Orientation

The line and surface integrals we meet in multi-variable calculus are signed, depending on orientations. Getting careless with these signs is an easy source for error in computations. In this note we give a precise definition of orientation and induced orientation on a boundary.

A line is oriented by choice of direction, a plane is oriented by a choice of 'sense': clockwise or counter-clockwise, a space is oriented by a choice of 'chirality': a right or left handed screw.

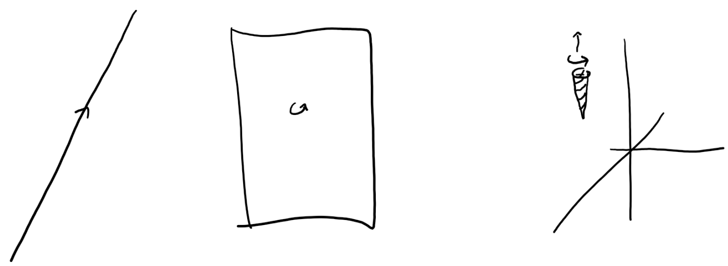

Figure 1. Lines, planes, and spaces may be oriented. An orientation of space may be represented by a 'right handed screw': the screw moves up when twisted counter-clockwise.

An *orientation* of a vector space, V, is a class of *ordered* bases for V. Two ordered bases, $v_1, ..., v_n$ and $w_1, ..., w_n$ give the same orientation of V when the linear map:

$$V \to V, v_j \mapsto w_j$$

has positive determinant.

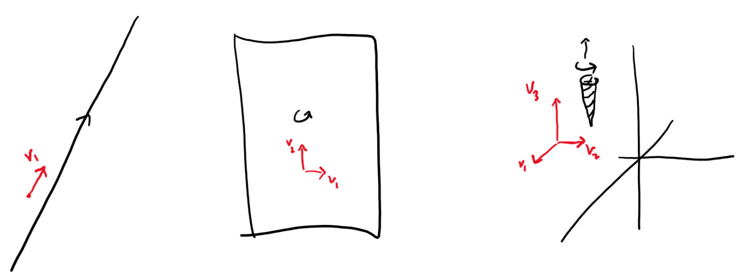

Figure 2. An orientation of a vector space may be represented by an ordered basis. For a line the basis vector is directed along the orientation, for a plane angles are measured from the first vector to the second vector, for a space when the screw is twisted from the first to the second vector it moves in the direction of the 3rd vector (right hand rule).

An orientation on a curve or surface is an orientation on their tangent spaces at each point. The orientation is assumed to vary continuously wrt the base point, that is in a neighborhood of each point may be represented by a continuous family of vector fields along the curve or surface. Orientations of surfaces in space are often represented by choice of a normal vector field, n, along the surface, directed according to v_1, v_2, n agreeing with the right handed orientation of space (here v_1, v_2 are tangent to the surface ordered according to the orientation of the surface).

Oriented curves, surfaces or regions, M, may have boundary, ∂M, which inherit orientations according to the following convention. First a point $p \in M$ is a boundary point when we may parametrize a neighborhood, U, of $p \in M$ by:

$$\varphi : U_+ \to U$$

with U_+ an open set (subspace topology) containing the origin in the upper half space $H_+ = \{(x_1, ..., x_{k+1}) : x_{k+1} \geq 0\}$ and $\varphi(0) = p$.

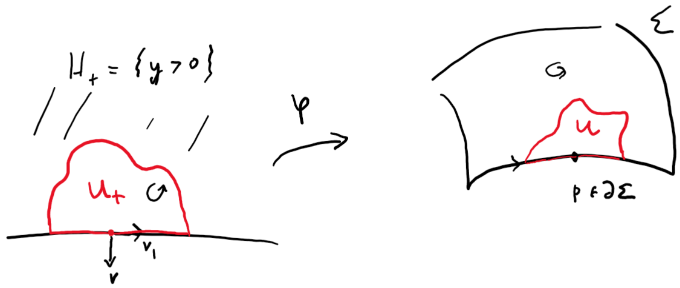

Figure 3. Boundaries inherit an orientation.

The ordered basis $v_1, ..., v_k$ for the tangent space to the boundary of a set represents its inherited orientation if the ordered basis $v, v_1, ..., v_k$ represents the orientation of the set where v is an *outward* pointing vector to the boundary. The direction for the vector v may be seen most clearly in a parametrization $\varphi : U_+ \to U$ around the boundary point, where v points outward to the half space (ie into $H_- = \{x_{k+1} < 0\}$).

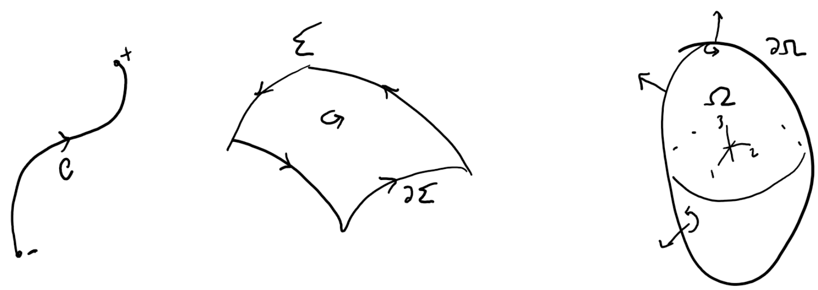

Figure 4. Inherited orientations on boundaries of curves, surfaces, and regions.