Notes on generalized functions (distributions)

In physics and pde's we encounter 'generalized functions'. Here, we briefly explain these objects [1].

Consider a 'physical quantity' given by a function $f : \mathbb{R}^3 \to \mathbb{R}$, eg $f(x)$ is some density of a substance at the point $x \in \mathbb{R}^3$. In practice one does not have the ability to localize ones measurements to arbitrarily small precision to measure the value $f(x)$ at a specific point. Rather one obtains average values: $\frac{\int_\Omega f\ dV}{\int_\Omega\ dV}$, over small regions Ω containing x. Actually, depending upon ones measuring device or process, one is more likely obtain a value:

$$\int_\Omega f\varphi\ dV$$

where $\varphi : \Omega \to \mathbb{R}$ is some function representing the properties of our measuring device. Thus, we may alternatively think of this physical quantity as represented by an operator:

$$\varphi \mapsto \int f\varphi\ dV$$

sending functions, φ, ('test functions' representing some measuring process) to numbers (the result of the measurement).

Now a *generalized function* or *distribution* may be defined by taking a class, $\mathscr{D}$, of *test functions* [2] and calling a (continuous) linear functional, $T \in \mathscr{D}'$:

$$T : \mathscr{D} \to \mathbb{R}, \quad T(a\varphi_1 + b\varphi_2) = aT(\varphi_1) + bT(\varphi_2),$$

a generalized function. For example a function [3] $f : \mathbb{R}^3 \to \mathbb{R}$ determines a generalized function:

$$T_f :\ \varphi \mapsto \int_{\mathbb{R}^3} f\varphi\ dV.$$

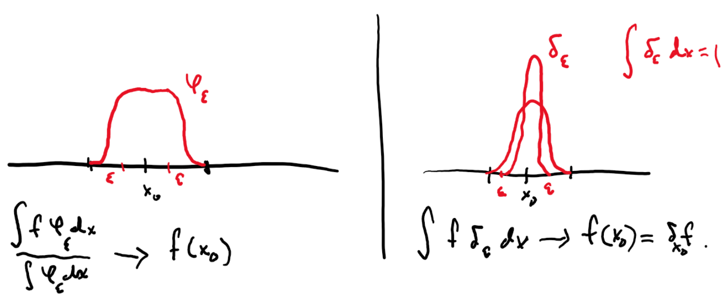

Figure 1. The values of a continuous function may be recovered from its action, $\varphi \mapsto \int f\varphi\ dx$, on test functions.

[1] See eg Lecture 9 of V.I. Arnold, *Lectures on partial differential equations.* Moscow, 2004. or ch. 2.1 of Y. Egorov, M. Shubin. *Foundations of the classical theory of partial differential equations.* Vol. 1. Springer Science & Business Media, 1998. or R. Strichartz, *A guide to distribution theory and Fourier transforms.* World Scientific Publishing Company, 2003.

[2] The standard choice is $\mathscr{D} = C_o^\infty(\mathbb{R}^3)$, consisting of smooth (infinitely differentiable) functions with compact support (vanish outside of some compact set). These test functions are a real vector space (with pointwise scaling and addition) and have a topology from $\varphi_n \to \varphi$ when there is some compact set K with $\varphi_n, \varphi \equiv 0$ on K^c and φ_n and all partial derivatives of φ_n converge uniformly to φ and all partial derivatives of φ on K. The linear functionals $\mathscr{D} \to \mathbb{R}$ which are called generalized functions are also required to be continuous wrt this topology, $\varphi_n \to \varphi \Rightarrow T(\varphi_n) \to T(\varphi)$.

[3] To determine a generalized function, f must satisfy some technical conditions. For example be *locally integrable*, ie $\int_K f\ dV$ is defined for any compact set K, eg whenever f is continuous. The space of generalized functions is itself a topological vector space (over $\mathbb{R}$) with the weak-$*$ topology: $T_n \to T$ when $T_n(\varphi) \to T(\varphi)$ for any $\varphi \in \mathscr{D}$. In particular it may be shown that the set of generalized functions $T_f \in \mathscr{D}'$ induced by usual smooth functions $f : \mathbb{R}^3 \to \mathbb{R}$ are dense in $\mathscr{D}'$.

EXAMPLES:

- The *dirac delta function* at x_o is the generalized function $\delta_{x_o}(\varphi) := \varphi(x_o)$. One usually writes δ_o or just δ for the dirac delta function at the origin, and for example $\delta(\varphi) = \int_{x\in\mathbb{R}^3} \delta(x)\varphi(x)\, dV = \varphi(0)$.

- The *Heaviside function*, $H : \mathbb{R} \to \mathbb{R}, x \mapsto \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$. Note that $T_H(\varphi') = -\varphi(0) = -\delta_o(\varphi)$.

DERIVATIVES: One may define derivatives of generalized functions using integration by parts. First, let $f : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function, with corresponding $T_f(\varphi) = \int_{\mathbb{R}^3} f\varphi\, dV$. Then $\partial_x T_f := T_{\partial_x f}$ and:

$$\partial_x T_f(\varphi) = \int_{\mathbb{R}^3} \partial_x f \ \varphi \ dV = -\int_{\mathbb{R}^3} f\partial_x\varphi \ dV = -T_f(\partial_x\varphi).$$

Using integaration by parts. In general we take:

$$\partial_x T(\varphi) := -T(\partial_x\varphi).$$

Then, for example:

$$H' = \partial_x T_H = \delta_o$$

for H the Heaviside function and δ_o the dirac-delta. Likewise, one has partial derivatives of any order, eg

$$\partial_x^2 T(\varphi) = T(\partial_x^2\varphi).$$

FUNDAMENTAL SOLUTIONS: Generalized functions are useful in studying differential equations. Consider a (non-homogeneous) linear pde:

$$Au = f$$

where the function f and operator $A = \sum_{k\leq m} a_{i_1\ldots i_k}\partial_{x_{i_1}}\ldots\partial_{x_{i_k}}, \quad a_{i_1\ldots i_k} \in \mathbb{R}$ are given.

One is interested in determining functions, u, with partial derivatives of order m satisfying $Au = f$. Since we may also take derivatives of generalized functions, we may also seek *weak solutions*: generalized functions T with $AT = f$. In particular, one may consider $T_u(\varphi) = \int u\varphi\, dV$ with u say only a continuous function as a weak solution when:

$$\int \sum_{k\leq m} a_{i_1\ldots i_k}(-1)^k u \ \partial_{x_{i_1}}\ldots\partial_{x_{i_k}}\varphi \ dV = \int f\varphi \ dV$$

for any smooth function φ with compact support.

In this way for example, given any continuous function $g : \mathbb{R} \to \mathbb{R}$, one may consider

$$u(x,t) := g(x-t)$$

as a weak solution of the wave equation, $u_{xx} = u_{tt}$. Whereas in the strict sense, only certain twice differentiable functions, $u(x,t)$, are 'true' solutions of this wave equation.

A *fundamental solution* to the pde $Au = f$ is a generalized function, T_o, solving:

$$AT_o = \delta_o$$

that is $AT_o(\varphi) = \varphi(0)$ for any smooth φ with compact support. Given a fundamental solution, one seeks a (weak) solution to the original problem by:

$$AT = f, \quad T = f * T_o$$

where $*$ is the convolution operation. For ordinary functions:

$$(f*g)(x) = \int_{y\in\mathbb{R}^3} f(x-y)g(y)\ dV.$$

To define a convolution for generalized functions, observe that

$$T_{f*g}(\varphi) = \int_{x\in\mathbb{R}^3}\int_{y\in\mathbb{R}^3} f(x)g(y)\varphi(x+y)\ dV_y dV_x.$$

By setting $\tau_y : \mathbb{R}^3 \to \mathbb{R}^3, x \mapsto x+y$, then for any $\varphi \in \mathcal{D}$ so too is $\varphi \circ \tau_y \in \mathcal{D}$. We take: $(T_f * T_g)(\varphi) := T_f(T_g(\varphi\circ\tau_y))$, where $y \mapsto T_g(\varphi\circ\tau_y) \in \mathcal{D}$. Then $T_f * T_g = T_{f*g}$. Now in general [4],

$$(T_1 * T_2)(\varphi) := T_1(T_2(\varphi\circ\tau_y)).$$

Then, when $T_1 * T_2 \in \mathcal{D}'$, one has:

$$T_1 * T_2 = T_2 * T_1, \quad \partial_x(T_1 * T_2) = (\partial_x T_1) * T_2 = T_1 * (\partial_x T_2)$$

Now, returning to a fundamental solution T_o to $Au = f$, with $f * T_o = T_f * T_o$, one then has:

$$A(f * T_o) = f * (AT_o) = f * \delta_o = f.$$

More examples:

- One may write solutions to a (non-homogeneous) linear ode:

$$\mathfrak{L}u = \frac{d^2u}{dx^2} + a\frac{du}{dx} + bu = f$$

using a *Green's function*: $G(x,y)$ such that $\mathfrak{L}G(x,y) = \delta(x-y)$. A particular solution is then:

$$u_p(x) = \int G(x,y)f(y)\ dy$$

Related to the discussion above, we have fundamental solutions $u(x) = G(x+y,y)$, with $Lu = \delta_o$ for each fixed y. Thus given a fundamental solution, $u_o(x)$, one may take a Green's function by $u_o(x) = G(x+y,y)$ or $G(x,y) := u_o(x-y)$. Then the formula above reads:

$$u_p = f * u_o.$$

We thus have an operator

$$\mathfrak{G}, \quad f \mapsto f * u_o$$

which is a right inverse of $\mathfrak{L}$:

$$\mathfrak{L}\mathfrak{G}f = f.$$

- The Laplacian $\Delta u = \partial_x^2 u + \partial_y^2 u + \partial_z^2 u$ on $\mathbb{R}^3$ has fundamental solution $u_o = -\frac{1}{4\pi r}$ as follows by first computing for any smooth f with compact support that:

$$(\Delta T_{u_o})(f) = \int_{\mathbb{R}^3} u_o \Delta f\ dV = \lim_{\varepsilon\to 0}\int_{S^2_\varepsilon} f\partial_\nu u_o - u_o\partial_\nu f\ dA$$

by using Green's formula (with S^2_ε a sphere of radius ε) and that $\Delta u_o = 0$ on $\mathbb{R}^3\backslash 0$. Now since $\partial_\nu u_o = \nabla u_o \cdot \nu = \frac{1}{4\pi\varepsilon^2}$ where ν is the outward normal to S^2_ε, we get:

$$(\Delta T_{u_o})(f) = \lim_{\varepsilon\to 0}\frac{1}{4\pi}\int_0^\pi\int_0^{2\pi} (f|_{r=\varepsilon} + \varepsilon\partial_\nu f|_{r=\varepsilon}\ )\sin\varphi\ d\theta d\varphi = f(0)$$

So that $\Delta u_o = \delta_o$ as claimed.

Thus we have solutions to the Poisson equation, $\Delta u = f$ by taking $u = f * u_o$, ie:

$$u(x) = -\frac{1}{4\pi}\int_{y\in\mathbb{R}^3}\frac{f(y)}{|x-y|}\ dV.$$

[4] As with convolution of ordinary functions, convolution of generalized functions may not always define a generalized function. However, if say T_1 has compact support, meaning $T_1(\varphi) = 0$ whenever $\varphi = 0$ in some fixed compact set K then convolution with T_1 yields a generalized function. In particular if f has compact support then $f * T = T_f * T \in \mathcal{D}'$ whenever $T \in \mathcal{D}'$.