Harmonic functions

We present some properties of harmonic functions ($\Delta u = 0$). First we consider the planar case – with relations to complex analysis– and then analogous results in the spatial case.

Planar harmonic functions: harmonic functions on the plane, $\mathbb{R}^2 = \mathbb{C}$, $(x, y) \longleftrightarrow z = x+iy$, relate to holomorphic functions[1] as their real (or imaginary) parts. Namely:

- If $f = u + iv$ is holomorphic then u (and v) are harmonic,
- if u is harmonic (on a simply connected region $D \subset \mathbb{C}$) then there exists a holomorphic function $f = u + iv$ on D with u as its real part.

One may obtain the *mean value property* of harmonic functions from Cauchy's integral formula [2] :

$$u(z_o) = \frac{1}{2\pi r} \int_{C_r(z_o)} u \; ds$$

where $C_r(z_o)$ is a circle of radius r centered at z_o.

From the mean value property follows the *maximum principle* [3] for harmonic functions:

A harmonic function on a compact region D attains its extremal values on the boundary, ∂D.

proof: Let $M := \max_{z \in D} u(z)$, and suppose that $u(z_o) = M$ for some interior point $z_o \in D^o$. Then by the mean value property,

$$M = u(z_o) = \frac{1}{2\pi r} \int_{C_r(z_o)} u \; ds \leq M$$

with equality iff $u|_{C_r(z_o)} \equiv M$. Hence $u \equiv M$ on a disk around z_o. By covering D with such disks we get that $u \equiv M$ on D and so the only way a harmonic function may attain a maximum at an interior point is when it is constant. □

The uniqueness of solutions to Dirichlet problems [4] on compact regions follows from this principle. To consider Dirichlet problems on unbounded regions, one needs certain conditions at infinity for uniqueness and a *removable singularities theorem*:

If u is harmonic and bounded on the region $D \backslash z_o$ then there exists a harmonic function $\hat{u}$ on D with $u = \hat{u}$.

Then by using a circle inversion, one finds unique solutions to Dirichlet problems on unbounded domains with the condition: $\lim_{|z| \to \infty} |u(z)| \leq C$ of being bounded at infinity.

As for existence, the results are more involved. One may consider the Dirichlet problem, $\Delta u = 0, u|_{S^1} = u_o$, inside the unit circle by expanding the boundary condition in Fourier series:

$$u_o = \frac{a_o}{2} + \sum a_n \cos n\theta + b_n \sin n\theta$$

and using linearity to solve for each term in the sum. In polar coordinates [5] :

$$u = \frac{a_o}{2} + \sum r^n (a_n \cos n\theta + b_n \sin n\theta).$$

[1] A function $f : \mathbb{C} \to \mathbb{C}$ is holomorphic (complex differentiable) when $f'(z_o) = \lim_{z \to z_o} \frac{f(z) - f(z_o)}{z - z_o}$ exists for each z_o.

[2] For f holomorphic, then $f(z_o) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_o} \, dz$ where C is a closed curve around z_o.

[3] This property may be seen from the open mapping theorem for holomorphic functions.

[4] $\Delta u = 0$ on D and $u|_{\partial D} = u_o$.

[5] This may be written in 'integral form' as $u(r, \theta) = \frac{1}{\pi} \int_0^{2\pi} \frac{1 - r^2}{r^2 - 2r\cos(\theta - \alpha) + 1} u_o(\alpha) \; d\alpha$ (the coefficient in front of u_o is called the *Poisson kernel*). To be complete, one needs to establish convergence and that the u so defined satisfies the original problem.

In the plane, one may apply a transformation, $z \mapsto f(z) = w$, to solve the Dirichlet problem on the image of the unit disk, $f(D) = D'$ with $u' \circ f = u$ and $u'_o \circ f = u_o$. A geometric description of the solutions may be given in the upper half plane. The function:

$$\alpha(z) = \text{Im}\left(\log(1 - \frac{1}{z})\right)$$

is harmonic and equal to the angle between the rays from z to the points $0, 1$ (in particular it is π in the interval of the real axis between these points and zero on the intervals of the real axis outside the points). By scaling and summing such α's one may realize given boundary conditions in the upper half plane (think approximation by step functions).

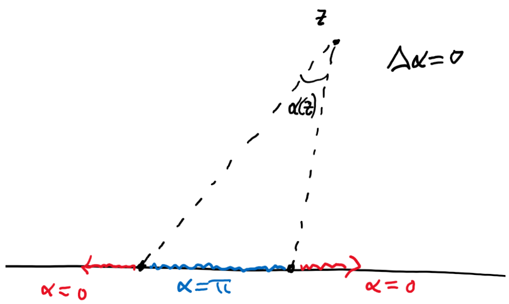

Figure 1. A harmonic function in the upper half plane restricting to a step function on the boundary.

A more general method for showing existence is to use a variational characterization of solutions to the Dirichlet problem. For functions $u : D \to \mathbb{R}$ with $u|_{\partial D} = u_o$ (and u_o given). Consider the functional:

$$u \mapsto \int_D |\nabla u|^2 \, dA.$$

A critical point (extremal) of this variational problem that is smooth is then a solution to the Dirichlet problem, $\Delta u = 0$ in D and $u|_{\partial D} = u_o$. One thus needs to establish that such a critical point exists, and this is done by showing that the functional has a minimizer and such minimizer is smooth.

In fact, there is a physical interpretation as equilibrium states of an elastic membranes. Namely, if one views the position of a membrane as given by a graph $z = u(x, y)$ with $(x, y) \in D$ with a fixed or 'clamped' boundary, $u|_{\partial D} = u_o$. Then the small oscillations of the membrane may be modeled by the pde:

$$\Delta u = k u_{tt}$$

where k is some constant and $u(x, y, t)$ gives the position of the membrane at time t. Solutions of the Dirichlet problem: $\Delta u = 0$ in D and $u|_{\partial D} = u_o$, thus model equilibrium configurations ($u_t \equiv 0$) of the elastic membrane with the given boundary condition (boundary 'clamped' to a fixed curve $z = u_o(x, y), \quad (x, y) \in \partial D$).

Harmonic functions in $\mathbb{R}^3$: Analogous properties hold for harmonic functions on $\mathbb{R}^3$. First, we have a mean value property:

$$u(x_o) = \frac{1}{4\pi r^2} \int_{S_r^2(x_o)} u \; dA$$

where $S_r^2(x_o)$ is a sphere of radius r centered at x_o.

proof: Let $\bar{u}(r) := \frac{1}{4\pi r^2} \int_{S_r^2(x_o)} u \; dA$. Then $\partial_r \bar{u} = \frac{1}{4\pi r^2} \int_{S_r^2(x_o)} \partial_r u \; dA = \frac{1}{4\pi r^2} \int_{S_r^2(x_o)} \nabla u \cdot \nu \; dA = \frac{1}{4\pi r^2} \int_{B_r(x_o)} \Delta u \; dV = 0$. So $\bar{u}$ is constant. Letting $r \to 0$, we have:

$$\bar{u}(r) = \lim_{r \to 0} \frac{1}{4\pi} \int_0^{\pi} \int_0^{2\pi} u|_{S_r^2(x_o)} \sin\varphi \; d\theta d\varphi = u(x_o).$$

□

The same proof used for the planar case gives us the maximum principle here from the mean value property. As well, we have uniqueness for Dirichlet problems on compact regions Ω.

To consider uniqueness on unbounded domains, one may use a removable singularities theorem [6]:

If u is harmonic and bounded on the region $\Omega \backslash x_o$ then there exists a harmonic function $\hat{u}$ on Ω with $u = \hat{u}$.

proof: Consider two spheres of radius R and ε centered at x_o and contained in Ω. Let $\hat{u}$ be a harmonic function on the ball of radius R with $\hat{u}|_{S_R^2} = u|_{S_R^2}$ (here we use *existence* of solutions to Dirichlet problem in compact regions). Since by assumption u is bounded, we have a bounded harmonic function $v := u - \hat{u}$ on $B_R(x_o) \backslash x_o$. We will show $v = 0$. Set $M := \max_{B_R(x_o)} |v|$ and

$$v_{\pm} := \pm M \frac{1/r - 1/R}{1/\varepsilon - 1/R}.$$

Then $v_{\pm} - v$ are harmonic on $B_R(x_o) \backslash x_o$, vanish on S_R^2, and satisfy:

$$v_+ \geq v \geq v_-$$

on S_ε^2. By the maximum principle, this inequality holds over the region $\varepsilon \leq r \leq R$, and sending $\varepsilon \to 0$ we have $v_{\pm} \to 0$ so that $v \equiv 0$. □

Using spherical inversion, one obtains uniqueness for Dirichlet problems on unbounded regions with the conditions at infinity [7]:

$$\lim_{|x| \to \infty} |x| u(x) \leq C$$

of vanishing at infinity to order $1/|x|$, ie $u(x) = O(\frac{1}{|x|})$ as $|x| \to \infty$.

[6] This theorem may be sharpened, namely it suffices to have $\lim_{x \to x_o} |x| u(x) = 0$.

[7] Using the sharper version of the removable singularities theorem, it suffices to have $\lim_{|x| \to \infty} u(x) = 0$ (it then follows that in fact $u = O(\frac{1}{|x|})$).