Final (electricity and magnetism 2022)

1. Determine the magnetic field, $\vec{B}$, produced by a constant current, I, flowing along an (infinite) straight line.

2. Consider a constant magnetic field, $\vec{B}$, and constant electric field, $\vec{E}$. With $\vec{B}$ directed 'into the page', and $\vec{E}$ directed 'downwards' as in figure 1.

   (a) Determine the total force on a point charge, q, moving with initial velocity $\vec{v}$ 'to the right'.

   (b) For what speed, $v = |\vec{v}|$, will a charge such as in part (a) experience no force (and so no deflection).

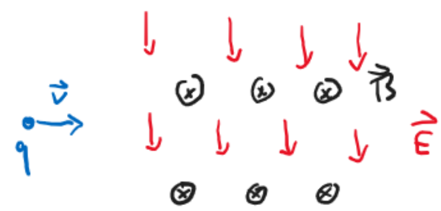

Figure 1. A constant magnetic field is directed into the page. A constant electric field is directed downwards. A point charge q moves with initial velocity $\vec{v}$ to the right.

3. Let $\mathcal{C}_1, \mathcal{C}_2$ be two disjoint closed curves in $\mathbb{R}^3$.

   (a) If a constant current, I_1, runs through $\mathcal{C}_1$, producing a magnetic field, $\vec{B}_1$, show that the resulting magnetic flux, Φ_2, through $\mathcal{C}_2$ is given by:

$$\Phi_2 = L_{12} I_1$$

   for some constant L_{12} (depending only on the curves).

   (b) For the analogous constant, L_{21} (with $\Phi_1 = L_{21} I_2$) relating a current, I_2, in the loop $\mathcal{C}_2$ and resulting magnetic flux, Φ_1 through $\mathcal{C}_1$, show that:

$$L_{21} = L_{12} =: L$$

   (the constant L is called the *mutual inductance* of the two loops, and is a magnetic analogue of capacitance of two conductors).

4. Consider Maxwell's equations in vacuum ($\rho = 0, \vec{J} = 0$). Show that $\vec{E}, \vec{B}$ satisfy the (vector) wave equations:

$$\Delta \vec{E} = \frac{1}{c^2} \partial_t^2 \vec{E}, \quad \Delta \vec{B} = \frac{1}{c^2} \partial_t^2 \vec{B},$$

   where $\Delta \vec{X} = \nabla(\nabla \cdot \vec{X}) - \nabla \times (\nabla \times \vec{X})$ is the vector Laplacian, and $c := \frac{1}{\sqrt{\varepsilon_o \mu_o}}$.

5. Consider the 1-dimensional wave equation, $c^2 u_{xx} = u_{tt}$.

   (a) Show that a linear transformation

$$x' = Ax + Bt, \quad t' = Cx + Dt$$

   preserves the wave equation iff

$$c^2 A^2 - B^2 = c^2, \quad D^2 - c^2 C^2 = 1, \quad c^2 AC = BD.$$

   (b) If $B = -Av$, show that the linear transformation satisfying (a) has the form:

$$x' = \gamma(x - vt), \quad t' = \gamma(t - \frac{v}{c^2} x)$$

   where $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$.