Final (differential geometry 2021, MSSG)

Each problem is 20 pts.

1. Let $t \mapsto \gamma(t) \in \Sigma$ parametrize a curve, γ, in the surface Σ. Show that if the acceleration of γ is normal to the surface at every instant, then the speed of γ is constant.

2. Let $\Sigma \subset \mathbb{E}^3$ be a surface of revolution. Show that the 'latitudes' of Σ (intersection of Σ with a plane containing the rotation axis) are geodesics and lines of curvature.

3. Let $\sigma(u, v)$ be local coordinates on Σ, with unit normal $n = \frac{\sigma_u \times \sigma_v}{|\sigma_u \times \sigma_v|}$. Derive the Weingarten equations:

$$n_u = \frac{(FM - GL)\sigma_u + (FL - EM)\sigma_v}{EG - F^2}, \quad n_v = \frac{(FN - GM)\sigma_u + (FM - EN)\sigma_v}{EG - F^2}$$

4. Let V be an n-dimensional (real) vector space with inner product $\langle \cdot, \cdot \rangle$, and β a symmetric bilinear form on V with $B : V \to V$ defined by $\langle Bv, w \rangle = \beta(v, w), \quad \forall v, w \in V$.

   (a *(10 pts)*) show that the critical values[1] of the function $V \backslash 0 \to \mathbb{R}, \quad v \mapsto \frac{\beta(v,v)}{\langle v,v \rangle}$ are the eigenvalues of B.

   (b *(5 pts)*) show the eigenvalues of B are real.

   (c *(5 pts)*) show the eigenvectors of B are orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle$.

5. Consider an (oriented, compact and without boundary) surface $\Sigma \subset \mathbb{E}^3$. For $\nu(p)$ the unit normal to Σ at $p \in \Sigma$, set

$$\tau_r(\Sigma) := \{p + t\nu(p) : p \in \Sigma, \ t \in [-r, r]\}.$$

   Show that the volume of this set, $\tau_r(\Sigma) \subset \mathbb{E}^3$, is given by:

$$2r Area(\Sigma) + \frac{4\pi r^3}{3} \chi(\Sigma)$$

   where $\chi(\Sigma)$ is the Euler-Characteristic[2] of Σ.

[1] A critical value of a function, f, is its value, $f(v_o)$, at a critical point v_o. The critical points being those points, v_o, in the domain of the function at which $d_{v_o} f = 0$.

[2] For this problem, you may freely make use of the Gauss-Bonnet theorem.