$ (since $v \times u$ is normal to ∂D).

   For the remaining term in the exterior derivative product rule, note that:

   $d\omega_w^1 = \omega_{curl(w)}^2$

   and by product rule for interior derivative:

   $0 = i_u(\omega_{curl(w)}^2 \wedge (i_v\omega_{vol})) = (i_u \omega_{curl(w)}^2) \wedge (i_v \omega_{vol}) + \omega_{curl(w)}^2 \wedge (i_u i_v \omega_{vol})$

   where we use that $\omega_{curl(w)}^2 \wedge (i_v\omega_{vol}) = 0$ since any 4-form on $\mathbb{R}^3$ is zero. That is:

   $-(i_u \omega_{curl(w)}^2) \wedge (i_v \omega_{vol}) = \omega_{curl(w)}^2 \wedge (i_u i_v \omega_{vol})$.

   Observe that $-i_u \omega_{curl(w)}^2 = curl(w) \cdot (u \times *) = -* \cdot (curl(w) \times u) \Rightarrow -i_u\omega_{curl(w)}^2 = \omega_{u \times curl(w)}^1$

   Now:

   $\langle [u, v], w\rangle = \int_D \omega_w^1 \wedge \omega_{[u,v]}^2 = \int_D \omega_{curl(w)}^2 \wedge (i_u i_v \omega_{vol}) = -\int_D (i_u \omega_{curl(w)}^2) \wedge (i_v \omega_{vol})$

   $= \int_D \omega_{u\times curl(w)}^1 \wedge \omega_v^2 = \langle u \times curl(w), v\rangle$

   (c) Using product rule, $d(\alpha\omega_v^2) = d\alpha \wedge \omega_v^2 + \alpha d\omega_v^2 = \omega_{grad(\alpha)}^1 \wedge \omega_v^2$, since $div(v) = 0$. Now:

   $\langle grad(\alpha), v\rangle = \int_D d\alpha \wedge \omega_v^2 = \int_D d(\alpha \omega_v^2) = \int_{\partial D} \alpha\omega_v^2 = 0$

   since v is tangent to ∂D so the flux of v (or αv) through ∂D is zero.

4. (a) We consider the velocity curve $\gamma(t) = e^t$ (so $\gamma(0) = x$ and $\dot{\gamma}(0) = x$). Then:
   $d_x f(x) = \frac{d}{dt}|_{t=0} f(e^t x) = \frac{d}{dt}|_{t=0} e^{\alpha t} f(x) = \alpha f(x)$.

   (b) Requiring that $q(t) = \lambda(t) q_o$ satisfy the equations of motion means:
   $\ddot{\lambda} M q_o = M\ddot{q} = \nabla_q U = \frac{\lambda}{|\lambda|^3} \nabla_{q_o} U$
   where M is the diagonal matrix weighting the positions by their masses. Rearranging we have:
   $\frac{\ddot{\lambda}|\lambda|^3}{\lambda} M q_o = \nabla_{q_o} U$
   where only the λ terms depend on t. Since q_o is fixed, we must have $\frac{\ddot{\lambda}|\lambda|^3}{\lambda} = k$ is constant.
   Note that $\nabla_{q_o} I = M q_o$, so a solution of this form necessarily has
   $$k \nabla_{q_o} I = \nabla_{q_o} U$$
   for some constant k.

   Finally, dotting both sides with q_o and using the homogeneity $U(rq) = r^{-1} U(q)$ we have by part (a) that $k = -U(q_o)/2I(q_o) < 0$, i.e.
   $$\ddot{\lambda} = -\mu \frac{\lambda}{|\lambda|^3}$$
   satisfies the Kepler problem with $\mu := -k > 0$.

5. (a) We consider the matrix representation, Ω, of ω in a basis: $\omega(\vec{u}, \vec{v}) = \vec{u} \cdot \Omega \vec{v}$. Then Ω is skew and $\det \Omega = \det \Omega^T = \det(-\Omega) = (-1)^n \det \Omega$. If n is odd then $\det \Omega = 0$, so Ω has a non-trivial kernel. In particular for $\vec{v} \neq 0$ in this kernel (so $\Omega \vec{v} = 0$) we have $\omega(\vec{u}, \vec{v}) = \vec{u} \cdot 0 = 0$ for all $\vec{u} \in V$, which contradicts the non-degeneracy of ω. Hence n is even.

   (b) Let us call a basis with the properties stated in the problem a symplectic basis. We use induction on k, where $dim(V) = 2k$ to prove the existence of symplectic bases.

   $k = 1$: Take any non-zero vector $e_1 \in V \backslash 0$. By nondegeneracy of ω, there exists $f \in V \backslash 0$ such that $\omega(e_1, f) = c \neq 0$. With $f_1 := f/c$ then e_1, f_1 is a symplectic basis.

   Now, suppose symplectic bases exist for symplectic vector spaces of dimension $2k$ and consider a symplectic vector space, (V, ω), of dimension $2k + 2$.

   First, let $e_0 \in V \backslash 0$. By non-degeneracy of ω there is $f_0 \in V \backslash 0$ such that $\omega(e_0, f_0) = 1$. Let $V_0 = span(e_0, f_0)$ be the 2-dimensional subspace spanned by e_0, f_0.

   Set $V_0^\perp := \{v \in V : \omega(v, u) = 0, \forall u \in V_0\}$.

   We claim that $V = V_0 \oplus V_0^\perp$. Indeed, first notice that for any $v \in V$ we may write $v = v_0 + v^\perp$ with $v_0 \in V_0$ and $v^\perp \in V_0^\perp$ by $v^\perp := v - \omega(v, f_0) e_0 + \omega(v, e_0) f_0$ and $v_0 = \omega(v, f_0) e_0 - \omega(v, e_0) f_0$. So $V = V_0 + V_0^\perp$. Next, if $v = a e_0 + b f_0 \in V_0 \cap V_0^\perp$ then $0 = \omega(v, f_0) = a$ and $0 = \omega(e_0, v) = b$ so that $v = 0$. So $V_0 \cap V_0^\perp = \{0\}$ and $V = V_0 \oplus V_0^\perp$. In particular $V_0^\perp$ is $2k$ dimensional.

   Next, we claim that the restriction of ω to $V_0^\perp$ is non-degenerate. If there where $v \in V_0^\perp$ with $\omega(v, u) = 0$ for all $u \in V_0^\perp$ then $\omega(v, w) = \omega(v, w_0 + w^\perp) = 0$ for all $w \in V$ contradicting the non-degeneracy of ω. So indeed, $V_0^\perp$ with $\omega|_{V_0^\perp}$ is a $2k$ dimensional symplectic vector space.

   Now, we apply the induction hypothesis to $V_0^\perp$ to obtain a symplectic basis $e_1, ..., e_k, f_1, ..., f_k$ of $V_0^\perp$. Then $e_0, e_1, ..., e_k, f_0, f_1, ..., f_k$ is a symplectic basis for V.

6. (a) The standard symplectic form is defined by $\omega(\vec{u}, \vec{v}) = \vec{u} \cdot J\vec{v}$. If $A \in \mathrm{Sp}(2n)$ then $\vec{u} \cdot J\vec{v} = A\vec{u} \cdot JA\vec{v} = \vec{u} \cdot A^T J A \vec{v}$ for all $\vec{u}, \vec{v} \in \mathbb{R}^{2n}$. Hence $J = A^T J A$.

   For the other ordering, we take inverse of both sides and use that $J^{-1} = -J$ so that:
   $-J = -A^{-1} J A^{-T} \Rightarrow A J A^T = J$.

(b) We will show that if λ is an eigenvalue of $A \in \mathrm{Sp}(2n)$ then also $\frac{1}{\lambda}$ is an eigenvalue of A. Since the determinant is the product of eigenvalues, this implies that $\det A = 1$.

Note that from $1 = \det J = \det AJA^T = (\det A)^2$, that $\det A = \pm 1$ and no eigenvalues are zero.

Suppose $0 = \det(A - \lambda I)$. Conjugating by J preserves the roots, so that $0 = \det(JAJ + \lambda I)$ (since $J^2 = -I$). Now $JAJ = J^2 A^{-T} = -A^{-T}$ so that λ satisfies as well: $0 = \det(\lambda I - A^{-T})$ or $0 = \det(\lambda A^t - I)$

Since $\lambda \neq 0$, we have $0 = \det(A^t - \frac{1}{\lambda} I)$, and as the determinant is invariant under transpose we have: $0 = \det(A - \frac{1}{\lambda} I)$, i.e. $\frac{1}{\lambda}$ is also an eigenvalue of A as claimed.

7. (a) We will determine a matrix B so that the linear transformation $q \mapsto Aq = Q, p \mapsto Bp = P$ is symplectic.

By 6a, we need to satisfy: $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} A^T & 0 \\ 0 & B^T \end{pmatrix} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} = J$, or:

$\begin{pmatrix} 0 & -AB^T \\ BA^T & 0 \end{pmatrix} = J \Rightarrow BA^T = I$, or $B = A^{-T}$.

Hence the transformation $Q = Aq$ 'lifts' or 'extends' to the symplectic transformation with $P = A^{-T} p$, or $A^T P = p$.

(b) Let the masses of the bodies be m_j. We consider the lift of $Q = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} q = Aq$ so that

$A^T P = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} P = p.$

Then $P_2 = p_2, P_3 = p_3, P_1 = p_1 + p_2 + p_3$. Note P_1 is the linear momentum.

To express the Hamiltonian in these symplectic coordinates, we first consider the potential term. Note that $Q_2 - Q_3 = q_2 - q_3$, so that

$$-V = \frac{m_1 m_2}{|Q_2|} + \frac{m_1 m_3}{|Q_3|} + \frac{m_2 m_3}{|Q_2 - Q_3|}.$$

For the kinetic term, we let $M = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}$, where $\mu_j = \frac{1}{m_j}$. Then $2K = Mp \cdot p = MA^T P \cdot A^T P = AMA^T P \cdot P$. We compute:

$$AMA^T = \begin{pmatrix} \mu_1 & -\mu_1 & -\mu_1 \\ -\mu_1 & \mu_1 + \mu_2 & \mu_1 \\ -\mu_1 & \mu_1 & \mu_1 + \mu_3 \end{pmatrix}$$

so that

$$K = \mu_1 \frac{|P_1|^2}{2} + (\mu_1 + \mu_2) \frac{|P_2|^2}{2} + (\mu_1 + \mu_3) \frac{|P_3|^2}{2} - \mu_1 P_1 \cdot (P_2 + P_3) + \mu_1 P_2 \cdot P_3.$$

The Hamiltonian is then $H = K + V$. Note that there is no Q_1 dependence reflecting that linear momentum P_1 is constant. When we take a frame with the center of mass fixed, $P_1 = 0$, then

$$H = (\mu_1 + \mu_2) \frac{|P_2|^2}{2} - \frac{m_1 m_2}{|Q_2|} + (\mu_1 + \mu_3) \frac{|P_3|^2}{2} - \frac{m_1 m_3}{|Q_3|} + \mu_1 P_2 \cdot P_3 - \frac{m_2 m_3}{|Q_2 - Q_3|}.$$

In particular, if one sets the mass of the 'sun' $m_1 = 1$ and rescales the masses of the 'planets' $m_2 \to \varepsilon m_1, m_3 \to \varepsilon m_3$, and so also correspondingly the momenta $P_2 \to \varepsilon P_2, P_3 \to \varepsilon P_3$, where ε is 'small' then we have:

$$\frac{1}{\varepsilon} H = \frac{|P_2|^2}{2m_2} - \frac{m_2}{|Q_2|} + \frac{|P_3|^2}{2m_3} - \frac{m_3}{|Q_3|} + \varepsilon \left(\frac{|P_2 + P_3|^2}{2} - \frac{m_2 m_3}{|Q_2 - Q_3|} \right)$$

reflecting that as $\varepsilon \to 0$, the dominant terms of the dynamics are two uncoupled Kepler problems.

8. Let $S(x, I)$ be our generating function, so that $p_x = S_x, \theta = S_I$ gives a symplectic change of coordinates. We want S to satisfy:

   $(\frac{dS}{dx})^2 + k^2x^2 = h(I) =: c.$

   Then $dS = \sqrt{c - k^2x^2}dx$ and the substitution $\sin y = \frac{kx}{\sqrt{c}}$ yields:

   $dS = \frac{c}{k}\cos^2 y dy \Rightarrow S = \frac{c}{2k}(\frac{\sin 2y}{2} + y).$

   Let us take $c = 2kI$ (here one has a choice). Then $S = I(\frac{\sin 2y}{2} + y)$ where $\sin y = \sqrt{\frac{k}{2I}}x.$

   By construction, we have
   $$kI = H = \frac{p_x^2 + k^2x^2}{2},$$
   while finding the relation for θ requires determining S_I.

   First compute $\cos y dy = -\sqrt{\frac{k}{2I}}\frac{x}{2I}dI$, so that $\frac{dy}{dI} = -\sqrt{\frac{k}{2I}}\frac{x}{2I\cos y} = -\frac{\tan y}{2I}$. Now:

   $\theta = S_I = \frac{\sin 2y}{2} + y + I(\cos 2y + 1)\frac{dy}{dI} = \sin y\cos y + y - I(2\cos^2 y)\frac{\tan y}{2I} = y$, so that
   $$\theta = \arcsin\sqrt{\frac{k}{2I}}x.$$

9. (a) The characteristic equations are $t' = t, x' = x, u' = 0$.

   The trajectories with initial condition $(t_o = 1, x_o, u_o = x_o)$ are given by $t = e^s, x = x_oe^s, u = x_o$. Eliminating x_o and s the solution with $u(x, 1) = x$ is $u = x/t$. The snapshots of the solution for various values of t are lines of varying slope (the slope becoming infinite as $t \to 0$).

   The trajectories with initial condition $(t_o = 1, x_o, u_o = x_o^2)$ are given by $t = e^s, x = x_oe^s, u = x_o^2$. Eliminating x_o and s the solution with $u(x, 1) = x^2$ is $u = (x/t)^2$. The snapshots with increasing t are flattening parabolas, while as $t \to 0$ the parabola approaches the upper y-axis.

   When we take an initial condition $u(x, 0) = f(x)$, the method of characteristics fails since $t' = 0$ so the characteristics stay constrained to the $t = 0$ plane failing to determine a graph $(x, t, u(x, t))$.

   (b) The characteristic equations are $t' = t, x' = x, u' = 1$. The trajectories with initial condition $(t_o = 1, x_o, u_o = x_o)$ are given by $t = e^s, x = x_oe^s, u = x_o + s$. Eliminating x_o and s the solution with $u(x, 1) = x$ is $u = x/t + \log t$.

   (c) The characteristic equations are $t' = x, x' = -t, u' = 0$.

   The trajectories with initial condition $(t_o = 1, x_o, u_o = x_o)$ are given by $t = x_o \sin s, x = x_o\cos s, u = x_o$. Eliminating x_o and s the solution with $u(x, 1) = x$ has $u^2 = x^2 + t^2$, where we choose the branches of the hyperbola to tend to the initial condition $u(x, 1) = x$, that is: $u = \sqrt{x^2 + t^2}$ for $x > 0$ and $u = -\sqrt{x^2 + t^2}$ for $x < 0$.

   The trajectories with initial condition $(t_o = 1, x_o, u_o = x_o^2)$ are given by $t = x_o \sin s, x = x_o\cos s, u = x_o^2$. Eliminating x_o and s the solution with $u(x, 1) = x^2$ is $u = x^2 + t^2$. The snapshots are vertically shifted parabolas.

10. (a) Since $\gamma(s)$ is unit speed, $0 = \frac{d}{ds}|\gamma'|^2 = 2\gamma' \cdot \gamma''$, so that γ'' is perpendicular to γ'. As we are in the plane, this means that γ'' is proportional to n, i.e. $\gamma''(s) = \kappa(s)n(s)$ for some $\kappa(s) \in \mathbb{R}$.

    Now $n' = i\gamma'' = i\kappa n = \kappa i(i\gamma') = -\kappa\gamma'$.

    (b) We will find $\kappa = \gamma'' \cdot i\gamma'$ by chain rule. First, $\dot\Gamma = \gamma'\dot s$, where $\dot s = |\dot\Gamma|$, that is $ds = |\dot\Gamma|dt$.

    Differentiating $\gamma' = \frac{\dot\Gamma}{|\dot\Gamma|}$ with respect to s gives:

    $\gamma'' = \alpha\dot\Gamma + \frac{1}{|\dot\Gamma|}\frac{d}{ds}\dot\Gamma = \alpha\dot\Gamma + \frac{1}{|\dot\Gamma|^2}\ddot\Gamma.$

    Hence, $\kappa = \gamma'' \cdot i\gamma' = \gamma'' \cdot \frac{i\dot\Gamma}{|\dot\Gamma|} = \frac{\ddot\Gamma \cdot i\dot\Gamma}{|\dot\Gamma|^3}$, since $\dot\Gamma \cdot i\dot\Gamma = 0$.

(c) The caustic of a parametrized curve is $\gamma(s) + \frac{n(s)}{\kappa(s)}$.

We use (b) to compute the curvature of the parabola via the parametrization $\Gamma : t \mapsto (t, t^2)$. Then $\dot{\Gamma} = (1, 2t), \ddot{\Gamma} = (0, 2)$. The unit normal is given by $n(t) = \frac{i\dot{\Gamma}}{|\dot{\Gamma}|} = \frac{(-2t,1)}{|\dot{\Gamma}|}$ so that $\kappa(t) = \frac{2}{|\dot{\Gamma}|^3}$, where $|\dot{\Gamma}|^2 = 1 + 4t^2$.

Hence the parabola's caustic is parametrized by:

$t \mapsto \Gamma(t) + \frac{n(t)}{\kappa(t)} = \Gamma(t) + \frac{|\dot{\Gamma}|^2}{2}(-2t, 1) = (t, t^2) + (1 + 4t^2)(-t, \frac{1}{2}) = (-4t^3, \frac{1}{2} + 3t^2)$.

In a different form, from $(y - \frac{1}{2})^3 = 27t^6, x^2 = 16t^6$, we have $16(y - \frac{1}{2})^3 = 27x^2$ gives the caustic.

11. (a) We apply the maximum principle. Let $x = \begin{pmatrix} q \\ v \end{pmatrix}$, so that $\dot{x} = \begin{pmatrix} v \\ u \end{pmatrix} = f(x, u)$.

Suppose $x_*(t), u_*(t)$ is an optimal trajectory.

The solutions of $\dot{p} = -d_x f(x_*(t), u_*(t))p$ are $p_1 = cst., p_2 = -tp_1 + p_2(0)$. Now there is some such solution $p(t)$ such that at each time t, an optimal trajectory satisfies: $f(x_*(t), u_*(t)) \cdot p(t) \geq f(x_*(t), u) \cdot p(t)$ for all $u \in [-1, 1]$. Written out:

$$v_*(t)p_1 + u_*(t)p_2(t) \geq v_*(t)p_1 + up_2(t) \Rightarrow u_*(t)p_2(t) \geq up_2(t), \forall u \in [-1, 1].$$

The solutions $p_2(t)$ are linear in t so (unless they are constant) have at most one zero. Apart from this one time we then have $u_*(t) = 1$ when $p_2(t) > 0$ and $u_*(t) = -1$ when $p_2(t) < 0$.

The optimal trajectories are then concactenations of solutions to $\ddot{q} = \pm 1$, which graph as parabolas in the (q, v) plane.

In fact, one can describe the concactenation process in the following way: we call the curve consisting of trajectories with $u = \pm 1$ going straight into $(0, 0)$: $v = \begin{cases} -\sqrt{2q} & q > 0 \\ \sqrt{-2q} & q < 0 \end{cases}$ the 'switching curve'. To construct an optimal trajectory, if ones initial position and velocity is below the switching curve one takes $u = 1$ until reaching the switching curve at which instant one takes $u = -1$. Likewise if ones initial position and velocity is above the switching curve, one takes $u = -1$ until reaching the switching curve at which point one takes $u = 1$.

it is interesting to observe that this optimal trajectory is not very smooth. One concactenates maximum acceleration and maximum braking

(b) We proceed similarly to part a. Take $f(x, u) = \begin{pmatrix} v \\ -q + u \end{pmatrix}$, so that solutions to $\dot{p} = -d_x f(x_*(t), u_*(t))p$ are $p_1(t) = A \sin(t + \phi), p_2(t) = A \cos(t + \phi)$. The condition $f(x_*(t), u_*(t)) \cdot p(t) \geq f(x_*(t), u) \cdot p(t)$ along an optimal trajectory again implies:

$$u_*(t)p_2(t) \geq up_2(t), \forall u \in [-1, 1].$$

Hence $u_*(t) = \pm 1$ depending on the sign of $p_2(t)$. Any non-zero solution $p_2(t)$ has zeroes at intervals of time seperated by π.

The optimal trajectories are then concactenations of solutions to $\ddot{q} = -q \pm 1$, which parametrize circles centered at ± 1 in the (q, v) plane. The motion along the circles is clockwise.

As before, the process for choosing the sign of u_* may be described with a switching curve. It consist of the upper halves $(v > 0)$ of circles of radius $1/2$ centered at $v = 0, q = -1, -2, \ldots$ and the lower halves $(v < 0)$ of circles of radius $1/2$ centered at $v = 0, q = 1, 2, 3, \ldots$. When one is above this switching curve one should set $u = -1$ and when below it $u = 1$.

12. (a) Any vector field bracketed with itself gives zero: $[X, X] = 0$. For some more interesting examples in the plane, one can check that the flows of $X = \partial_x$ (translation along x-axis) and $Y = \partial_y$ (translation along y-axis) commute so have Lie bracket zero. Also, the flows of rotating $\partial_\theta = x\partial_y - y\partial_x$, and dilating $r\partial_r = x\partial_x + y\partial_y$ commute so also have zero Lie bracket.

(b) We may just give an example on $\mathbb{R}$, since this also provides an example in the plane (e.g. a 'y' independent vector field). We seek 'simple' vector fields whose flows we can solve explicitly.

Consider $X = \partial_x, Y = x\partial_x$. The flow of X is $\phi^t(x_o) = x_o + t$ and the flow of Y is $\psi^s(x_o) = e^s x_o$.

Then the flows of X and Y do not commute to first order: $\frac{d^2}{dsdt}|_{t=s=0}\phi^{-t} \circ \psi^s \circ \phi^t(x_o) = 1$, so that their Lie bracket is not zero. In fact the constant vector field '1' corresponds to translations along the x-axis so that $[X, Y] = X = \partial_x = [\partial_x, x\partial_x]$.

One could also verify this using the commutating formula for vector fields thought of as operators that we derived in class:

$[X, Y] = XY - YX = (\partial_x(x) - x\partial_x(1))\partial_x = \partial_x$, since $\partial_x(x) = 1$ and $\partial_x(1) = 0$.

Yet another alternate method to compute this Lie bracket is to use its 'derivation' property (that we didn't cover in class, but is still useful!). This says in general one has: $[U, fV] = (Uf)V + f[U, V]$, for f a function and vector fields U, V. Then:

$[\partial_x, x\partial_x] = \partial_x(x)\partial_x + x[\partial_x, \partial_x] = \partial_x$.

13. We will rephrase tracking the first digit of 2^k as tracking an orbit of an irrational rotation on the circle. Observe that:

$k \log_{10} 2 = \log_{10}(d_k.d_1...d_{n_k}) + n_k$

where d_k is the leading digit of 2^k, and $n_k + 1 \in \mathbb{N}$ is the number of digits in 2^k.

Viewed on the circle, $\mathbb{R}/\mathbb{Z} = \{x : x \equiv x + m, m \in \mathbb{Z}\}$, tracking the leading digits of powers of 2 is to examine the orbit of the point 0 under the irrational rotation map $f : x \mapsto x + \log_{10} 2$.

For example if $f^k(0) \in [0, \log_{10} 2)$ then the leading digit of 2^k is 1. Likewise, for $m \in \{1, 2, ..., 9\}$, if $f^k(0) \in [\log_{10} m, \log_{10}(m + 1))$ then the leading digit of 2^k is m.

Now, we apply the theorem stated in class on irrational rotations: the time an orbit spends in an interval is proportional to the length of the interval. Hence the limits:

$$
\begin{aligned}
p_m = \lim_{n\to\infty} & \frac{\#\{\text{ times } f^k(0) \text{ lies in } [\log_{10} m, \log_{10}(m+1)) \text{ for } 0 \le k \le n\}}{n} \\
& = \text{length}[\log_{10} m, \log_{10}(m+1)) = \log_{10}(1 + \frac{1}{m}).
\end{aligned}
$$

Numerically, $p_1 \approx 30.1\%, p_2 \approx 17.6\%, p_3 \approx 12.5\%, p_4 \approx 9.7\%, p_5 \approx 7.9\%, p_6 \approx 6.7\%, p_7 \approx 5.8\%, p_8 \approx 5.1\%, p_9 \approx 4.6\%$.