Midterm (differential geometry 2021, MSSG)

Each problem is 20 pts, with parts (a), (b), (c) of problem 3 worth 8, 4, 8 respectively.

1. Let $\mathcal{C}$ be a plane curve with constant curvature κ. Use the Frenet-Serret equations to show that $\mathcal{C}$ is a circle (of radius $1/|\kappa|$).

2. Let $s \mapsto c(s) \in \mathbb{E}^3$ be a smooth arc-length parametrized spatial curve with $c''(0) \neq 0$. Show that there is a system of Cartesian coordinates centered at $c(0)$ in which:

$$c(s) = (s - \frac{s^3}{6}\kappa_o^2,\ \frac{s^2}{2}\kappa_o + \frac{s^3}{6}\kappa_o',\ \frac{s^3}{6}\kappa_o\tau_o) + O(s^4)$$

where κ_o, τ_o are the curvature and torsion at $c(0)$ and $\kappa_o' = \frac{d}{ds}|_{s=0}\kappa(s)$.

3. Consider a plane curve given in polar coordinates by $r(\theta)$.

(a) Show that the (signed) curvature is given by:

$$\kappa = \frac{2(r')^2 + r^2 - rr''}{((r')^2 + r^2)^{3/2}}$$

where $r' = \frac{dr}{d\theta}, r'' = \frac{d^2r}{d\theta^2}$.

(b) Calculate the curvature of the logarithmic spiral: $r = ae^{b\theta}$.

(c) Consider a convex plane curve given in support coordinates (see figure) with respect to some interior point by $p(\theta)$. Show that the curvature is given by:

$$\kappa = \frac{1}{p + p''}.$$

4. Show that the evolute of a tractrix curve is a catenary curve.[1]

5. Consider the Roulette curve, γ, generated by tracing a point fixed to a line as the line rolls without slipping along the plane curve $\mathcal{C}$. Show that γ is an involute of $\mathcal{C}$.

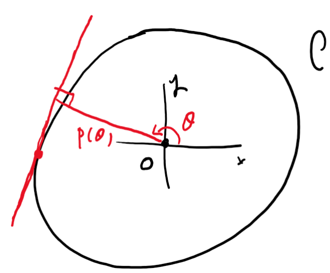

Figure 1. Given a direction $n(\theta) = (\cos\theta, \sin\theta)$ there is precisely one tangent line to the convex curve with $n(\theta)$ as outward normal at a point $c(\theta) \in \mathcal{C}$. Let $p(\theta)$ be the distance from this tangent line ($\cos\theta x + \sin\theta y = p(\theta)$) to the origin. One finds that the curvature is $\kappa = \frac{1}{|c'(\theta)|}$.

[1] You may use here the parametrization $t \mapsto (t - \ell\tanh\frac{t}{\ell},\ \frac{\ell}{\cosh\frac{t}{\ell}})$ of a tractrix curve, and formula $y = b + c\cosh\frac{x+a}{c}$ for a catenary curve as a graph (where ℓ, a, b, c are constants). For extra credit derive these equations from the bicycling and variatonal definitions of the tractrix and catenary respectively.