Superficies abstractas

Bolas

La n-bola:

$$D^n = B^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$$

La n-bola abierta:

$$\mathring{D}^n = \mathring{B}^n = \{x \in \mathbb{R}^n : |x| < 1\}$$

homeomorfismos

Definición. X, Y espacios; $h : X \to Y$ función.

La función h se llama *homeomorfismo* si y sólo si h es continua, biyectiva y h^{-1} es continua.

Se escribe $X \cong Y$.

homeomorfismos

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Se considera (de momento) *que dos espacios homeomorfos son idénticos.*

homeomorfismos

(Los espacios que nos interesan están hechos de la misma sustancia que $\mathbb{R}^n$)

homeomorfismos

(Los espacios que nos interesan están hechos de la misma sustancia que $\mathbb{R}^n$)

“$\mathbb{R}^n$ es un material muy elástico.”

ejemplo

Sea $\varepsilon > 0$.

$$B(0,1) \to B(0,\varepsilon)$$
$$w \mapsto \varepsilon \cdot w$$
$$\frac{z}{\varepsilon} \leftarrow z$$

Luego

$$B(0,1) \cong B(0,\varepsilon)$$

Lema. *Sean* $x, y \in \mathring{B}^n$, $x \neq y$. *Entonces existe un homeomorfismo* $h : B^n \to B^n$ *tal que*

$$h(x) = y \text{ y } h|\partial B^n = 1_{B^n}.$$

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$$h(x) = y \ \text{ y } \ h|\partial B^n = 1_{B^n}.$$

Dem. Tenemos

$$\varphi : \mathring{B}^n \to \mathbb{R}^n; w \mapsto \frac{w}{1 - |w|},$$

con inversa

$$\varphi^{-1} : \mathbb{R}^n \to \mathring{B}^n; z \mapsto \frac{z}{1 + |z|}.$$

Luego φ es un homeomorfismo.

En $\mathbb{R}^n$ tenemos la traslación

$$T(v) = v - \varphi(x) + \varphi(y)$$

con inversa

$$T^{-1}(u) = u + \varphi(x) - \varphi(y)$$

Luego T también es un homeomorfismo y

$$T(\varphi(x)) = \varphi(y).$$

Luego si $h = \varphi^{-1} \circ T \circ \varphi$,

$$h : \overset{\circ}{B}{}^n \to \overset{\circ}{B}{}^n$$

es un homeomorfismo tal que $h(x) = y$.

$$\begin{array}{ccc}
\overset{\circ}{B}{}^n & \xrightarrow{h} & \overset{\circ}{B}{}^n \\
{\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\
\mathbb{R}^n & \xrightarrow[T]{} & \mathbb{R}^n
\end{array}$$

Pero queremos $h : B^n \to B^n$.

Para $z \in \partial B^n$, definimos $h(z) = z$
(o sea, forzamos $h|\partial B^n = 1$.)

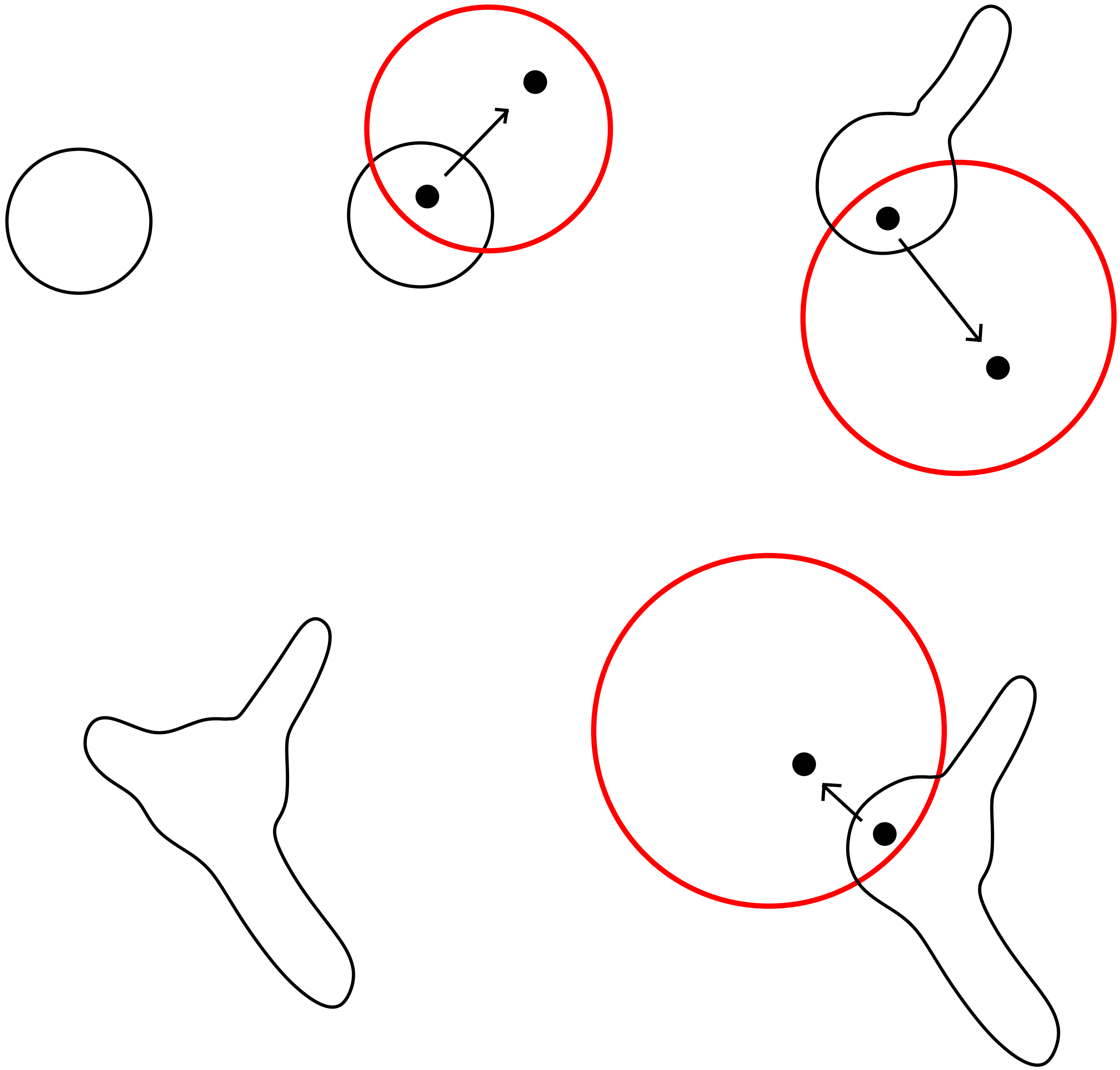
Falta demostrar que h es continua:

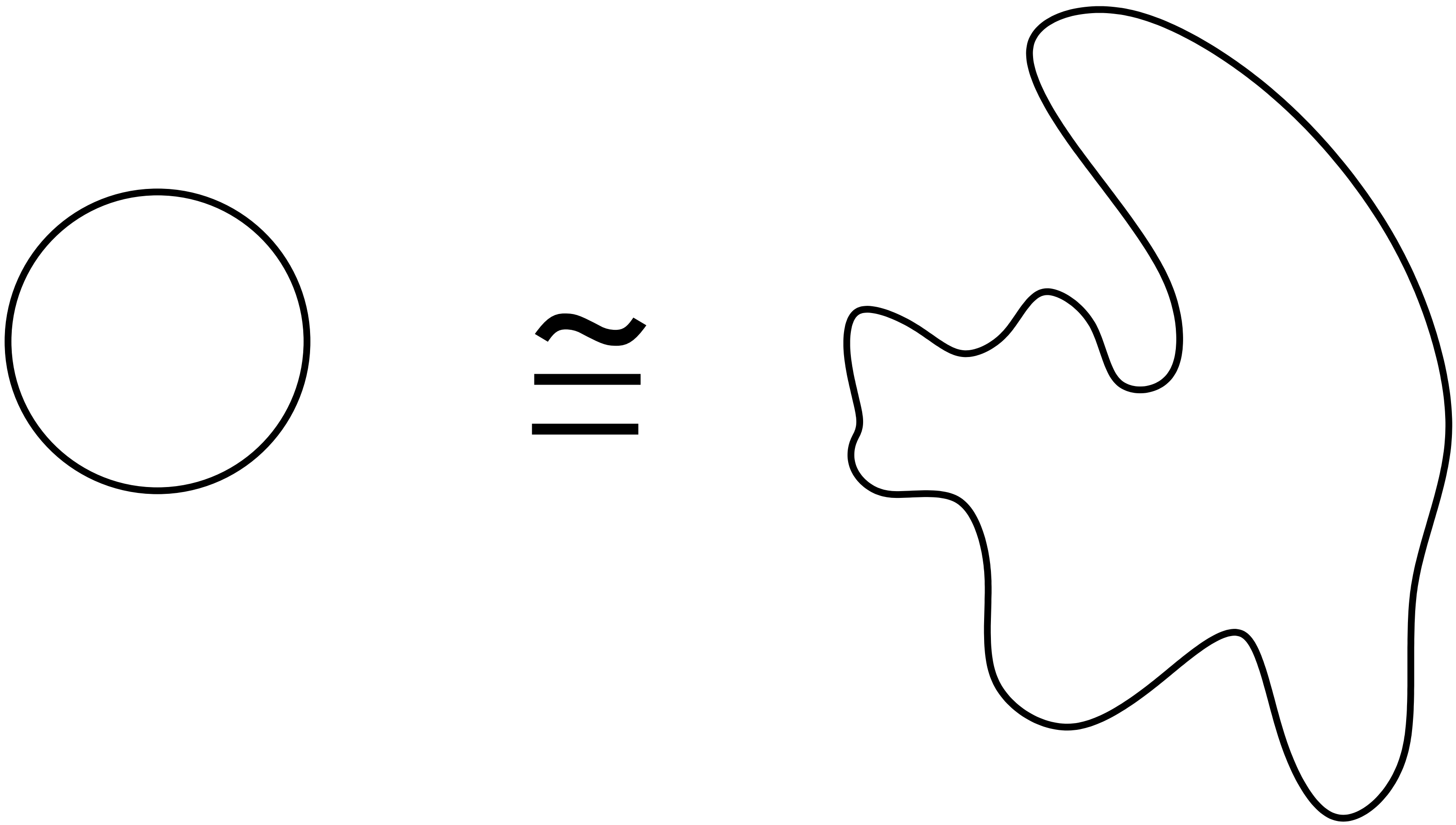
P. D. $\lim\limits_{|w|\to 1} h(w) = w$. O sea:

P. D. $$\lim_{|w|\to 1} \left(\frac{\frac{w}{1-|w|} - \varphi(x) + \varphi(y)}{1 + \left| \frac{w}{1-|w|} - \varphi(x) + \varphi(y) \right|} \right) = w.$$

$\square$

Ejemplo



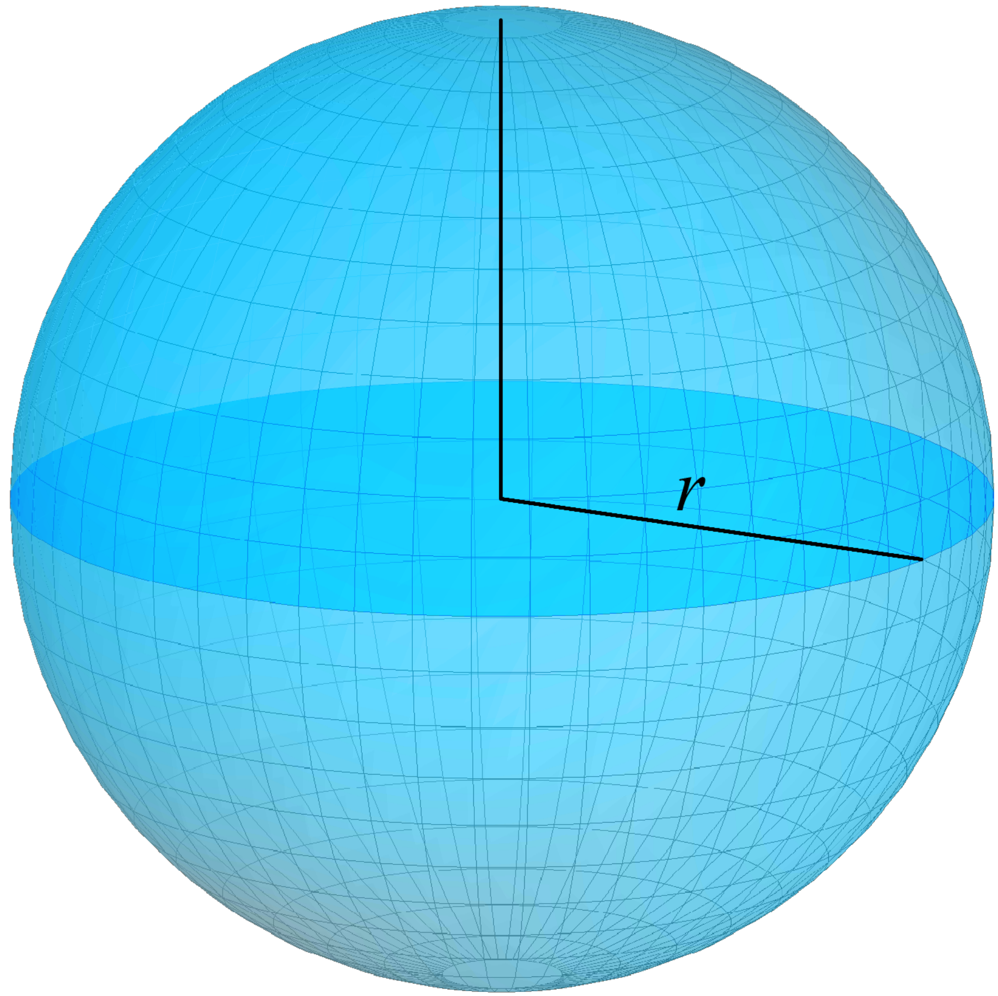


Superficies

Superficies

Una *superficie* es un espacio X tal que X es de Hausdorff y cada punto de X tiene una vecindad homeomorfa a B^2.

Ejemplos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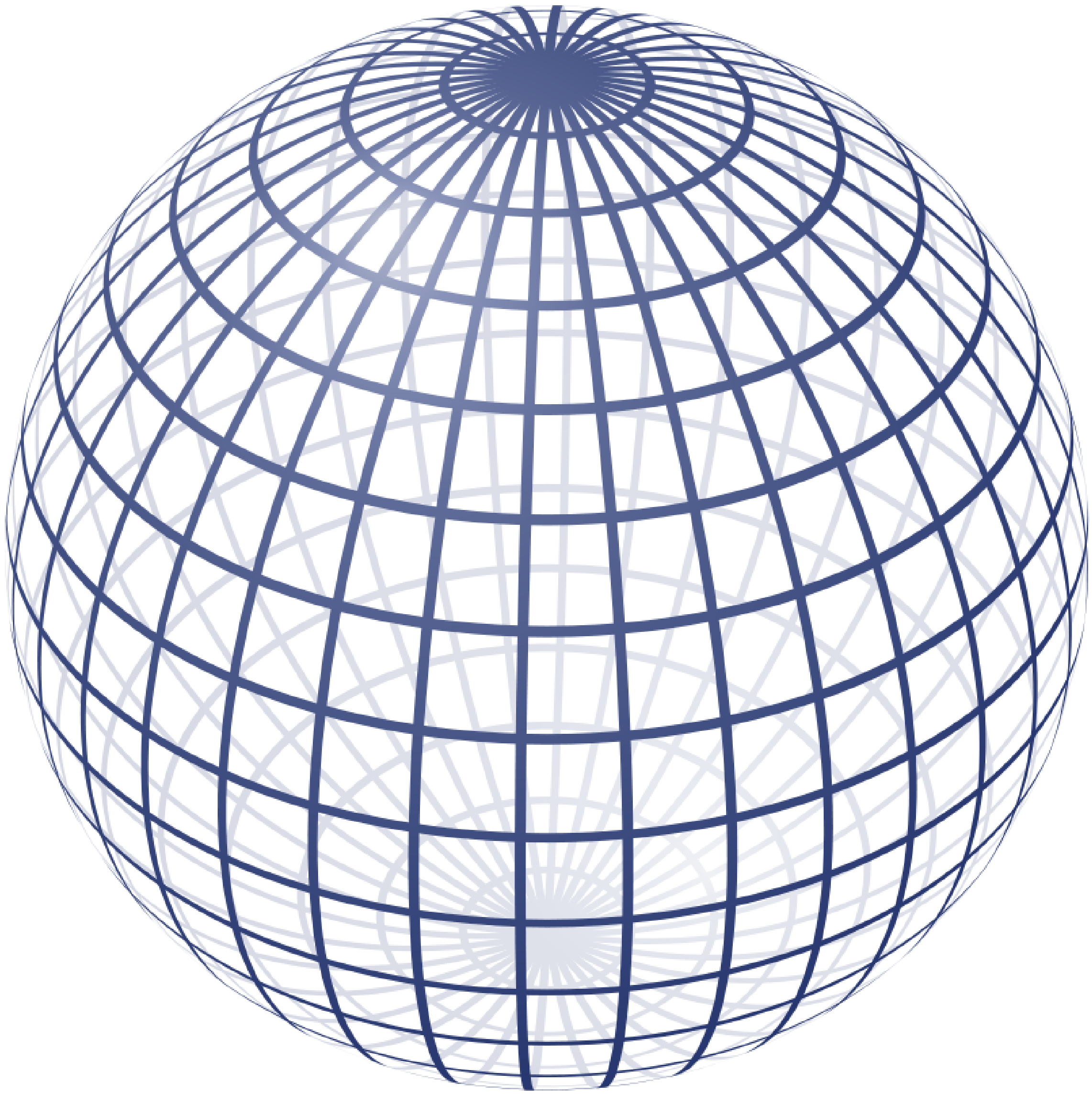
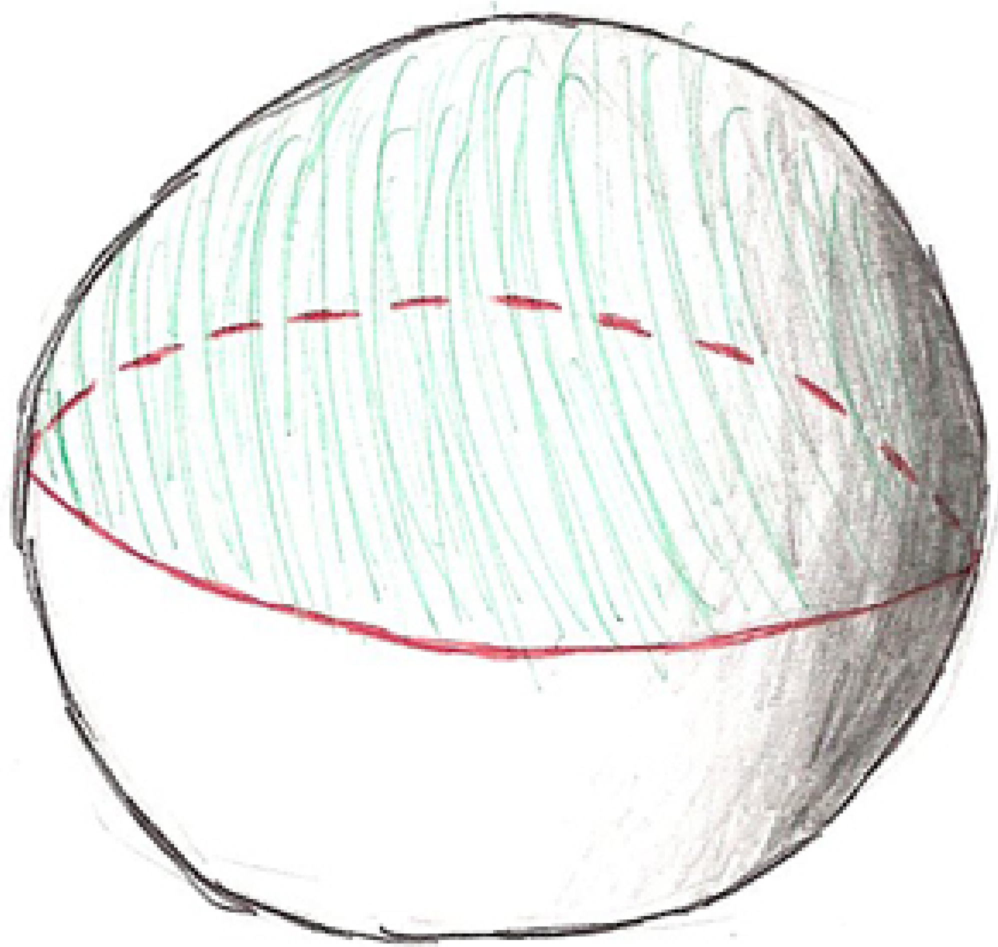
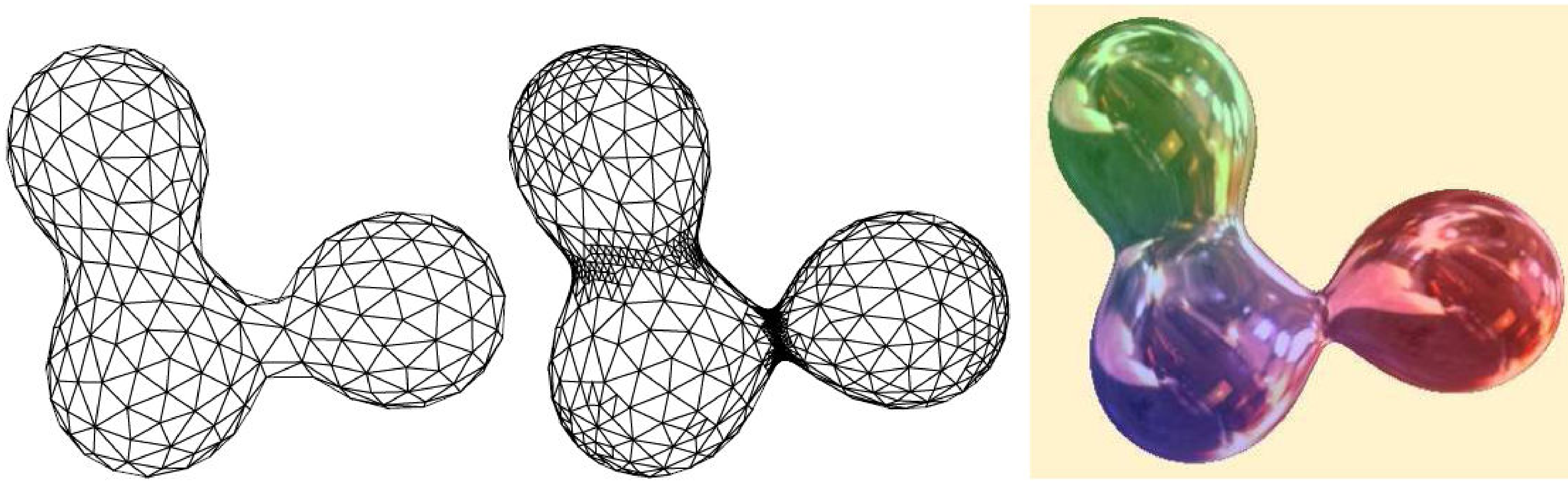
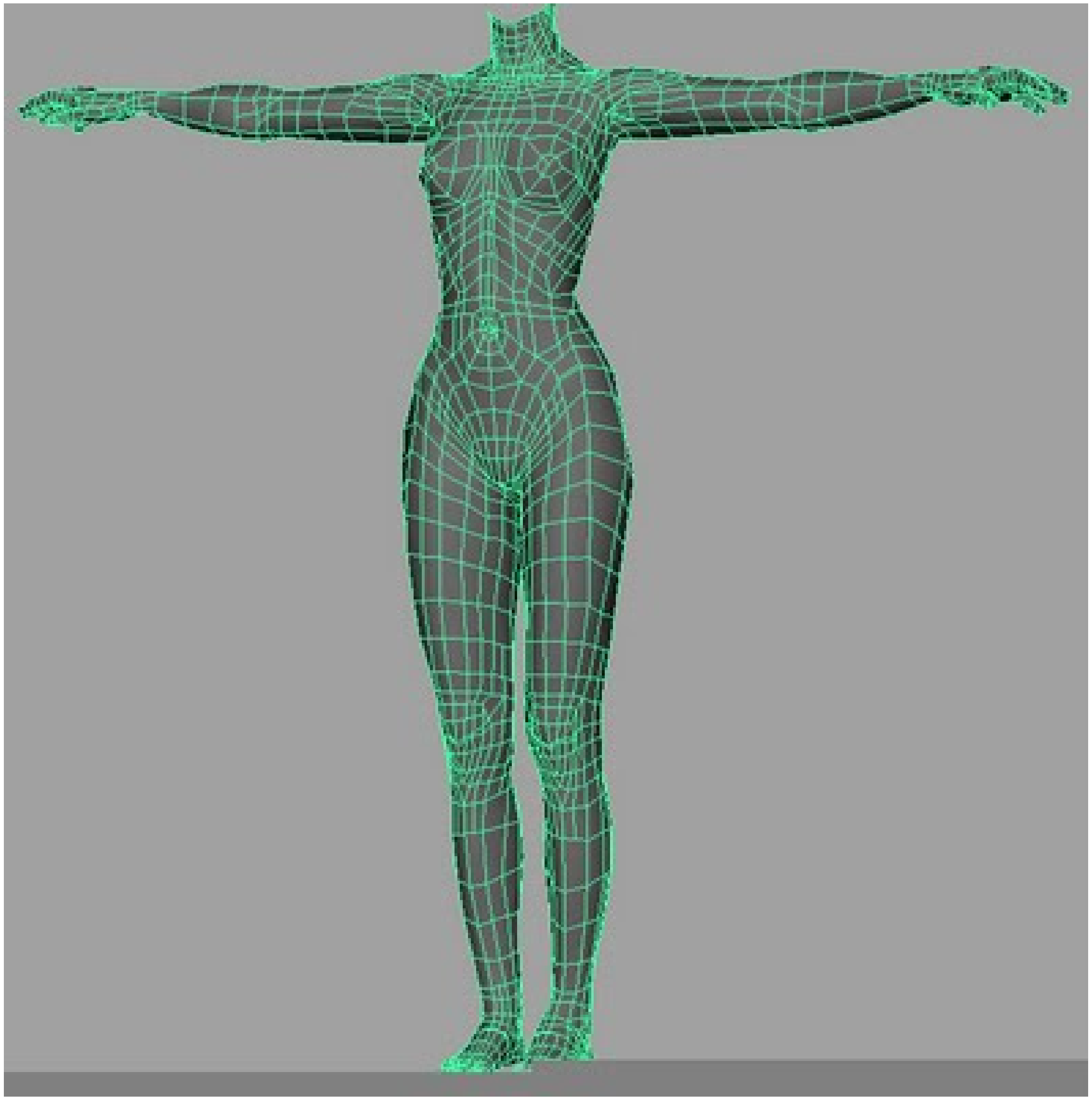
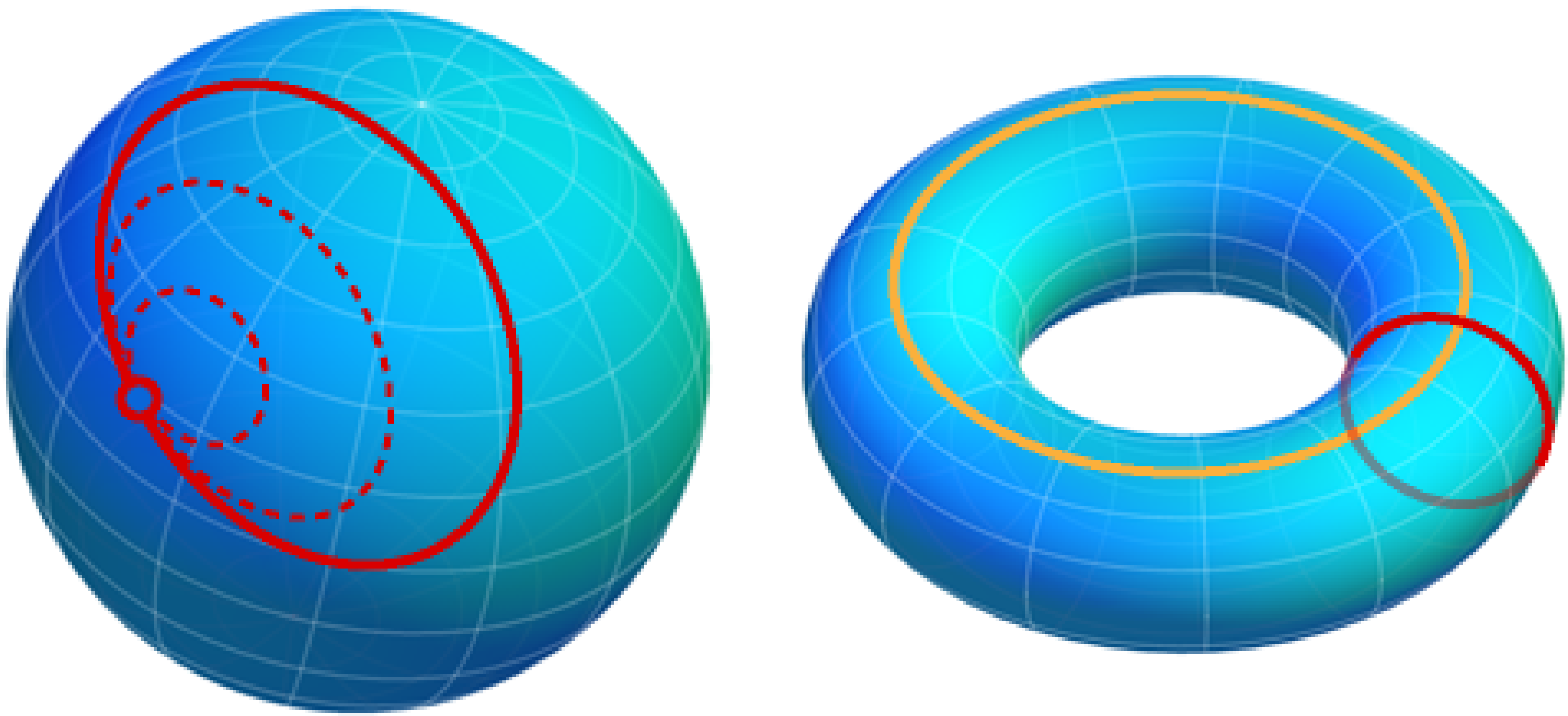
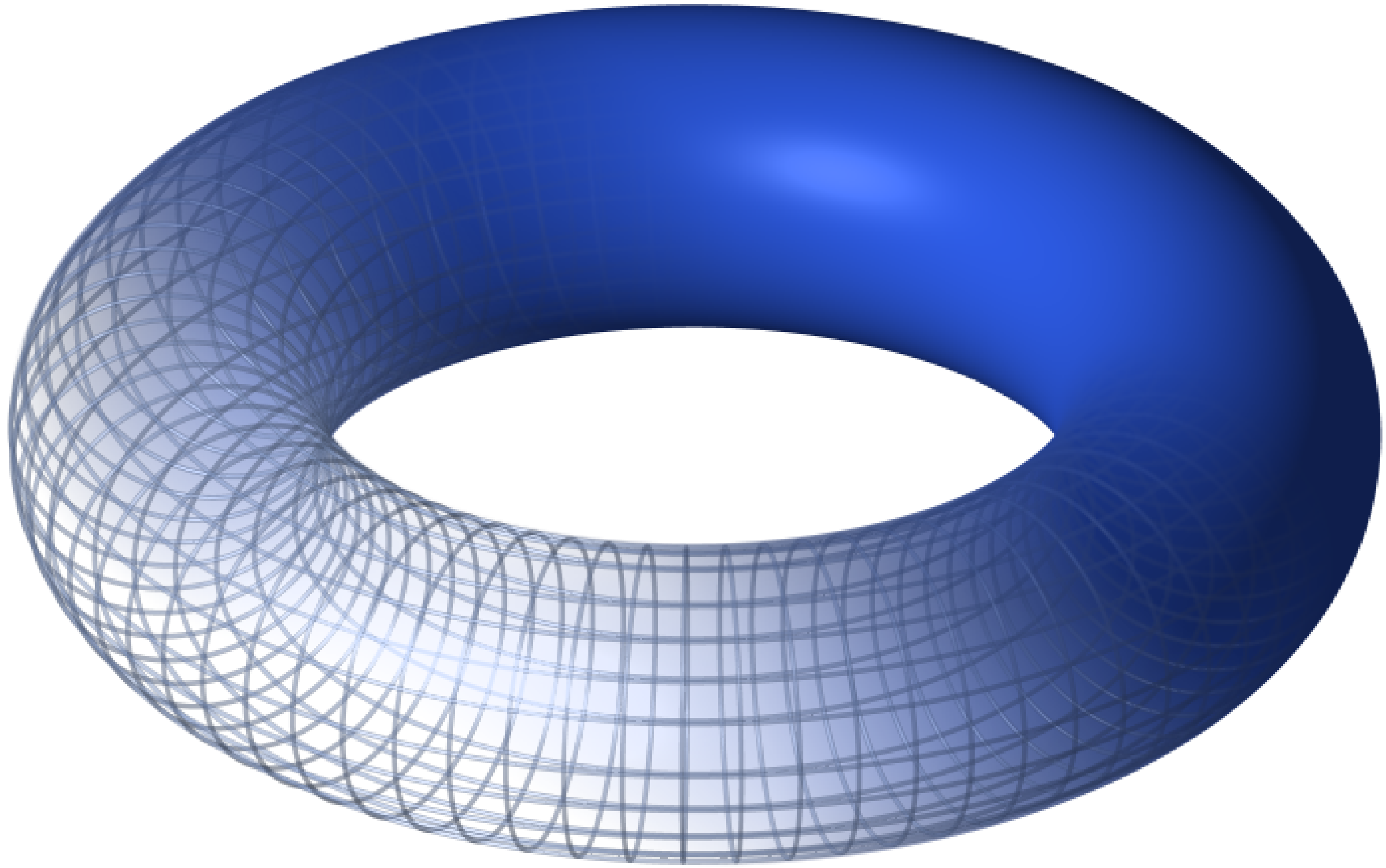


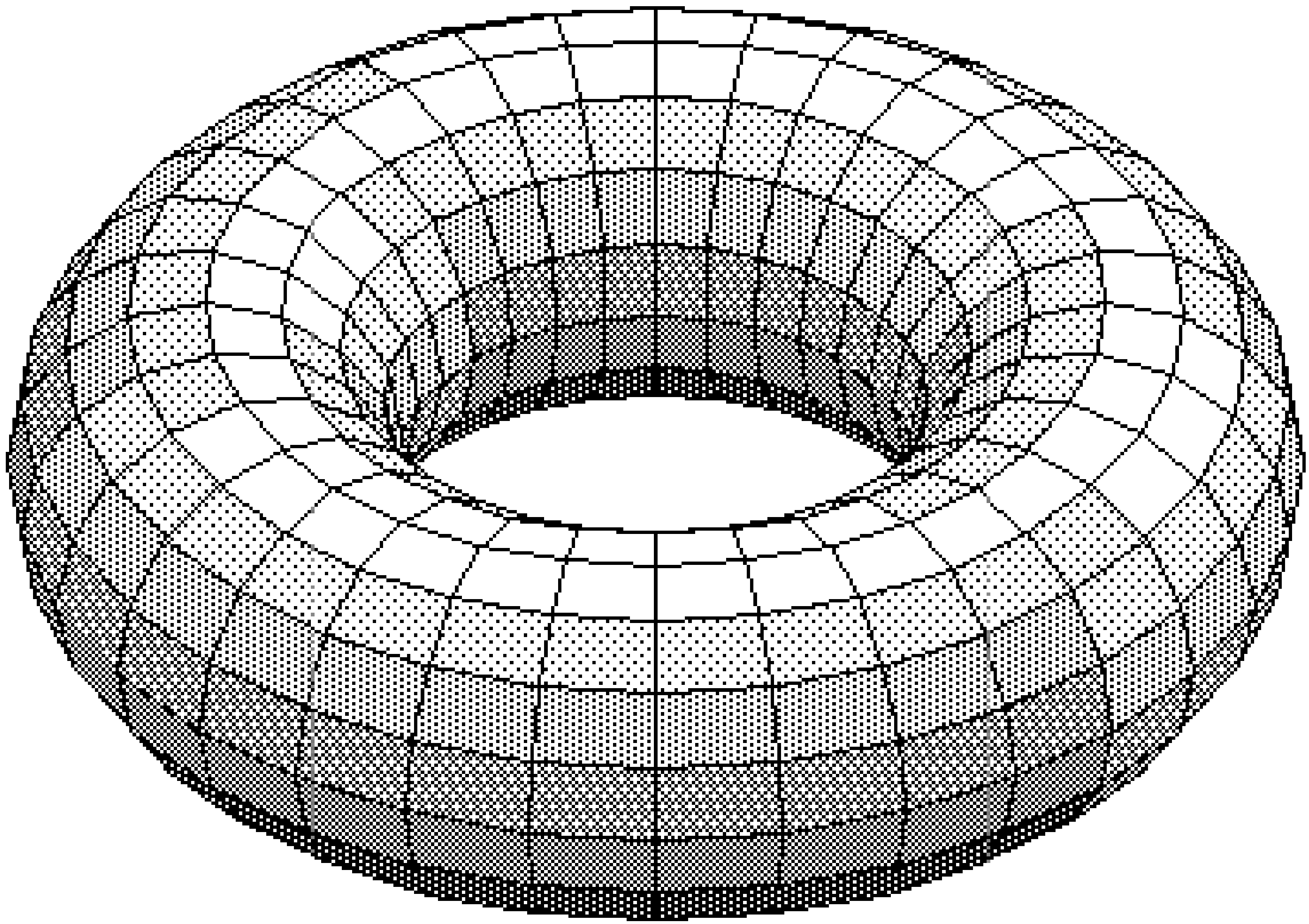


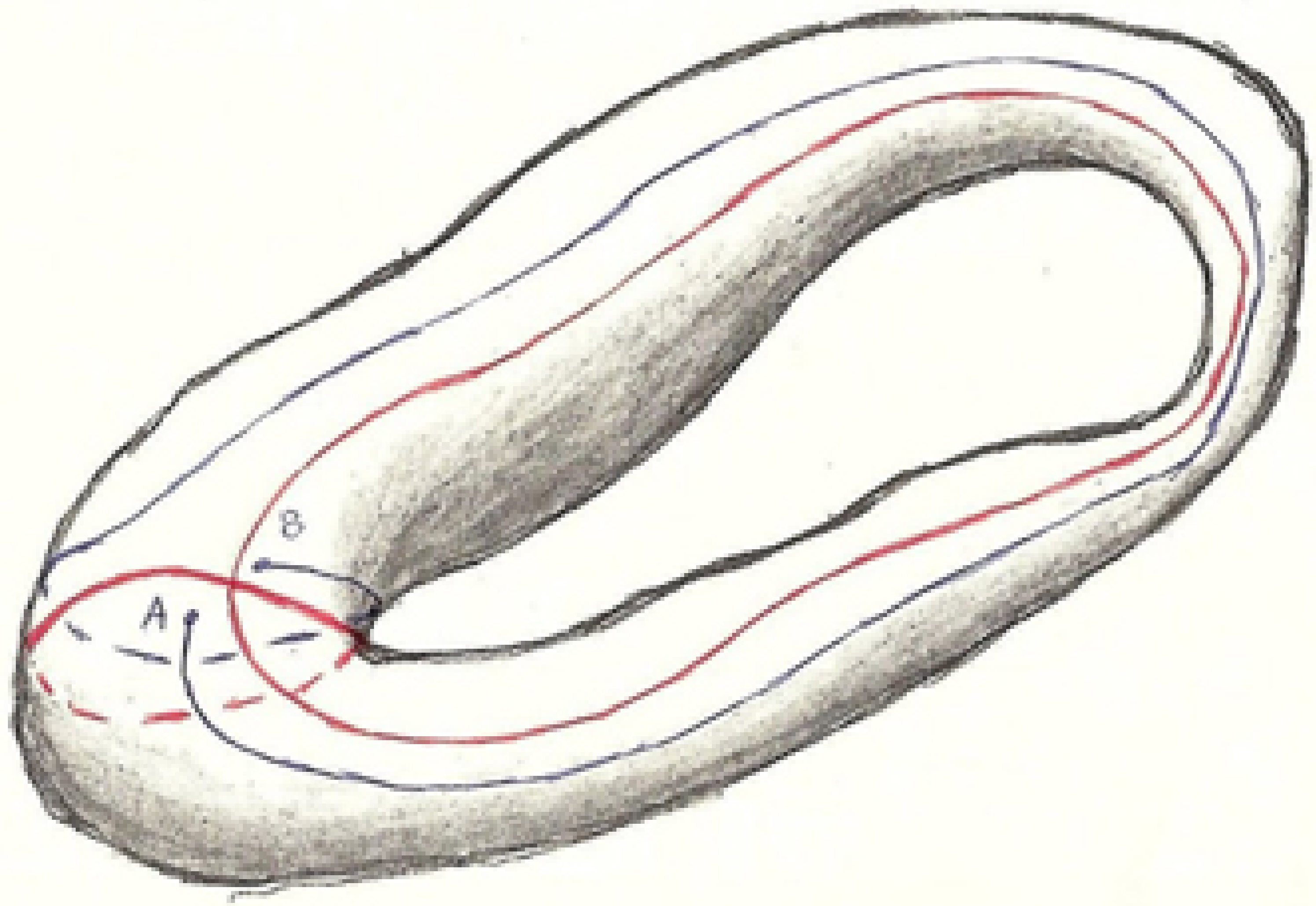


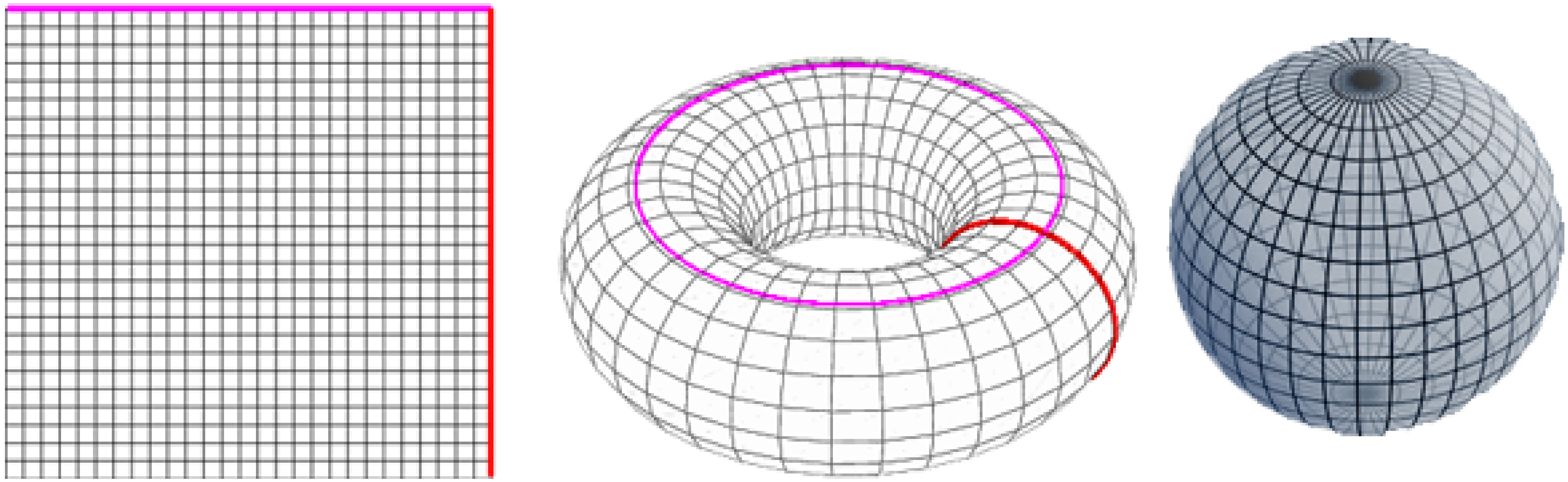


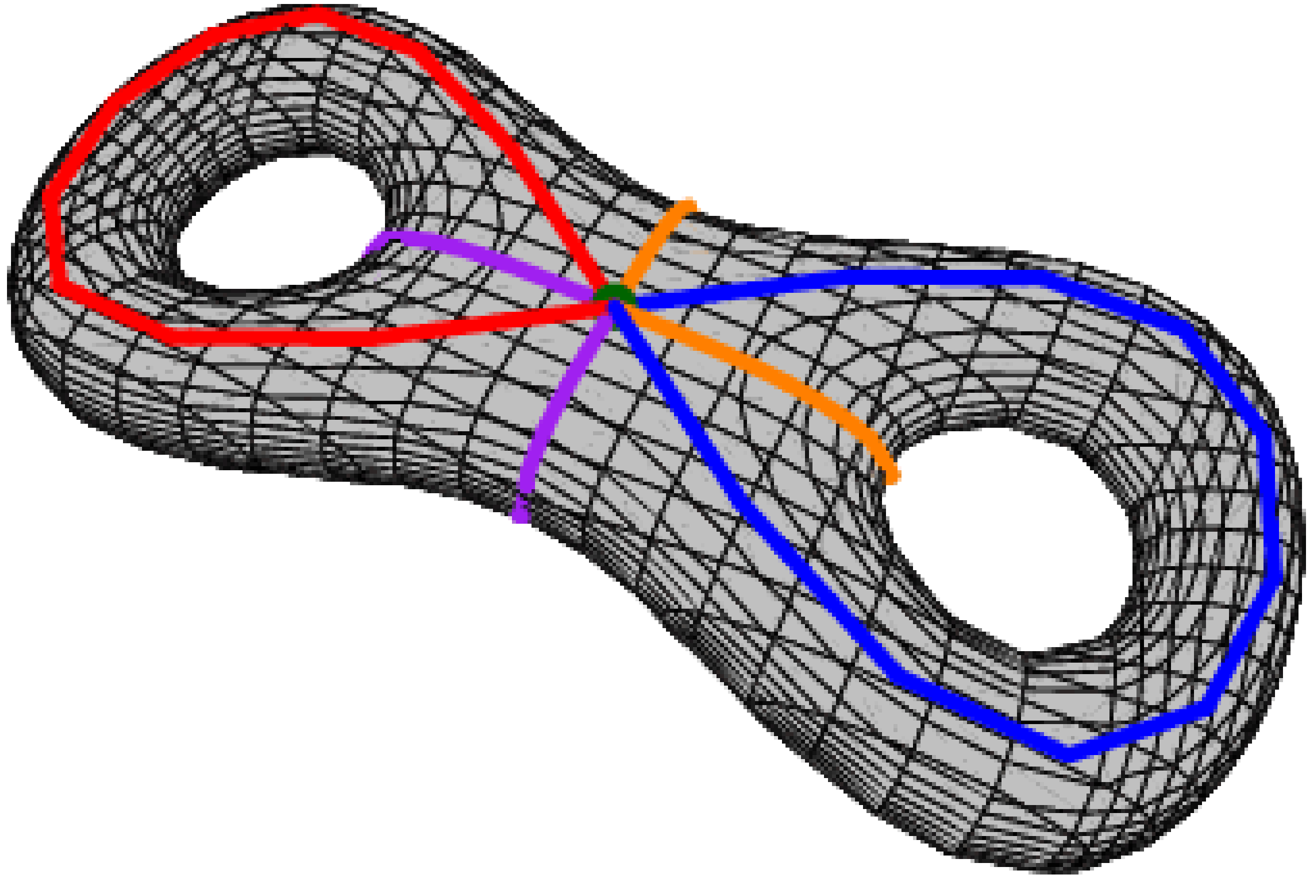


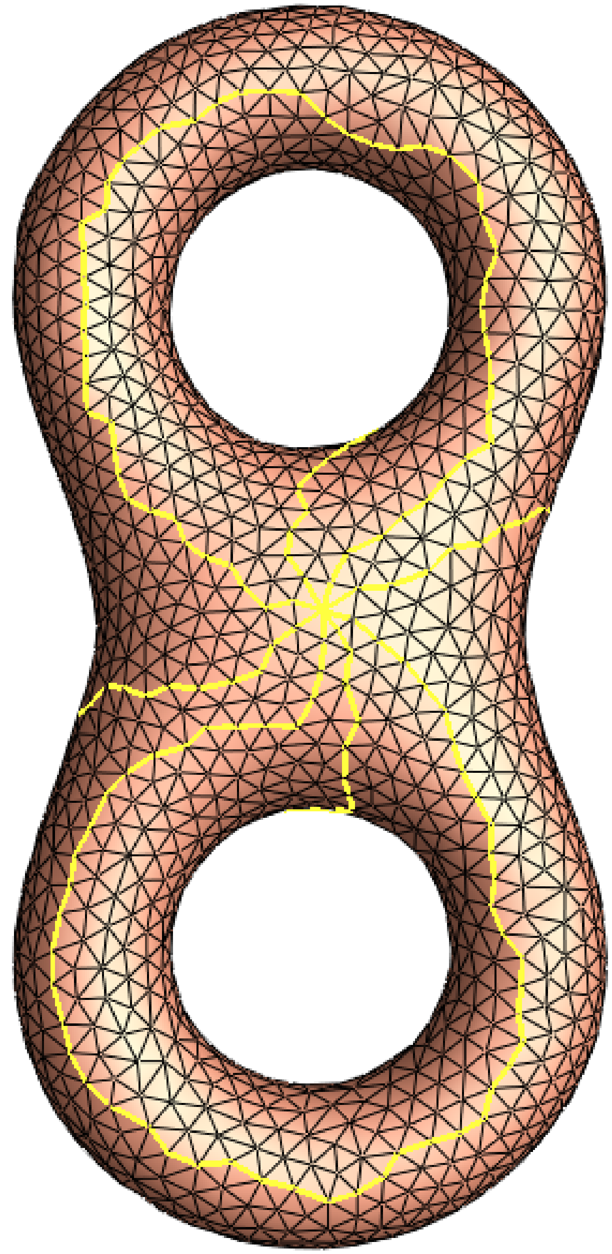


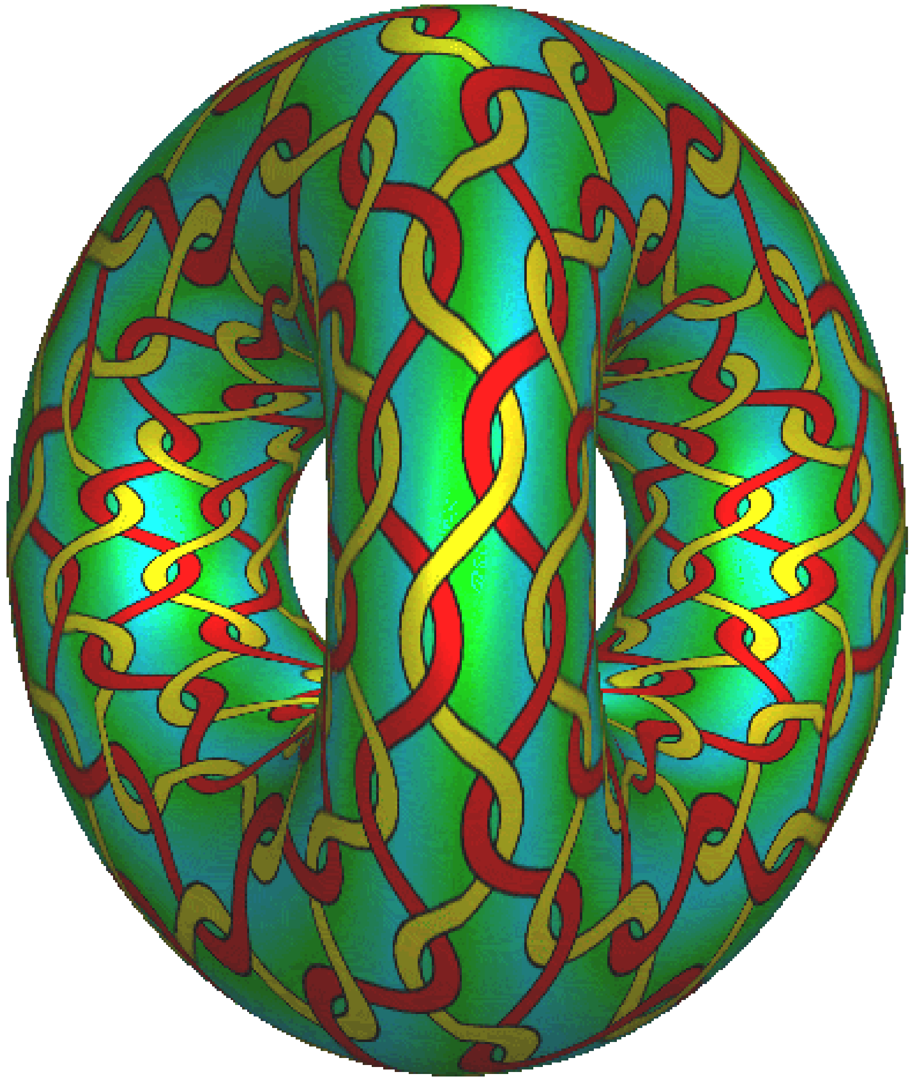


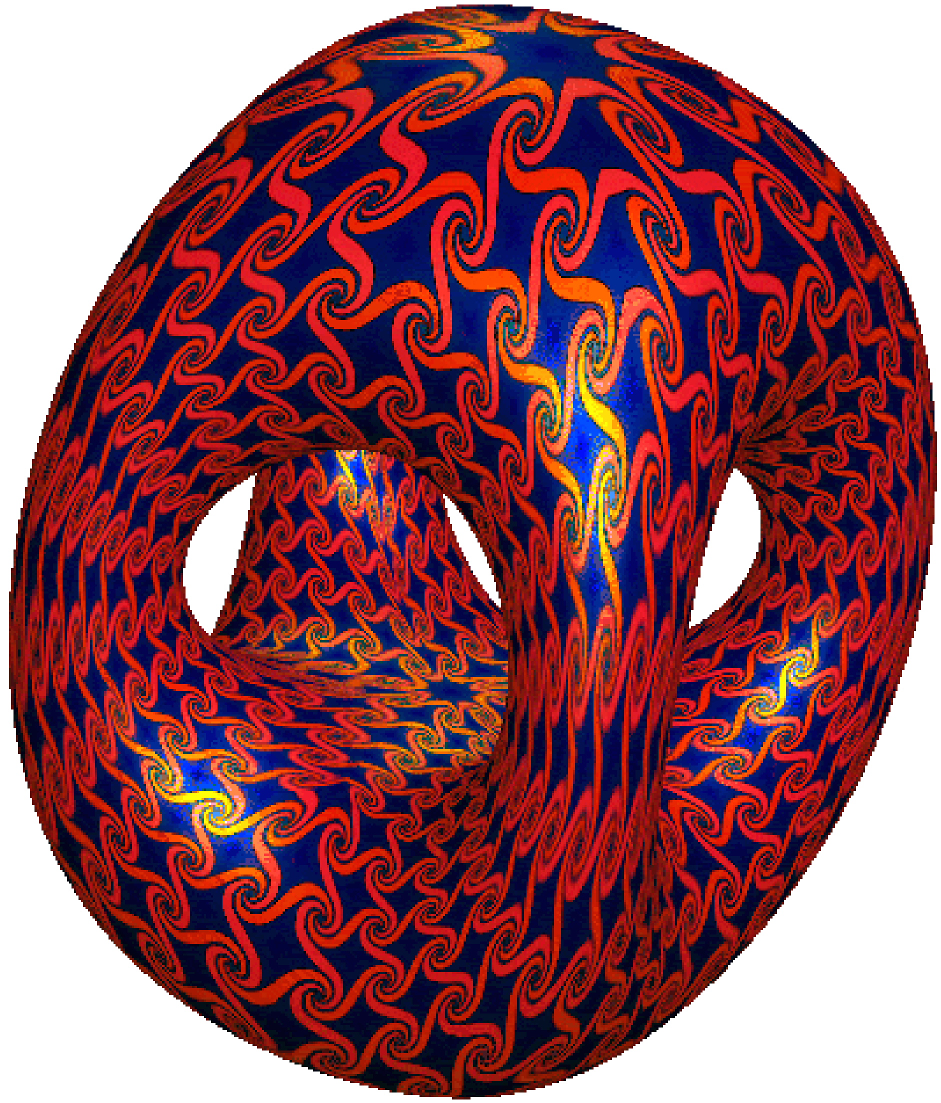

B
A

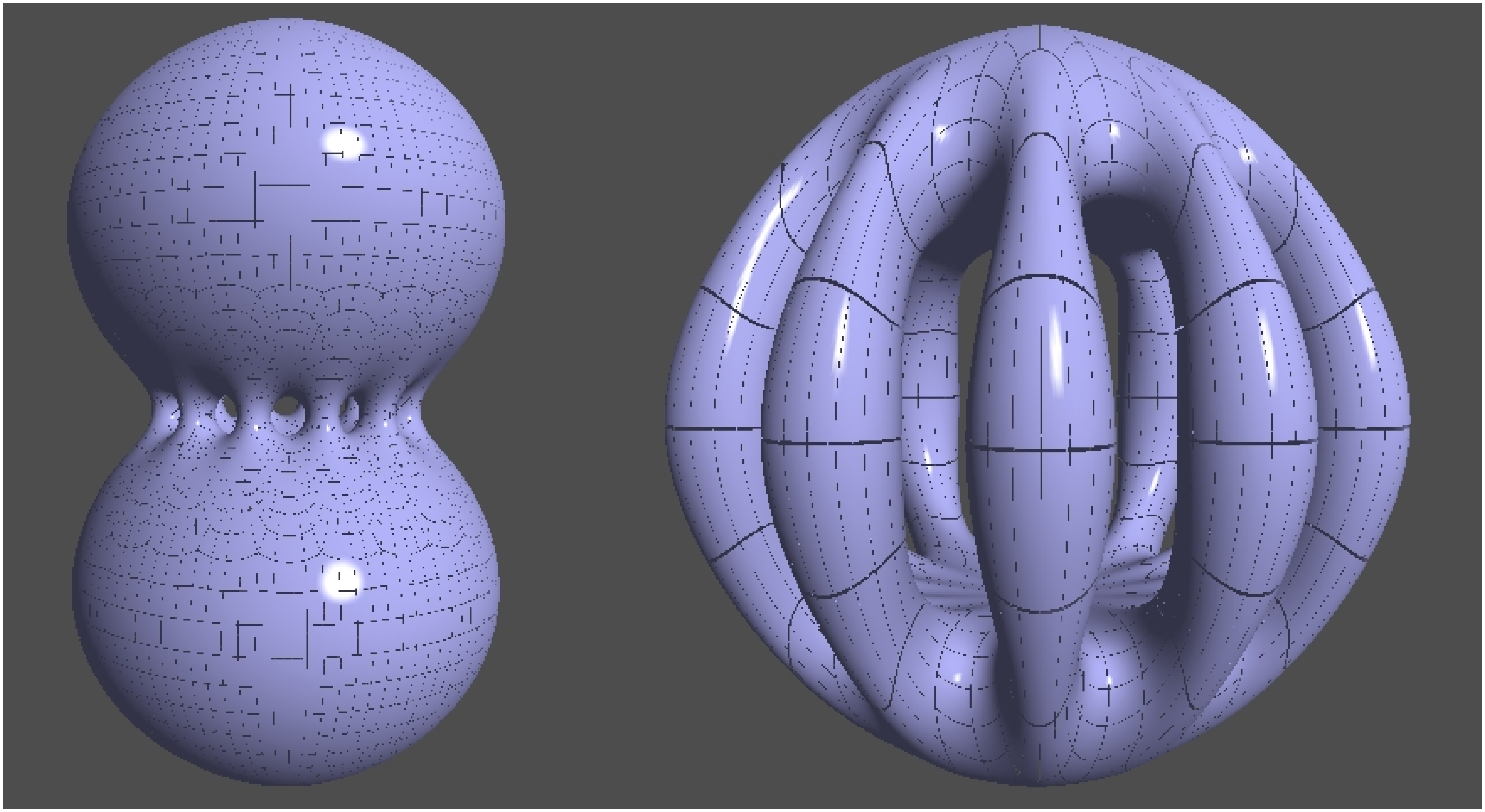


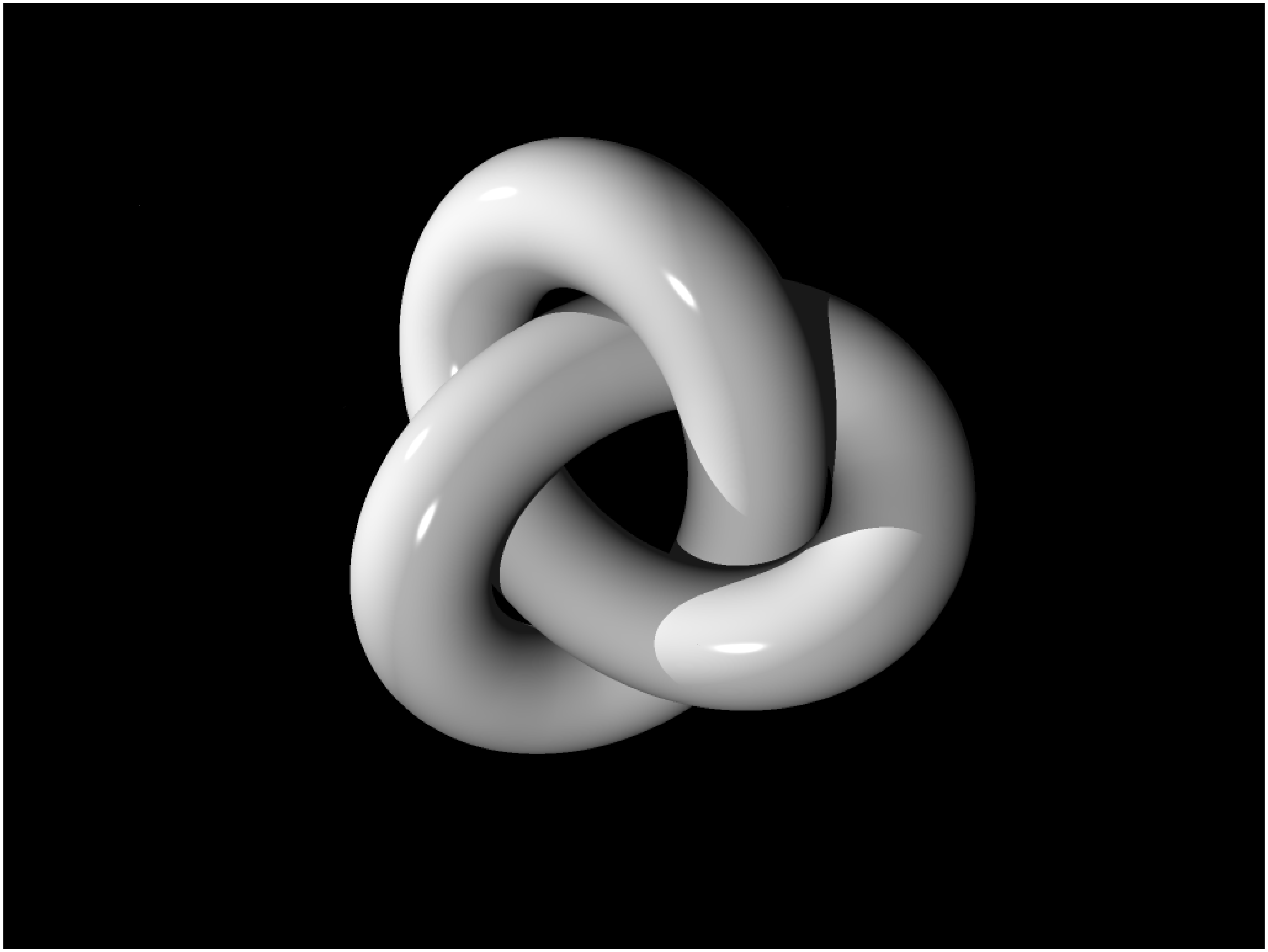


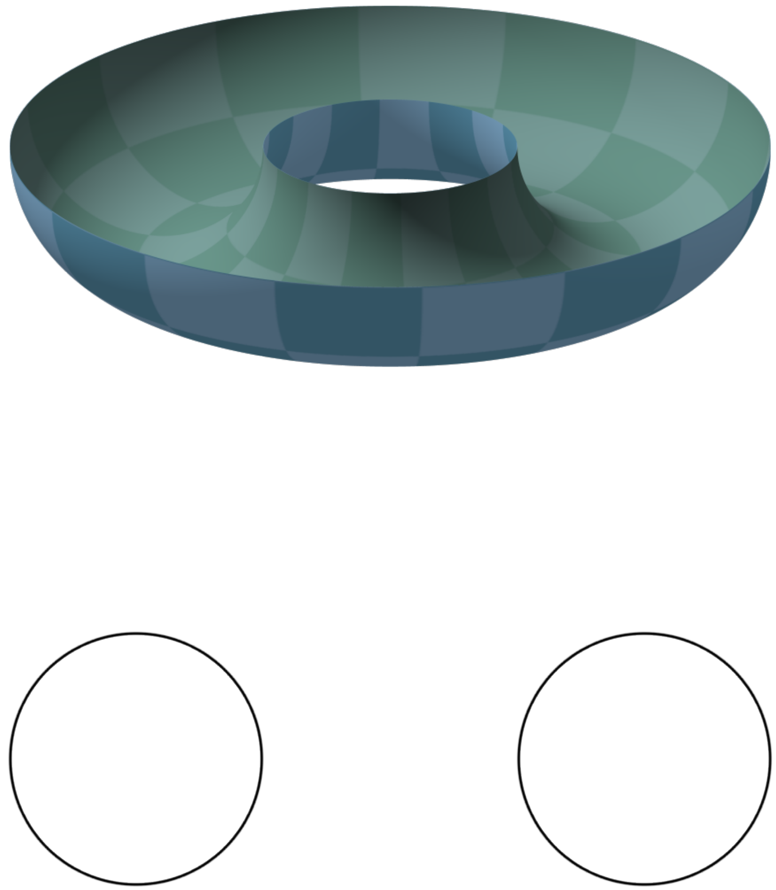


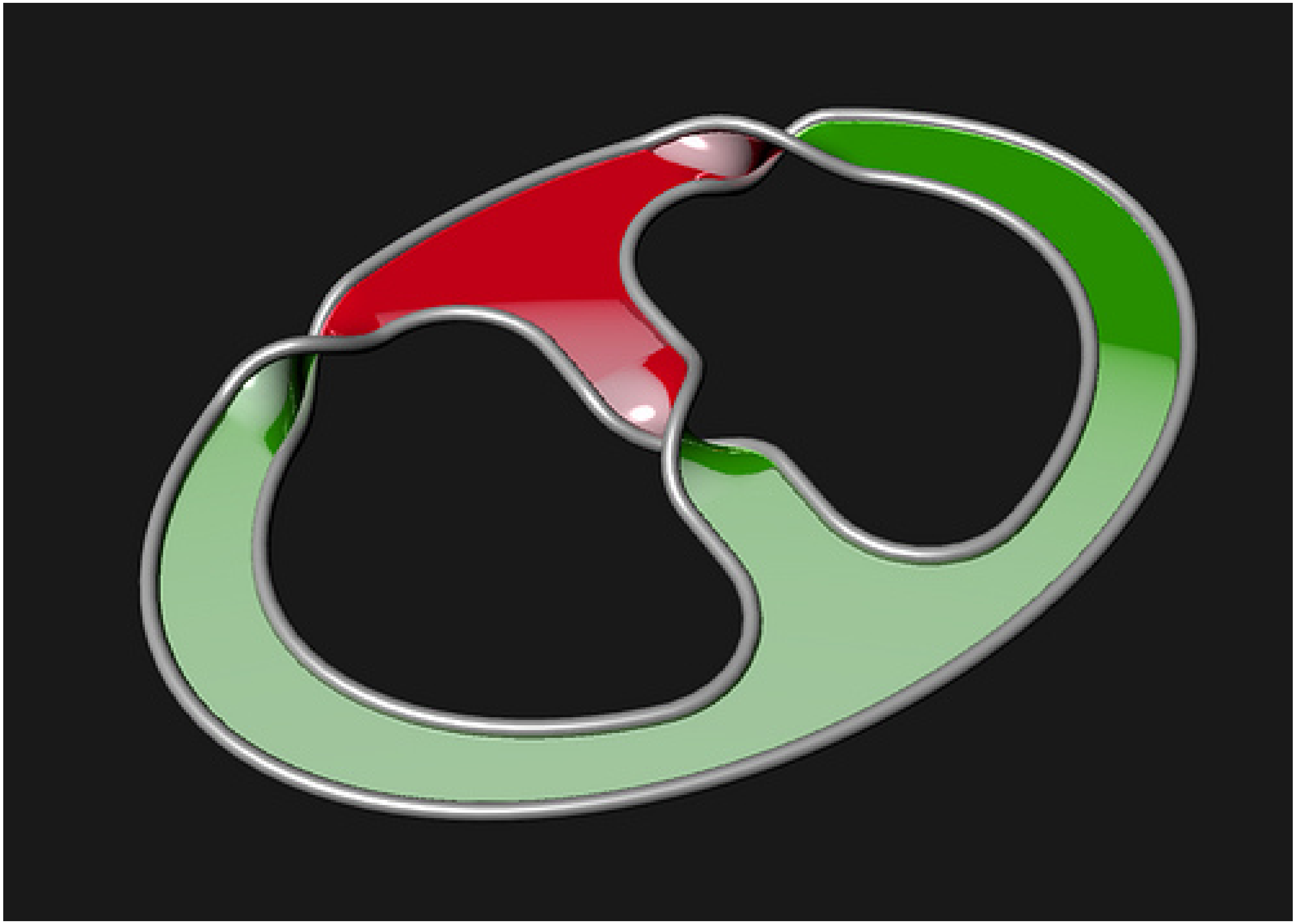


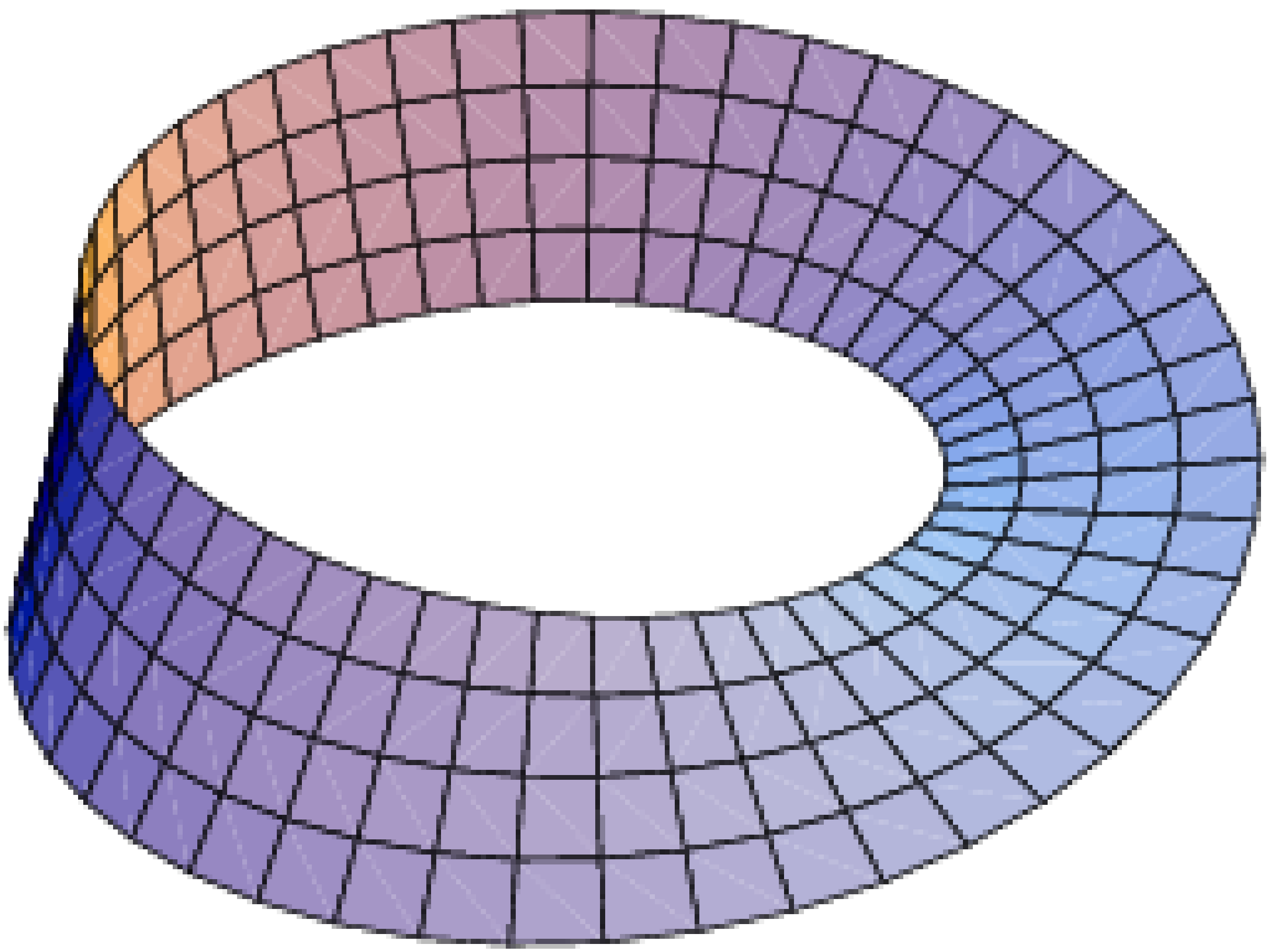


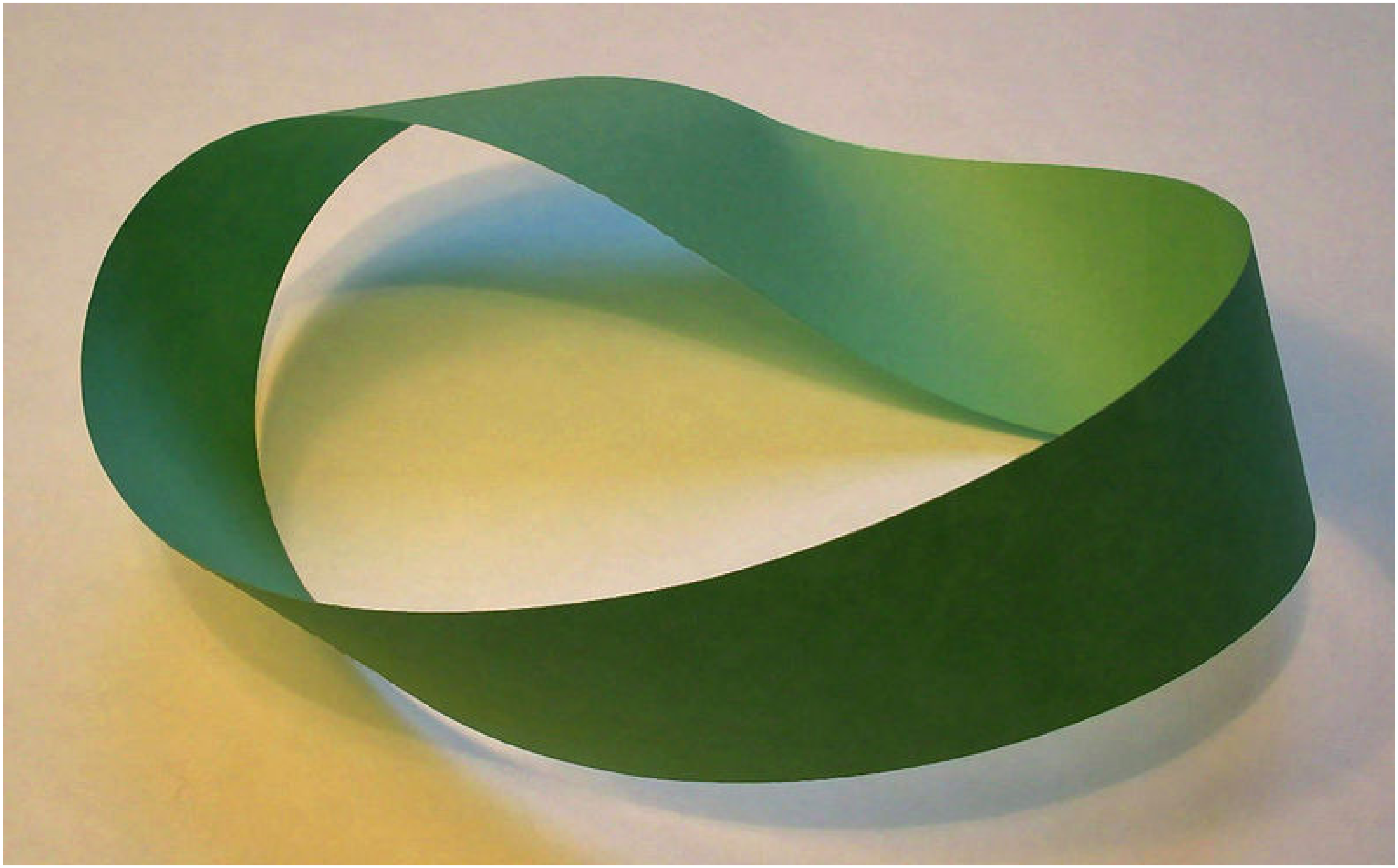


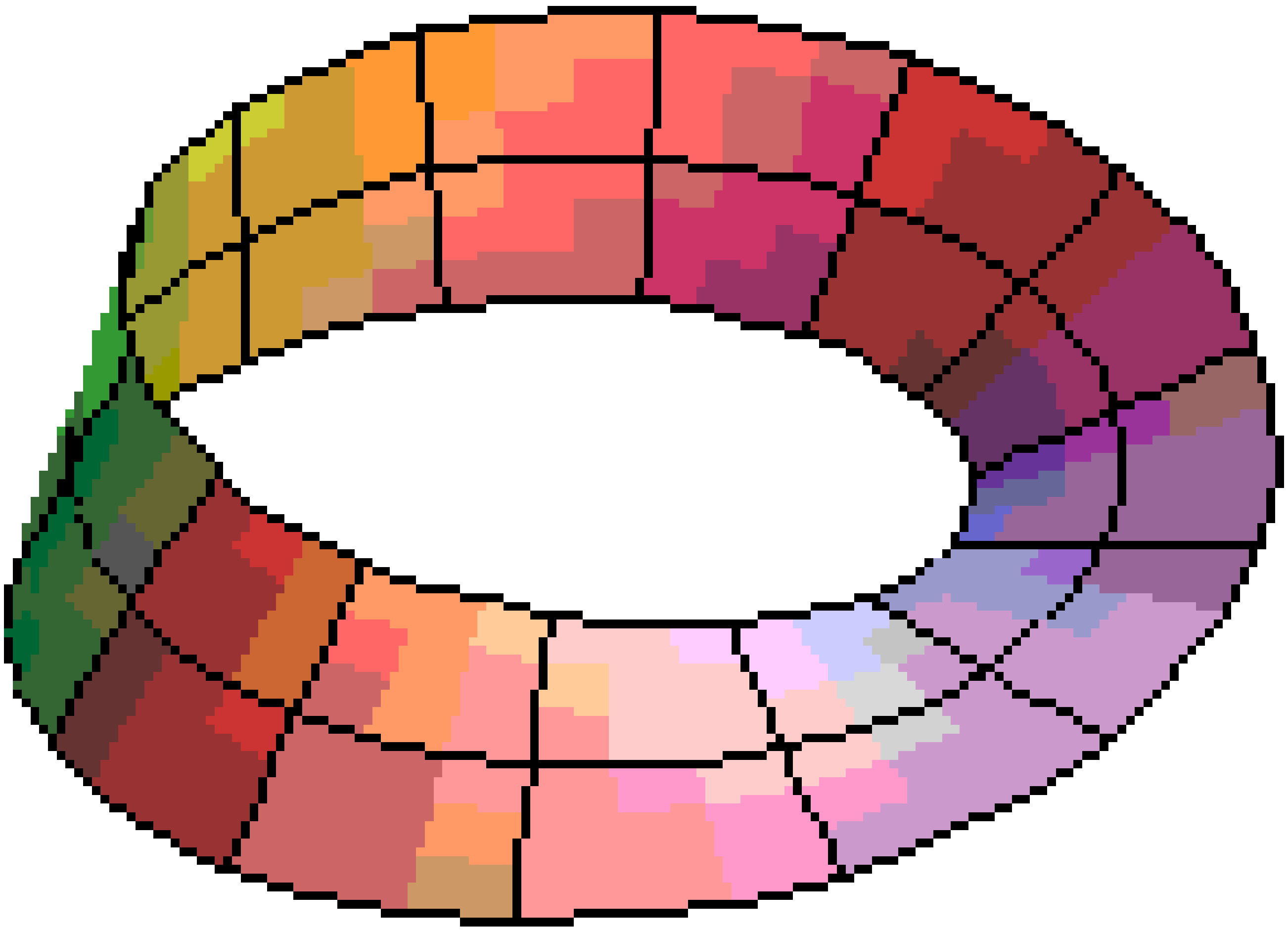


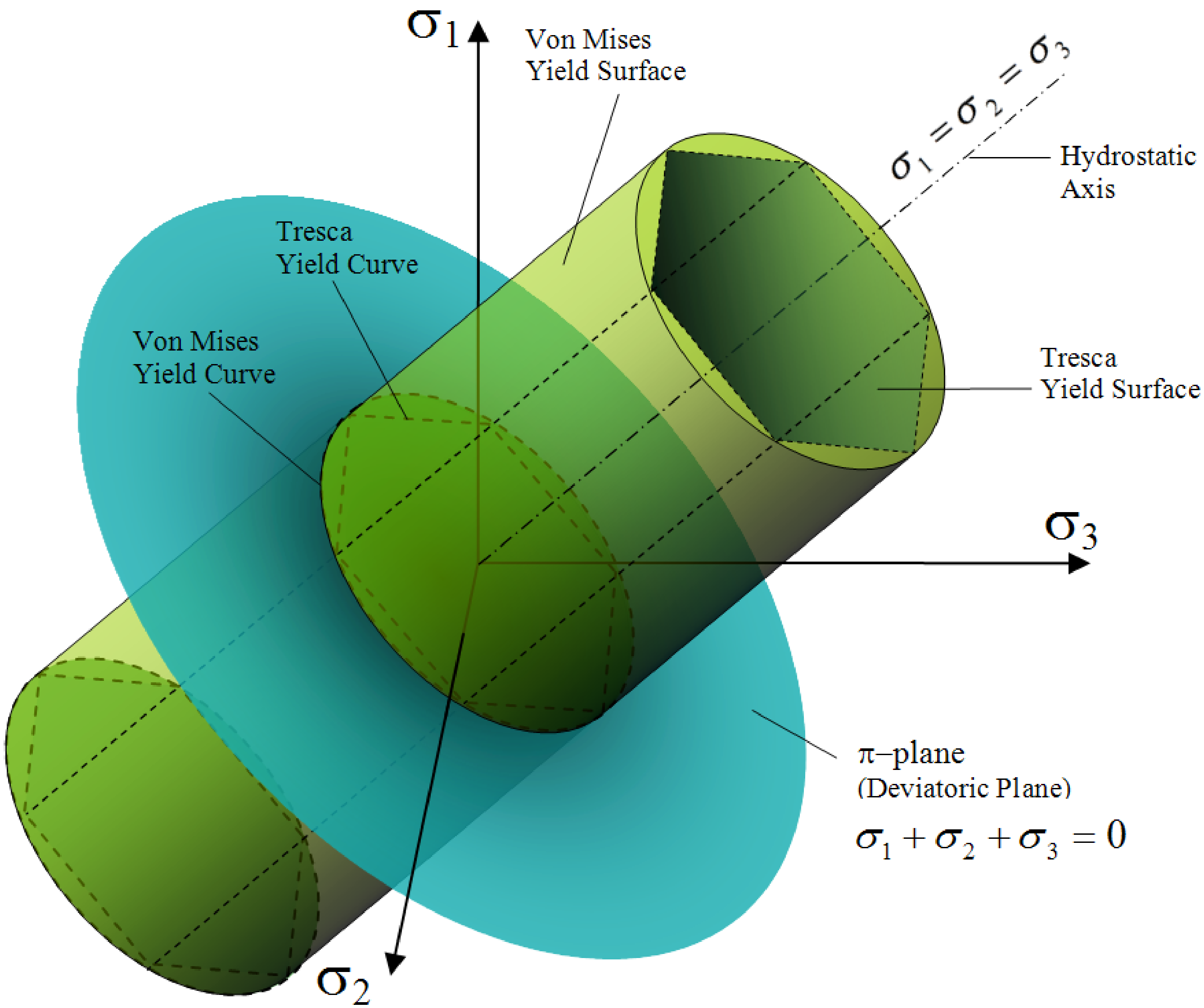




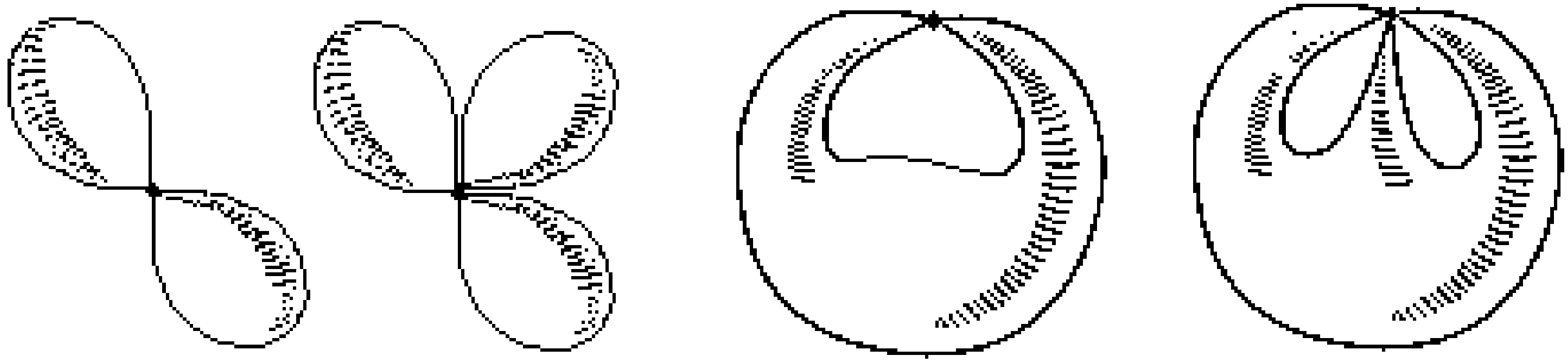




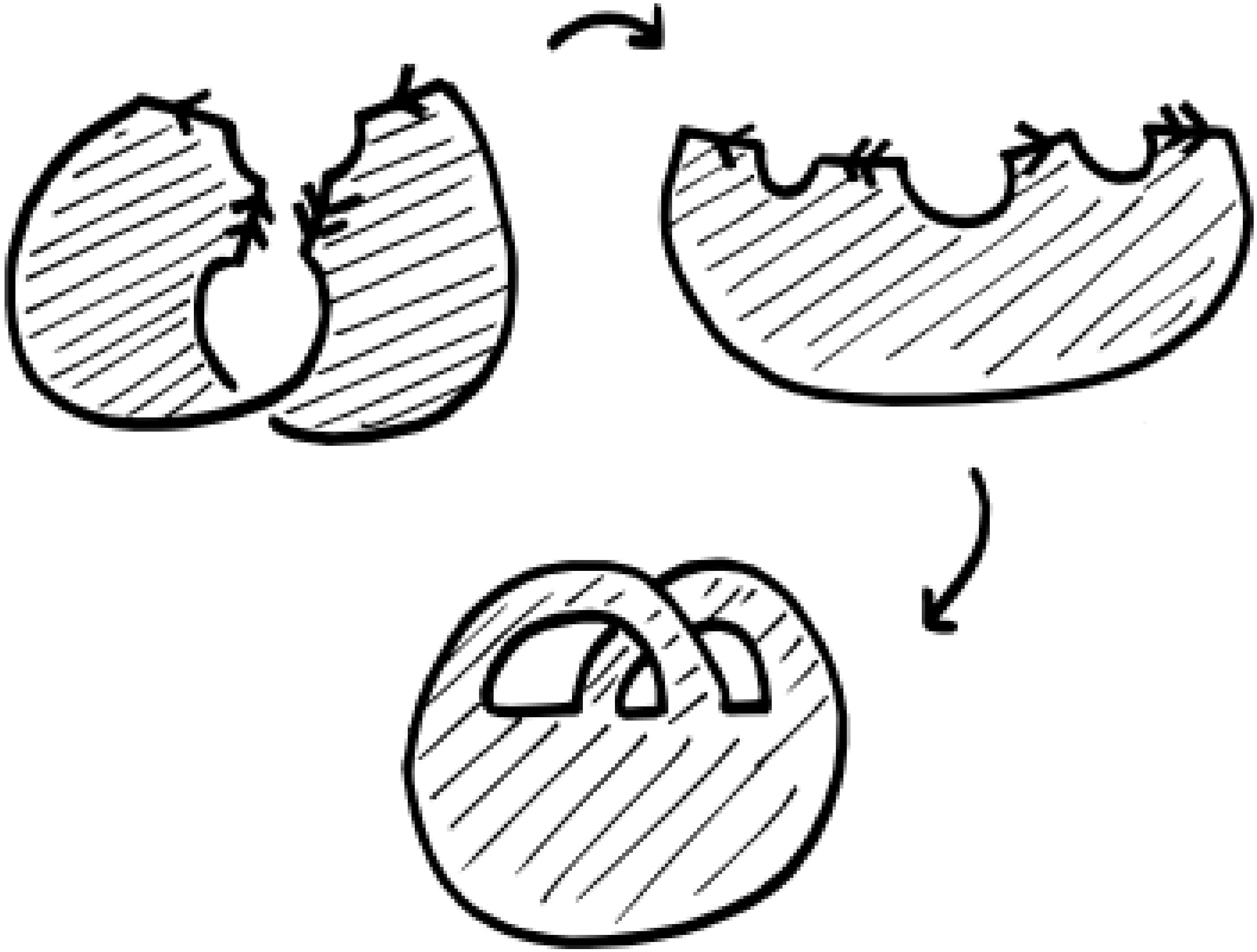


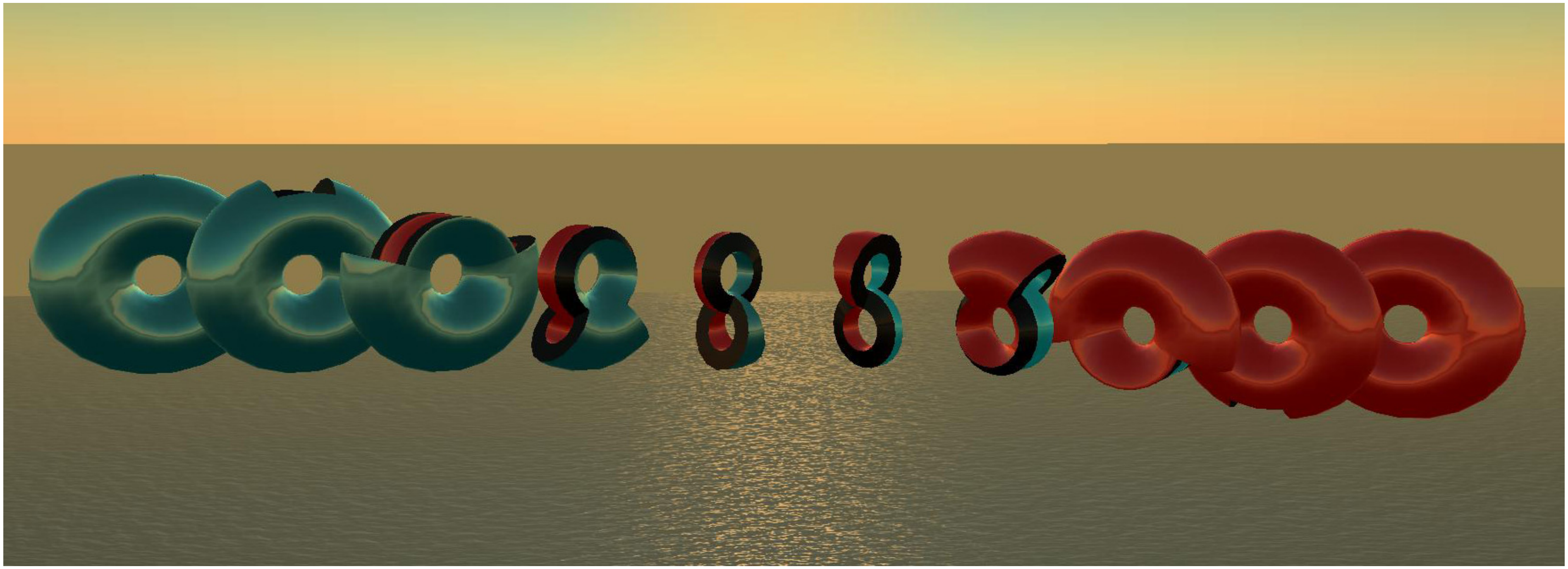


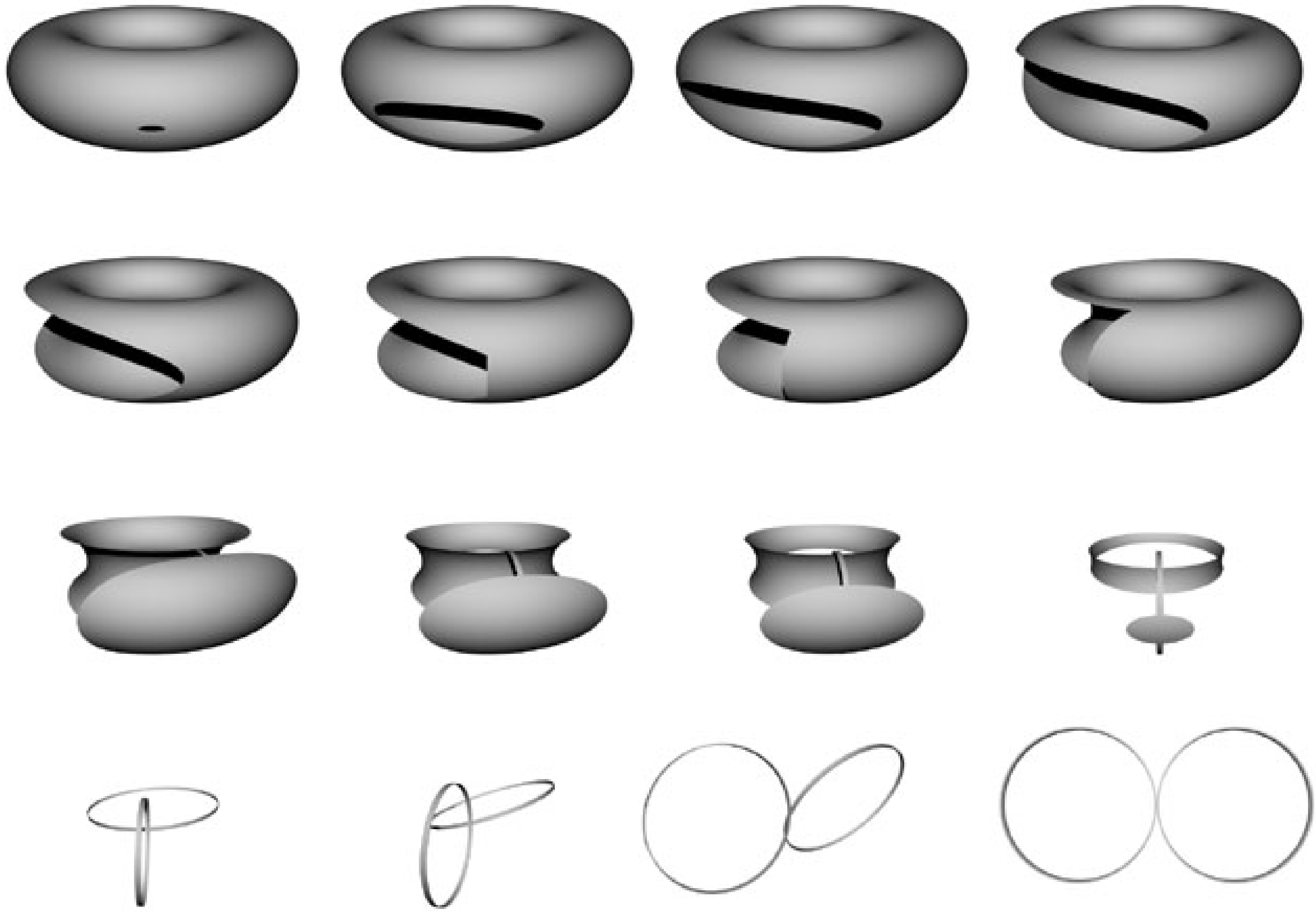
σ_1
Von Mises
Yield Surface
$\sigma_1 = \sigma_2 = \sigma_3$
Hydrostatic
Axis
Tresca
Yield Curve
Von Mises
Yield Curve
Tresca
Yield Surface
σ_3
π–plane
(Deviatoric Plane)
$\sigma_1 + \sigma_2 + \sigma_3 = 0$
σ_2

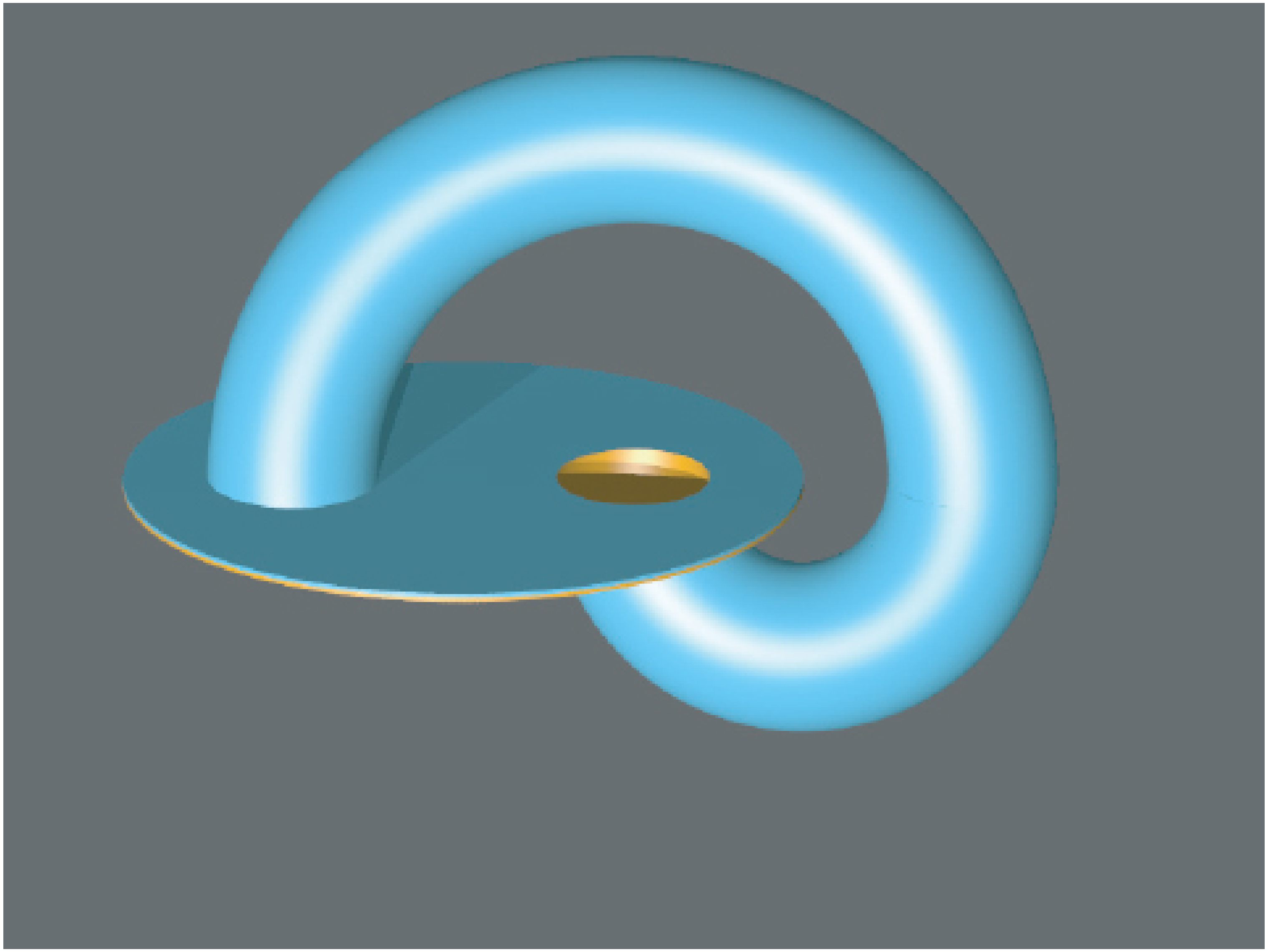


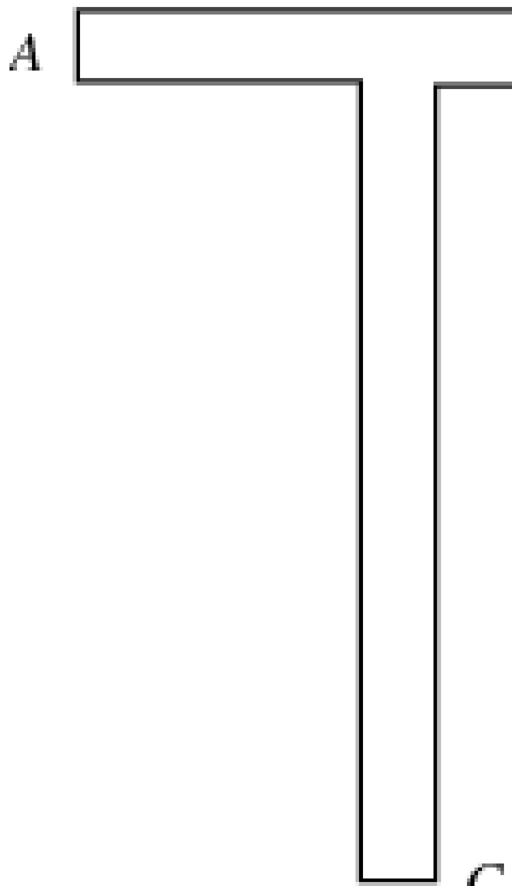
$\chi=3$ $\chi=4$ $\chi=1$ $\chi=0$

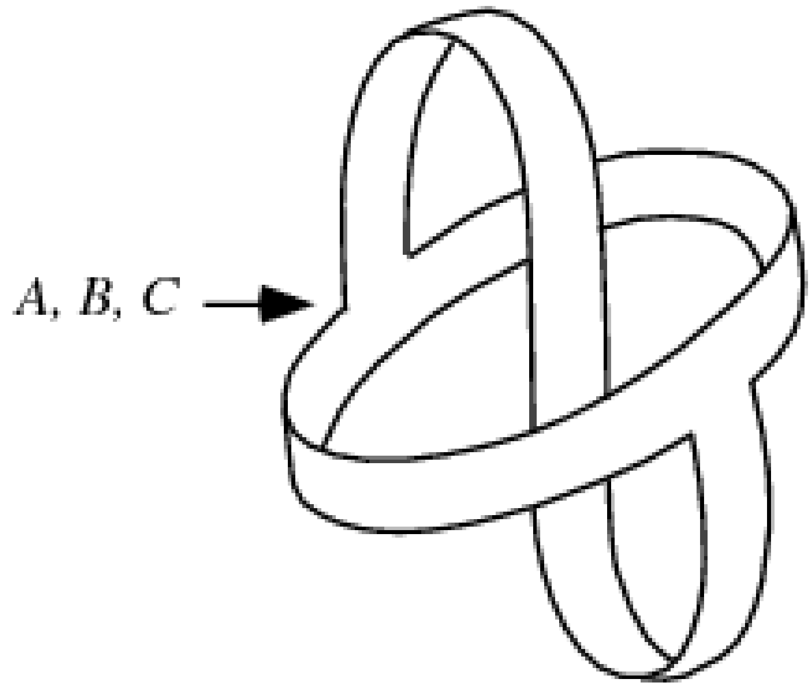









A
B
C


A, B, C

Superficies

Una *superficie* es un espacio X tal que X es de Hausdorff y cada punto de X tiene una vecindad homeomorfa a B^2.

(Sólo nos interesan las superficies conexas)

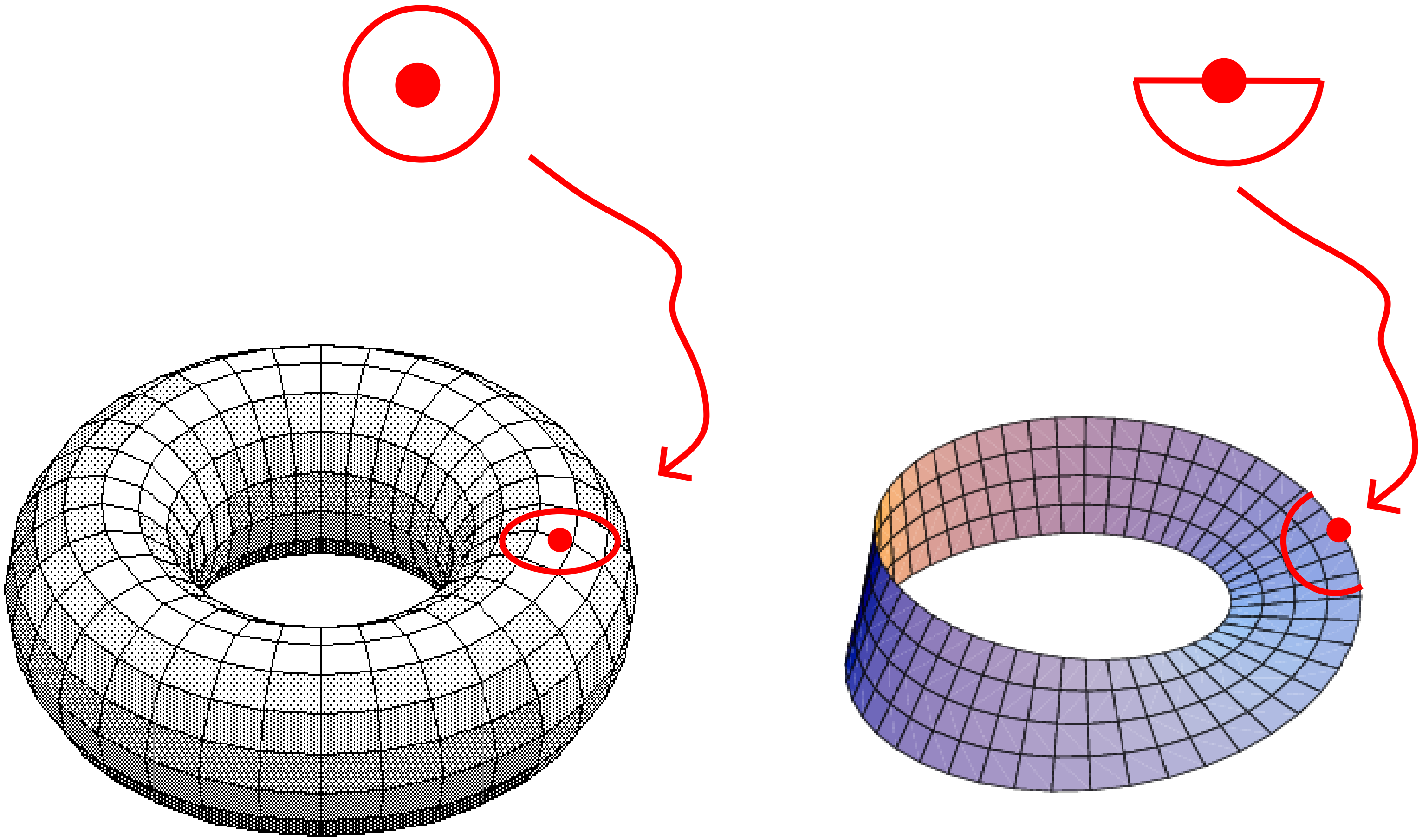
Si X es una superficie, entonces

$$int(X) = \mathring{X} = \{a \in X | \exists U \in \mathcal{N}_a, U \cong \mathring{B}^2\}$$

es el *interior* de X.

$$\partial X = X - int(X)$$

es la *frontera* de X.



Una superficie X se dice que es *no orientable*, si X contiene una cinta de Möbius.

Se dice que X es *orientable*, si X no es no orientable.

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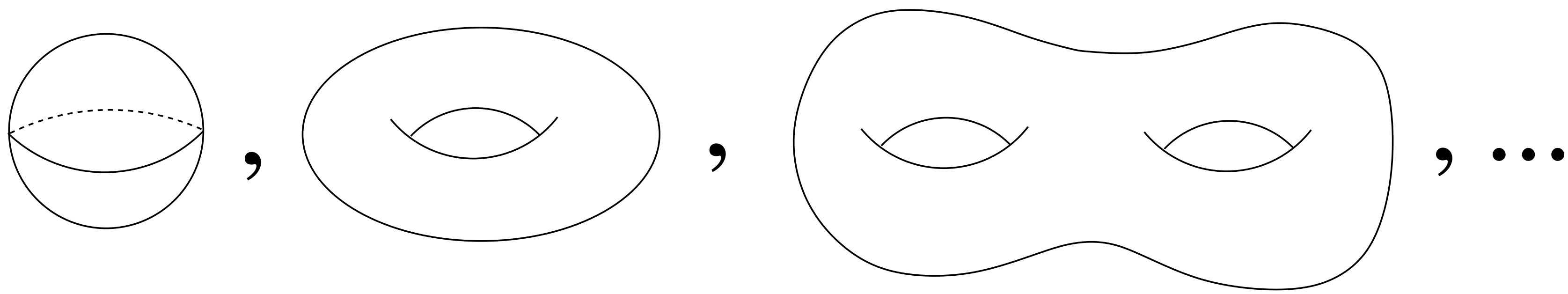
Se dice que X es *orientable*, si X no es no orientable.

(cosas de matemáticos)

Una superficie X se dice que es *cerrada* si es compacta y tiene frontera vacía.

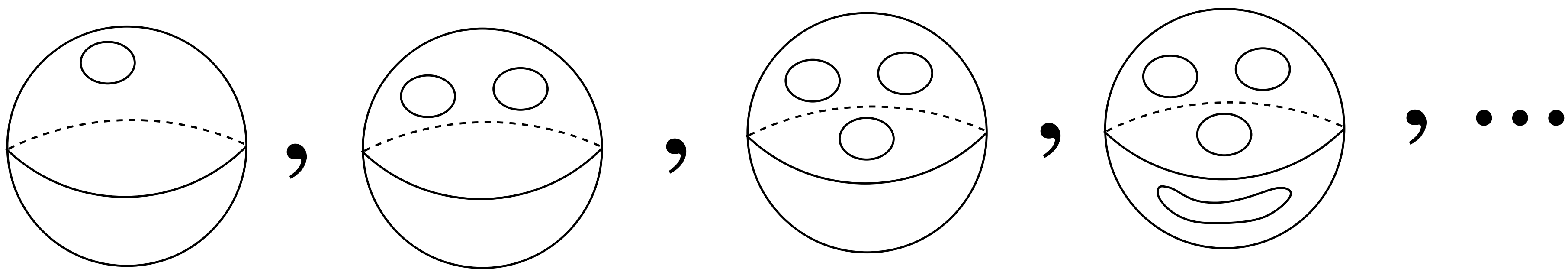
Las Superficies cerradas y orientables (y conexas)



Las Superficies cerradas y no orientables (y conexas)

$$P^2,\ P^2 \# P^2,\ P^2 \# P^2 \# P^2,\ \ldots$$

Las Superficies compactas con frontera:



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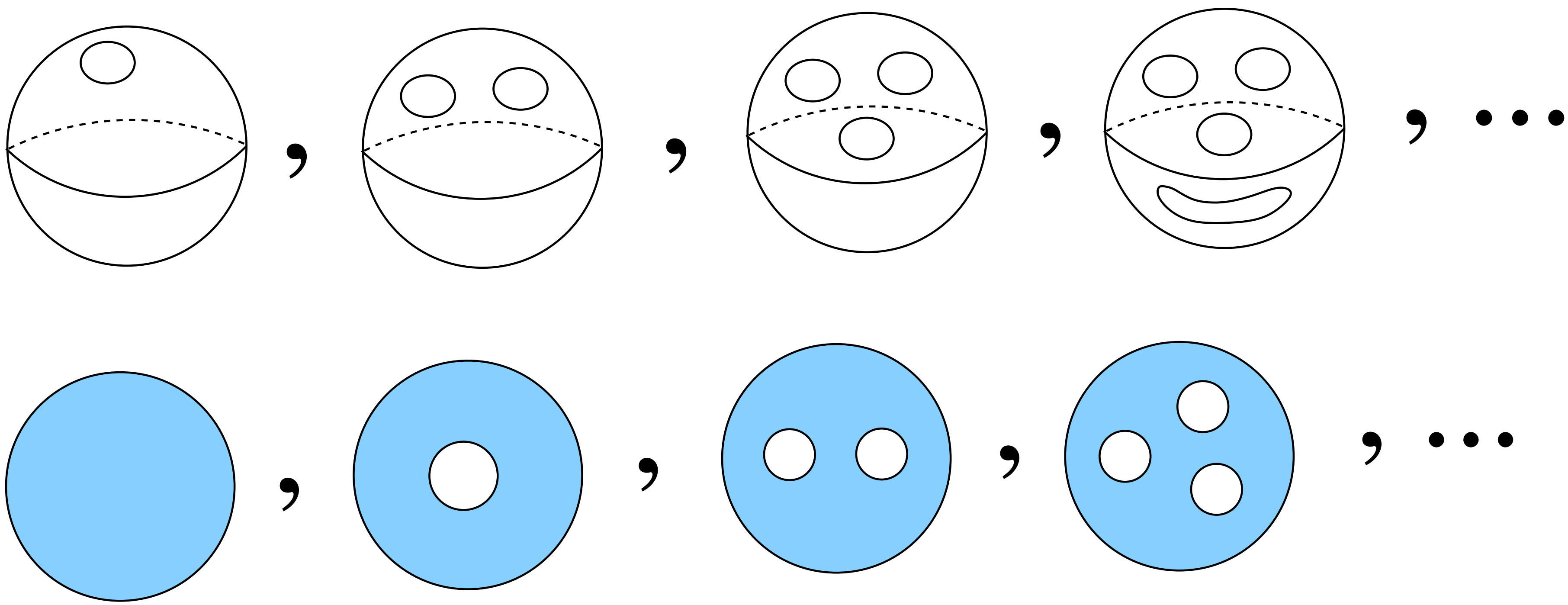
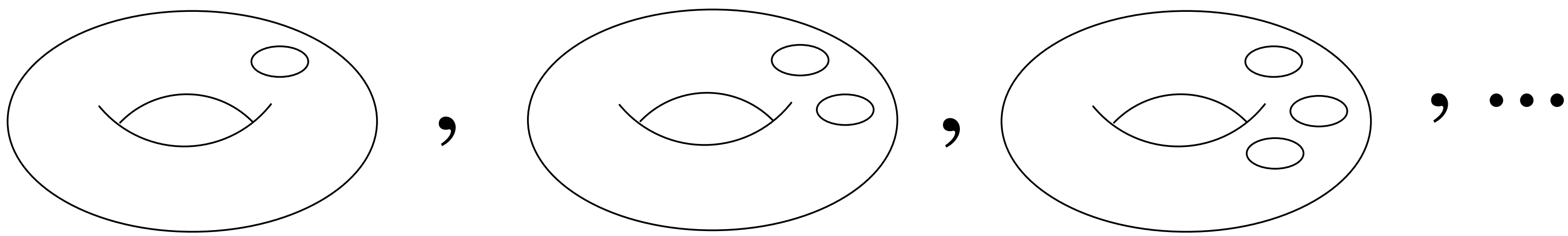
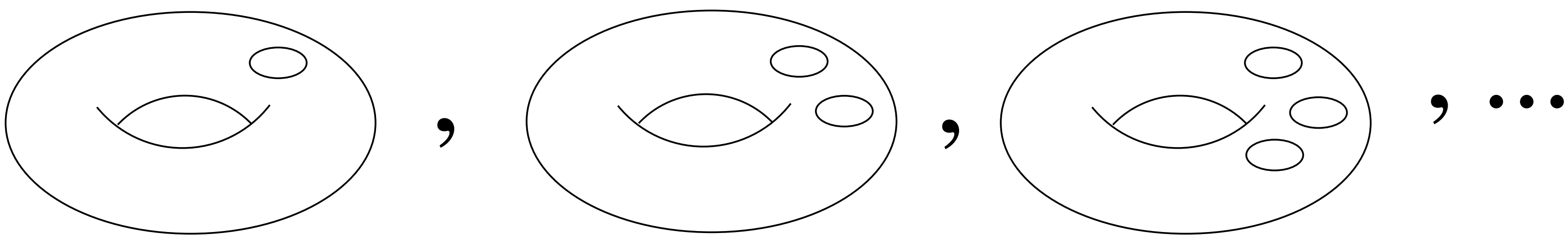
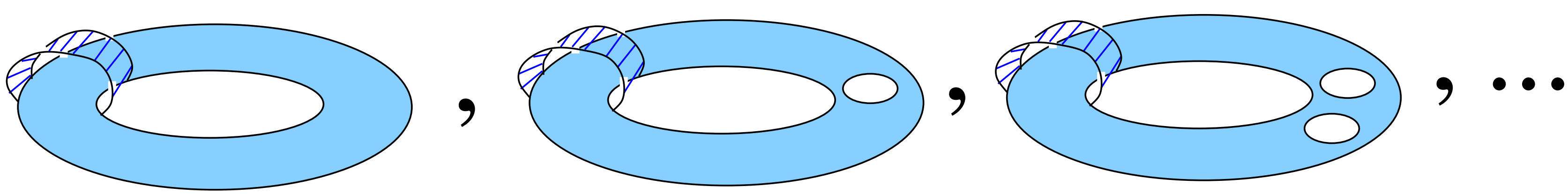


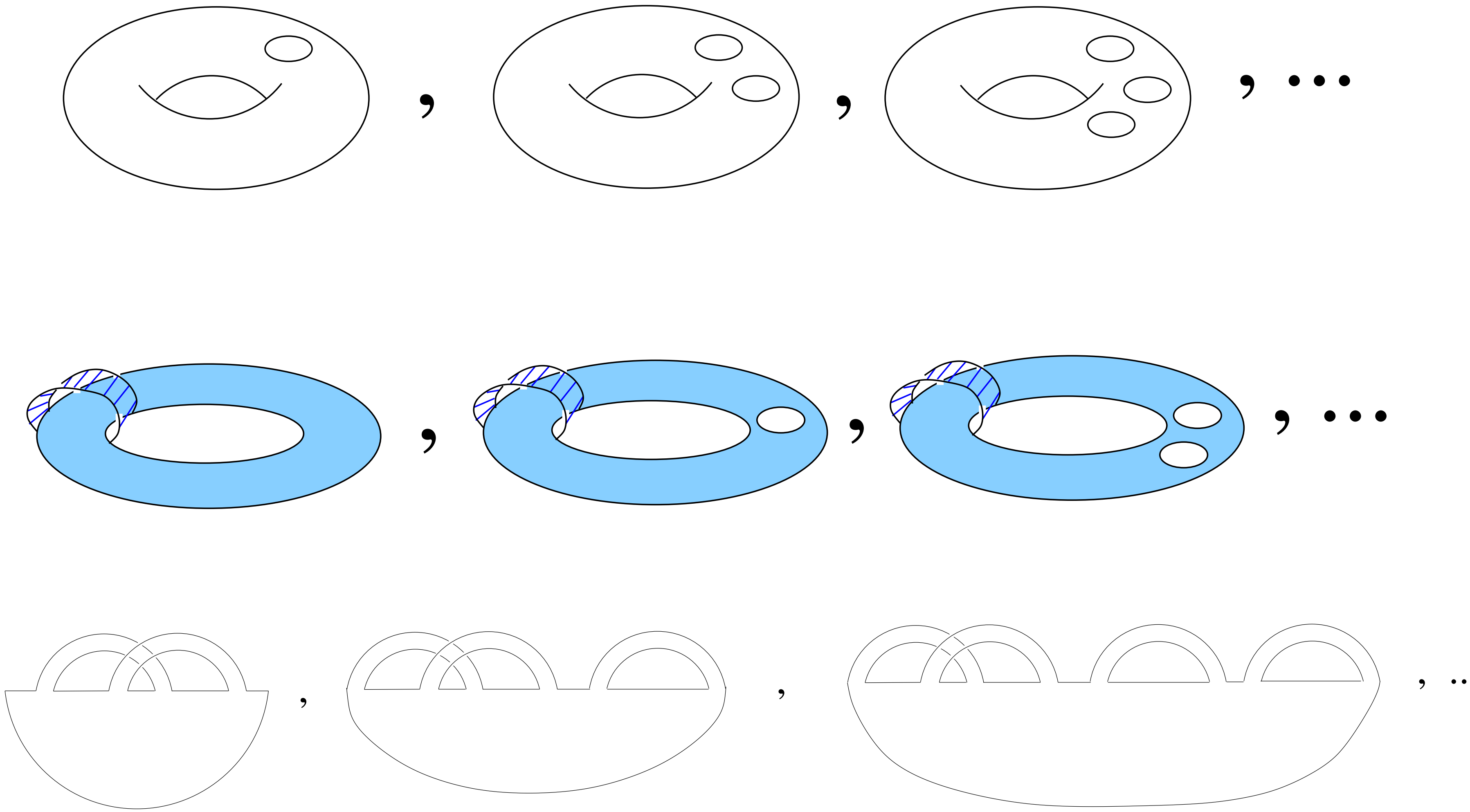
Las Superficies compactas con frontera:

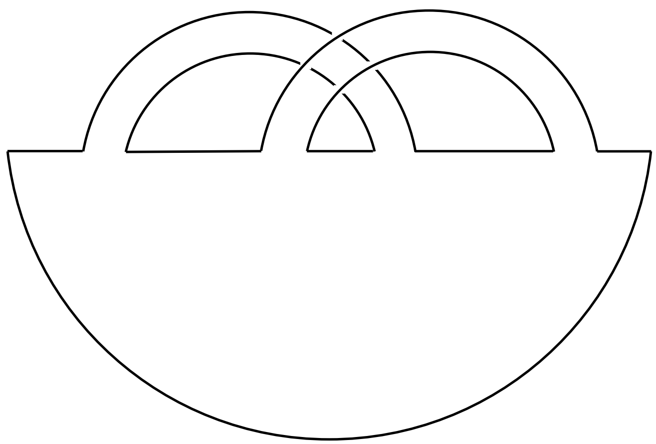


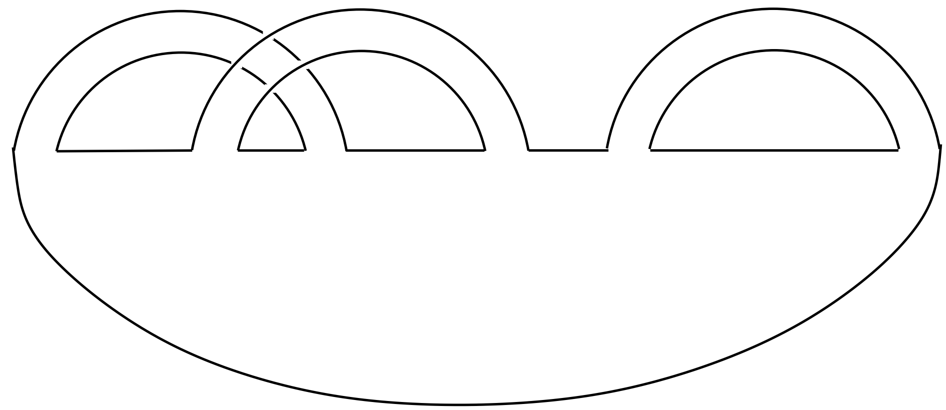
Las Superficies compactas con frontera:

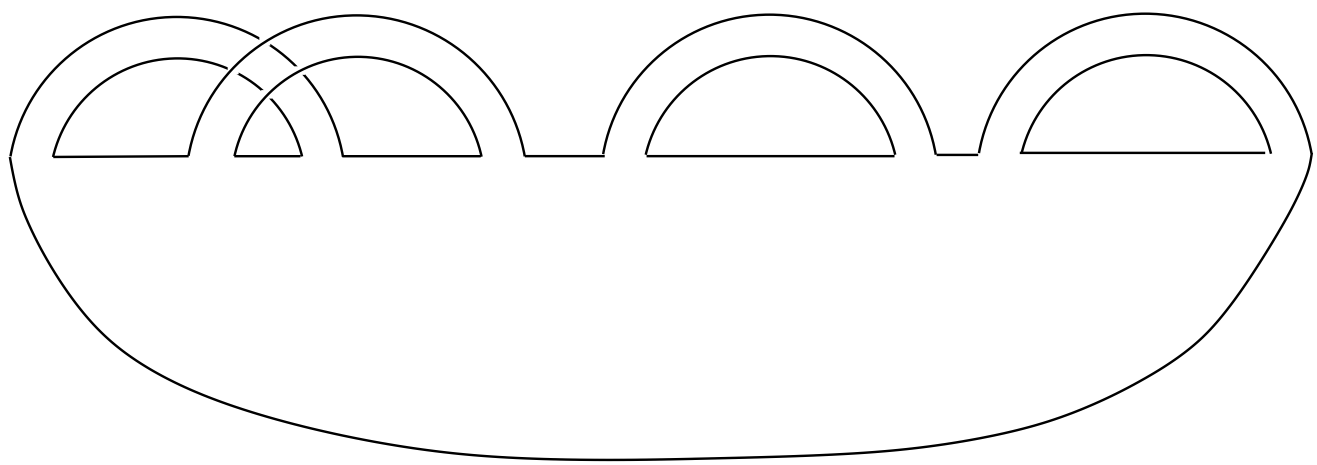


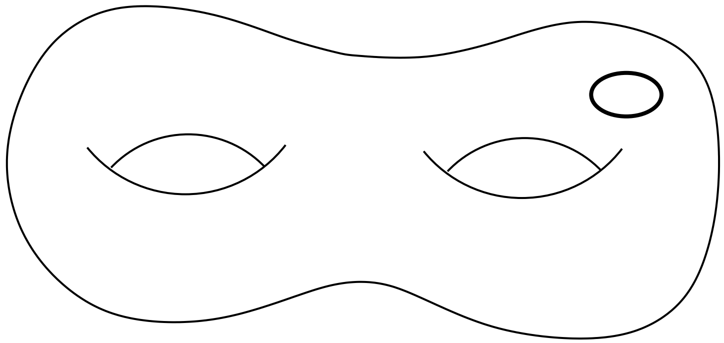


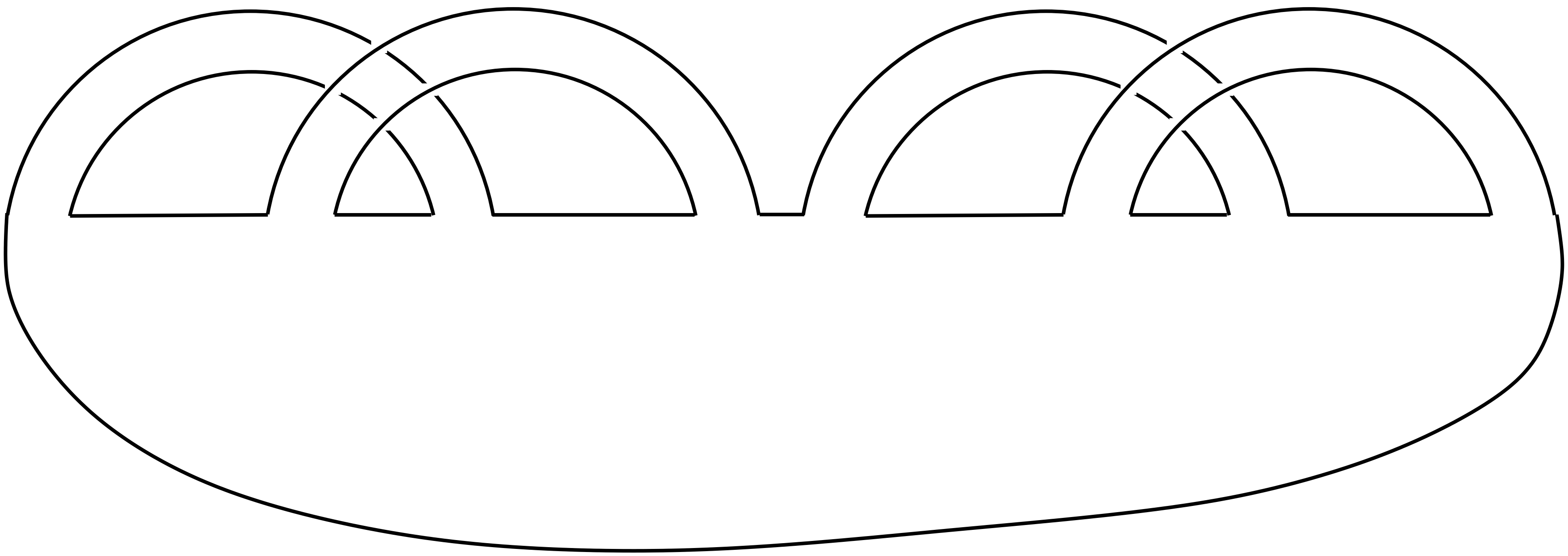


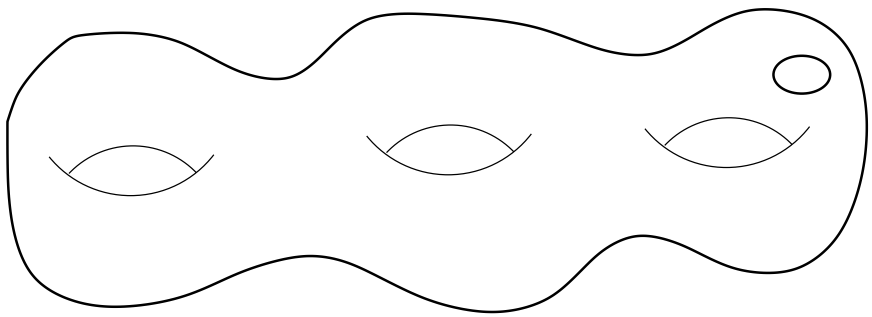


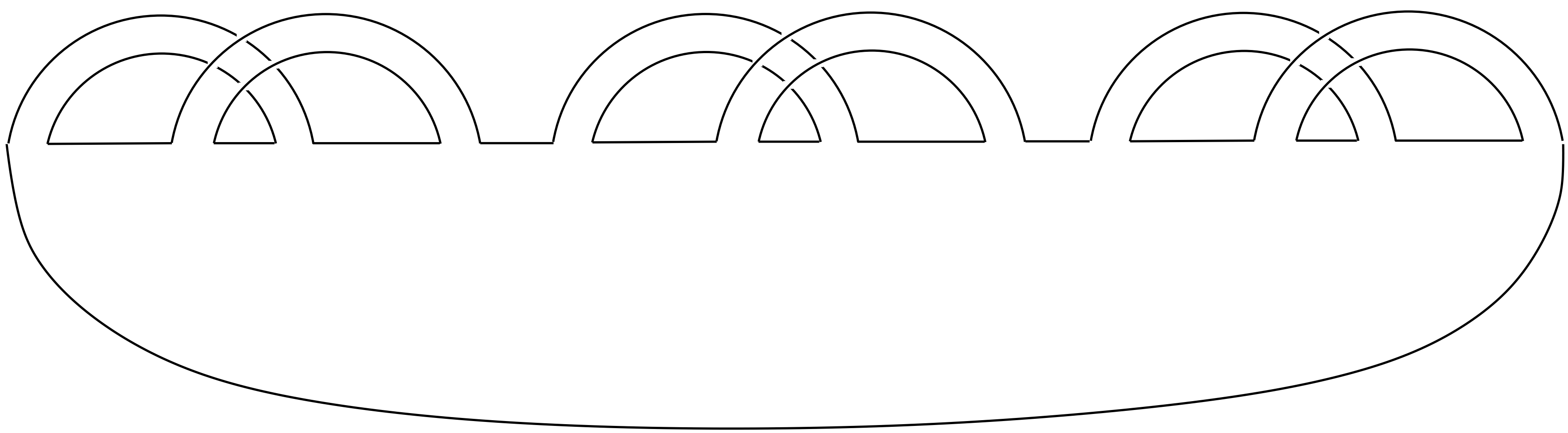


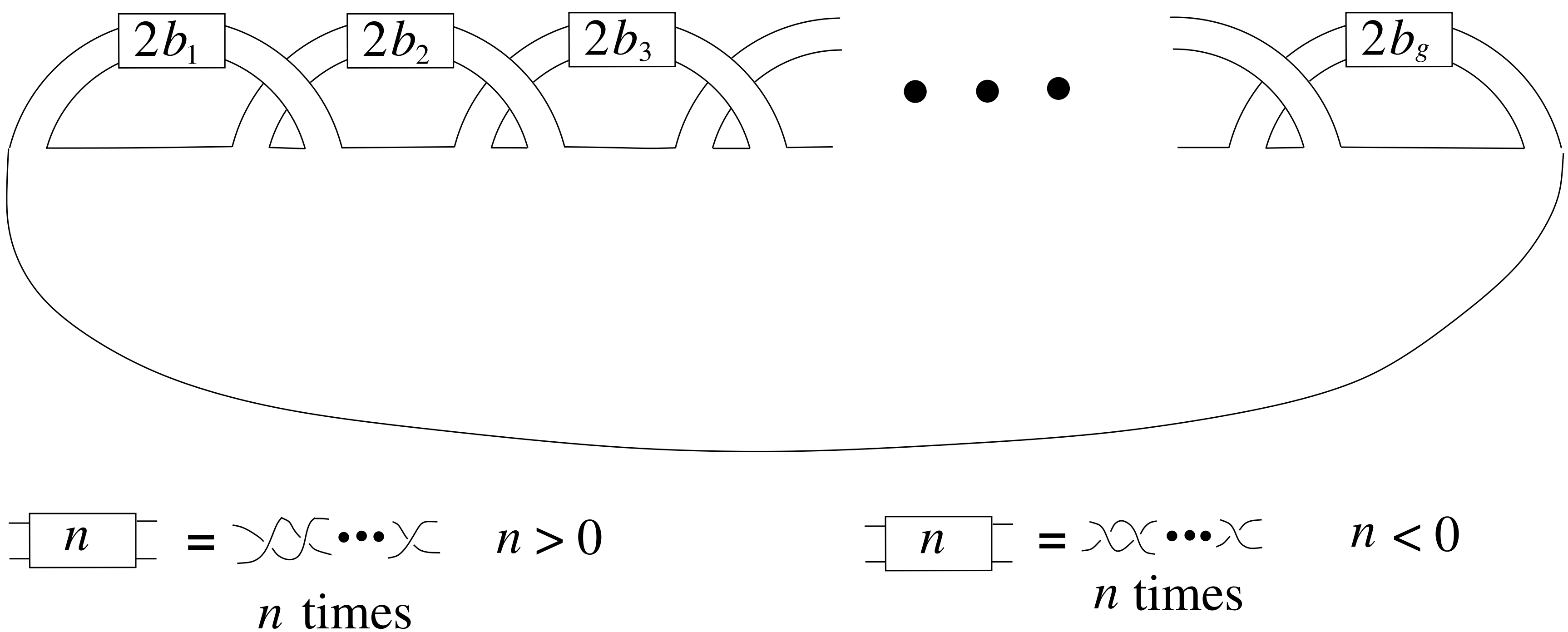






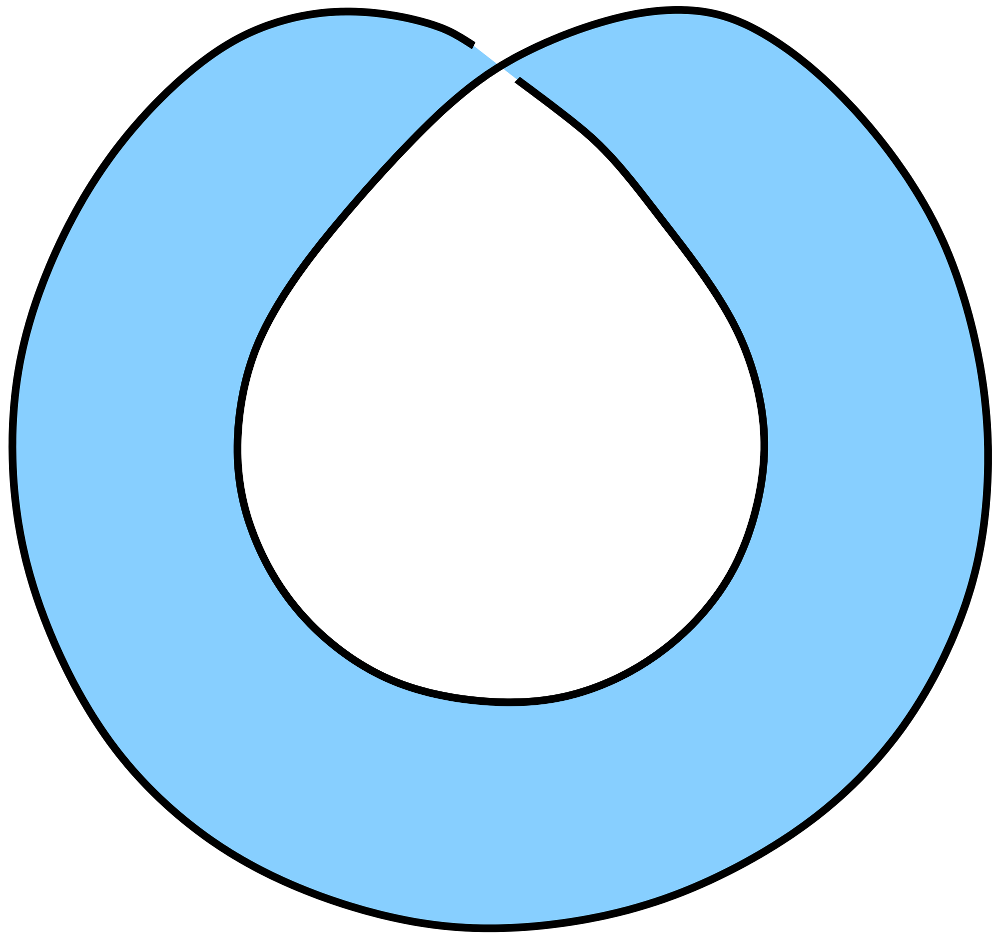


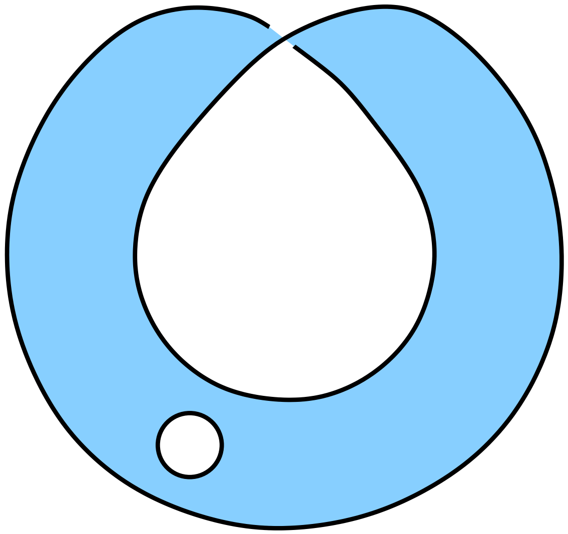
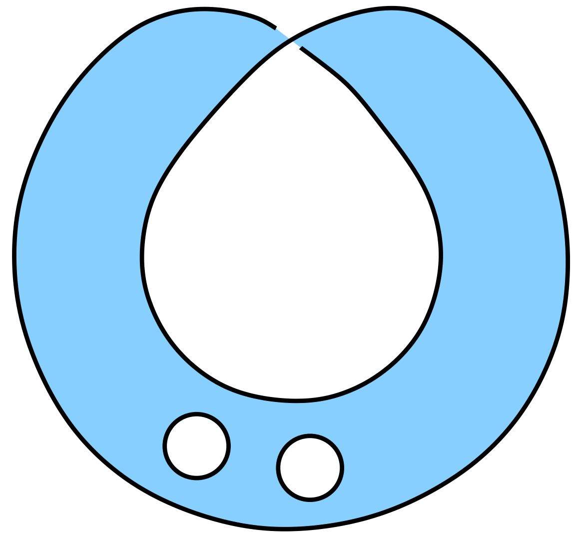
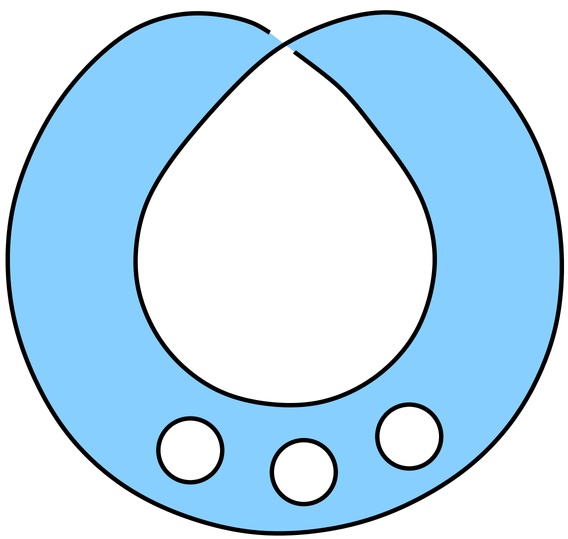




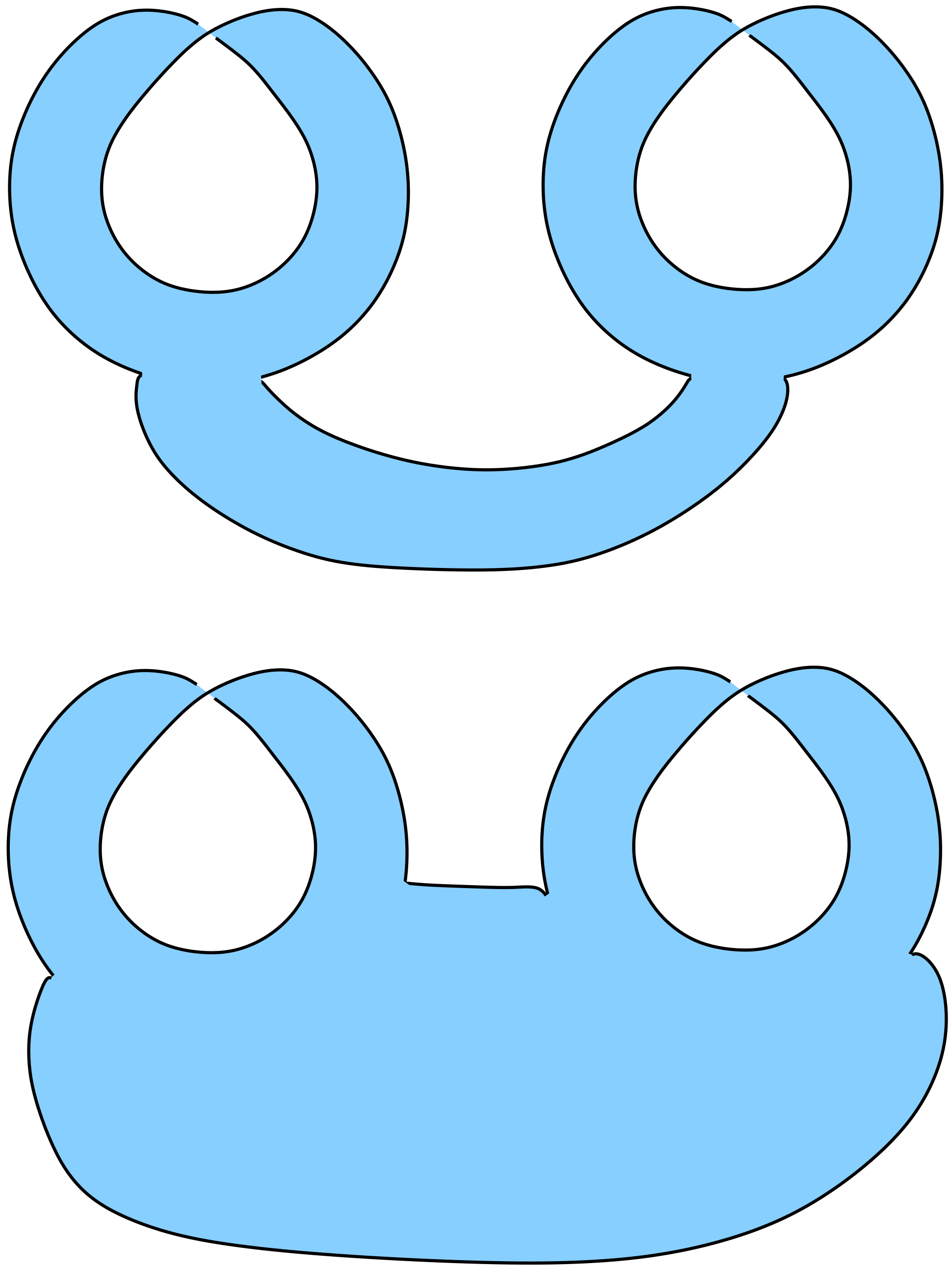
$2b_1$
$2b_2$
$2b_3$
$2b_g$
n
=
$n > 0$
n times
n
=
$n < 0$
n times

$P^2_\circ$

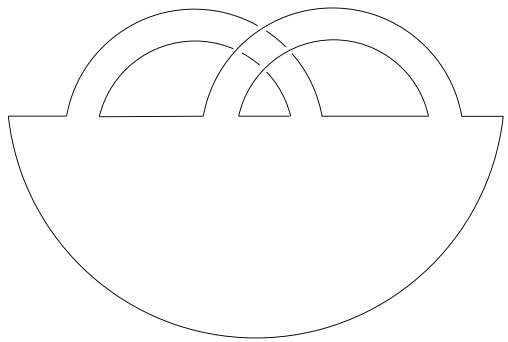


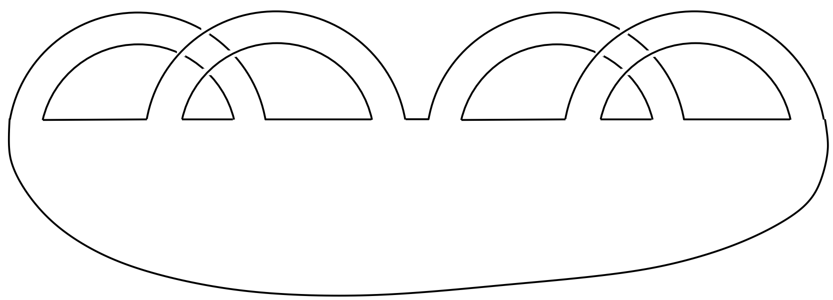




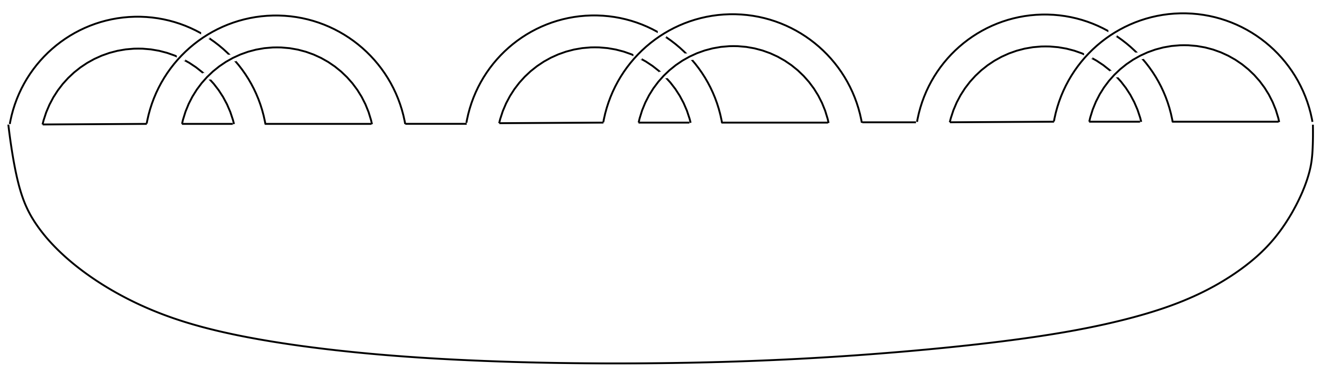
$(P^2 \# P^2)_\circ$



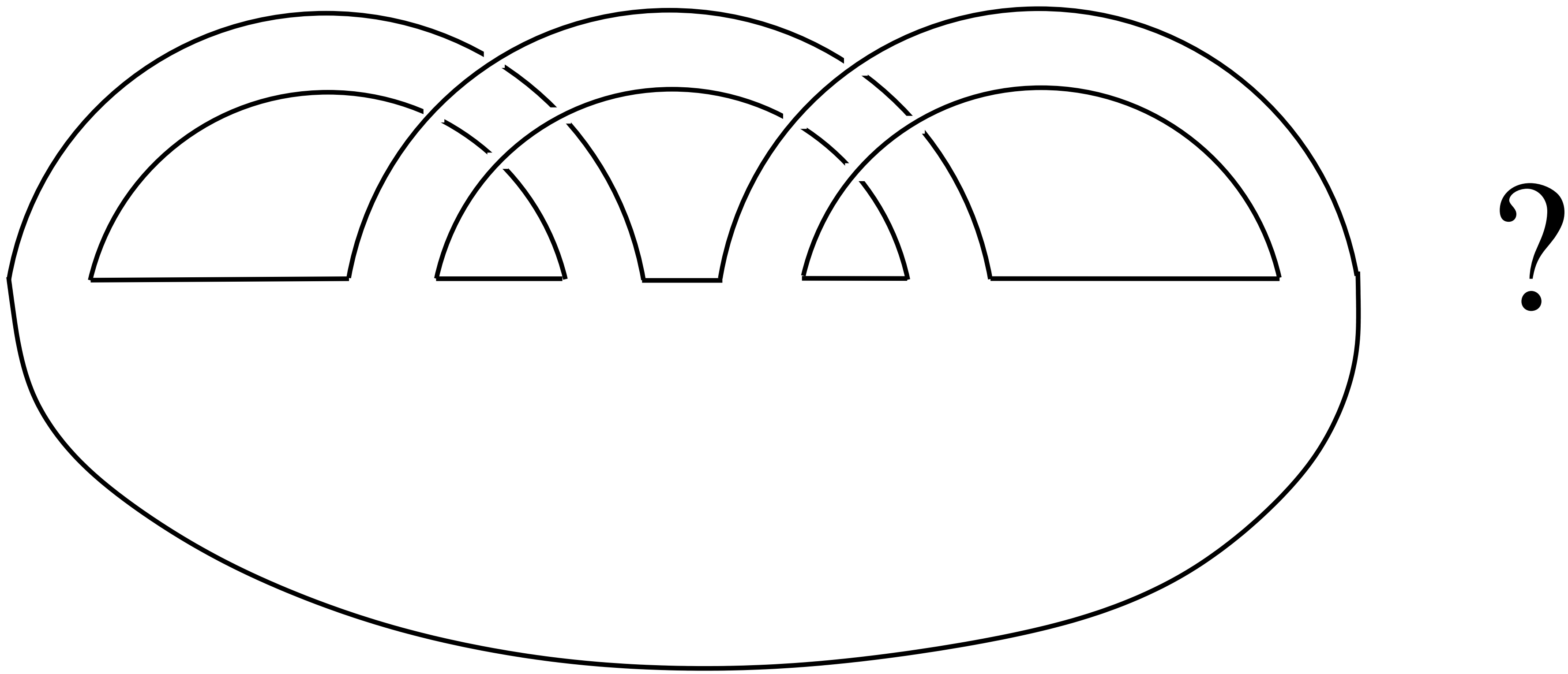
Detengámonos un momento.



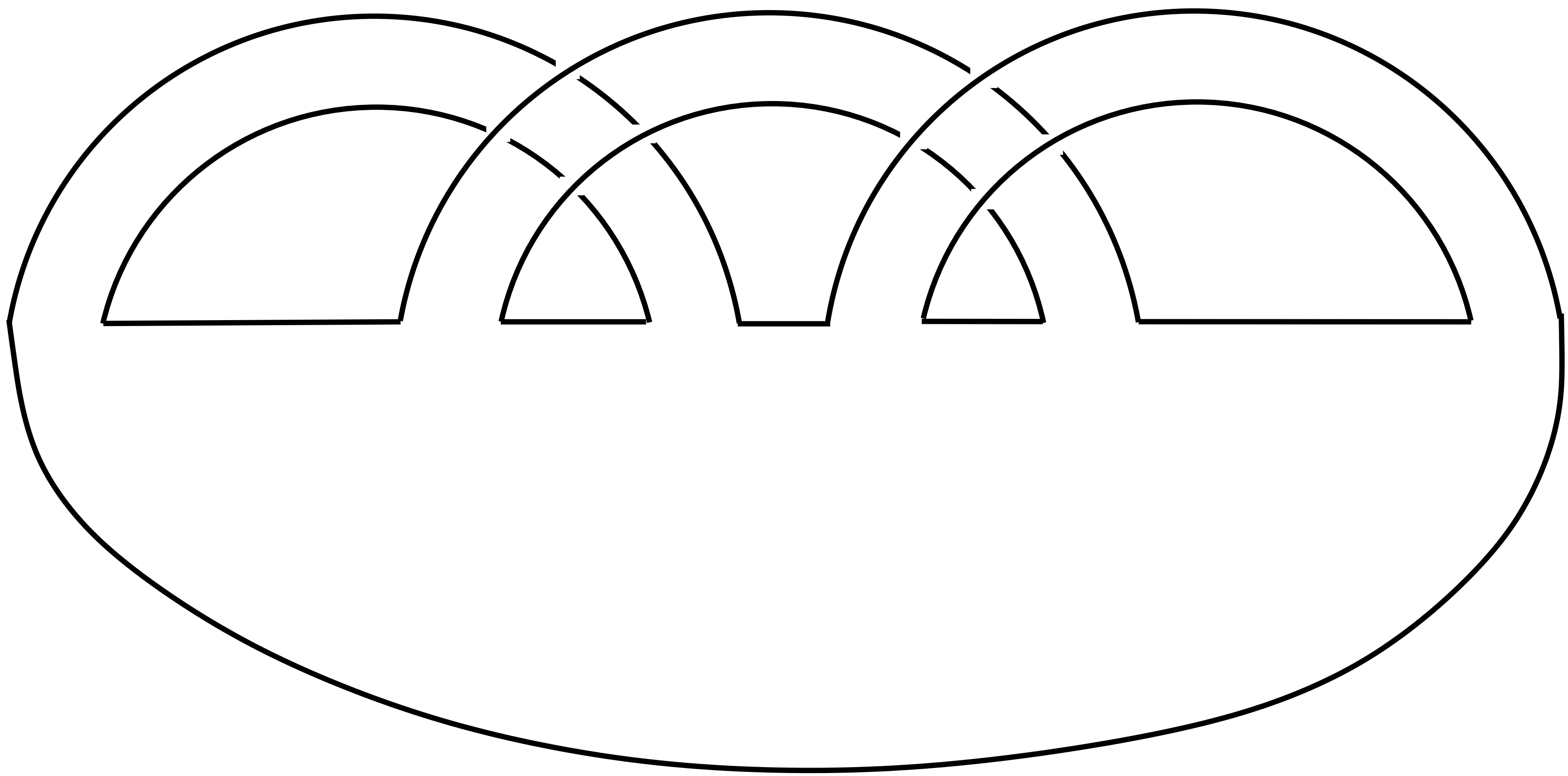


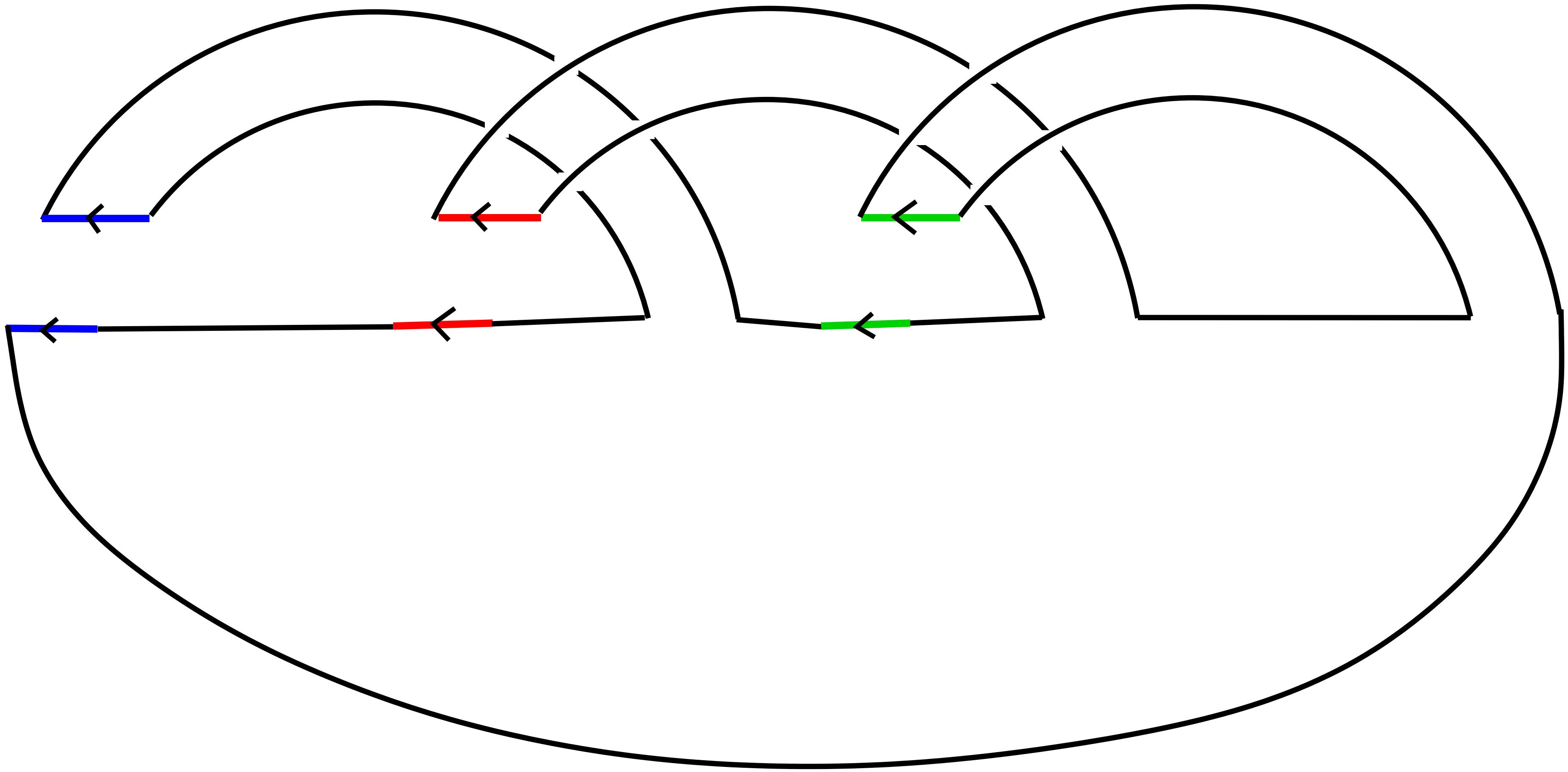


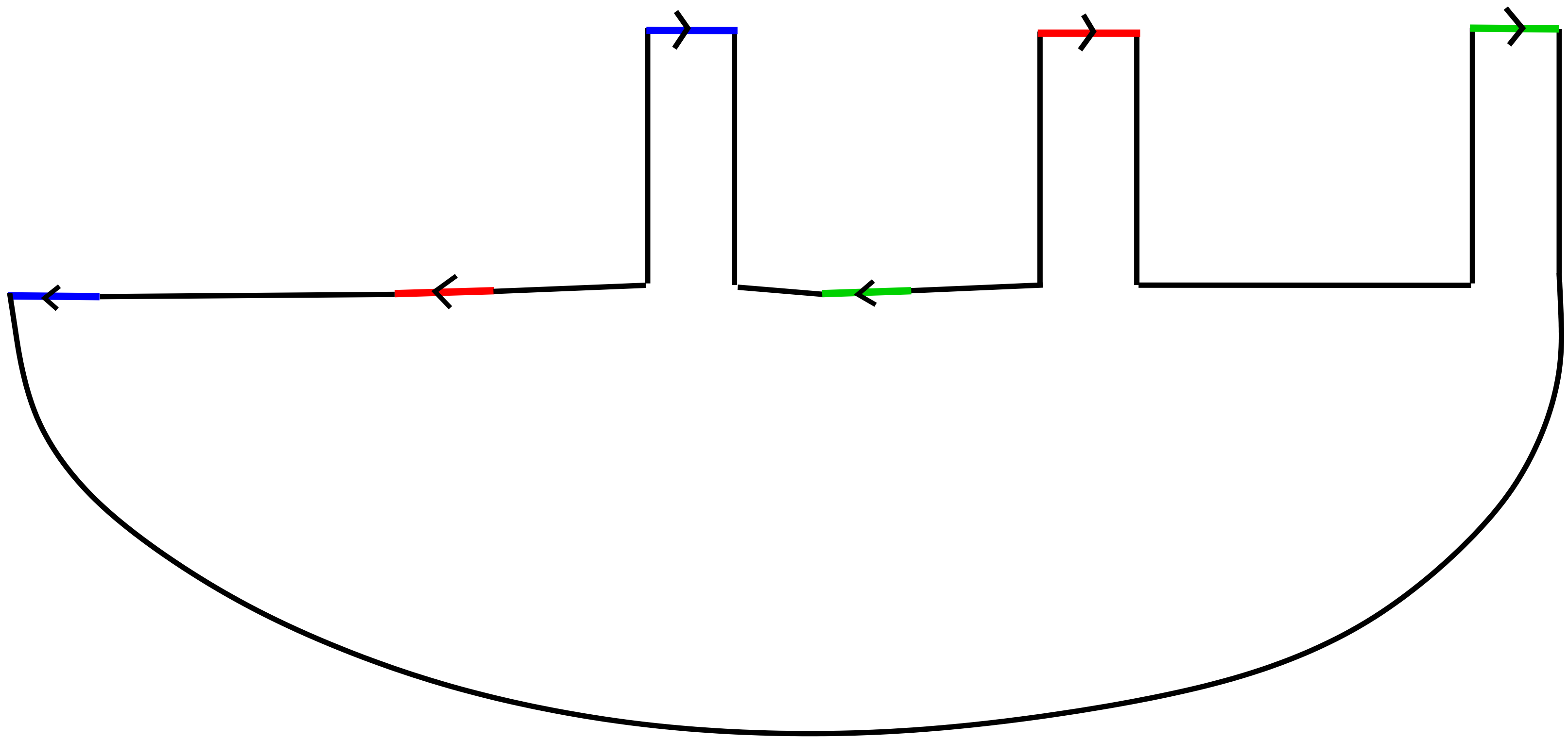
¿Dónde quedó

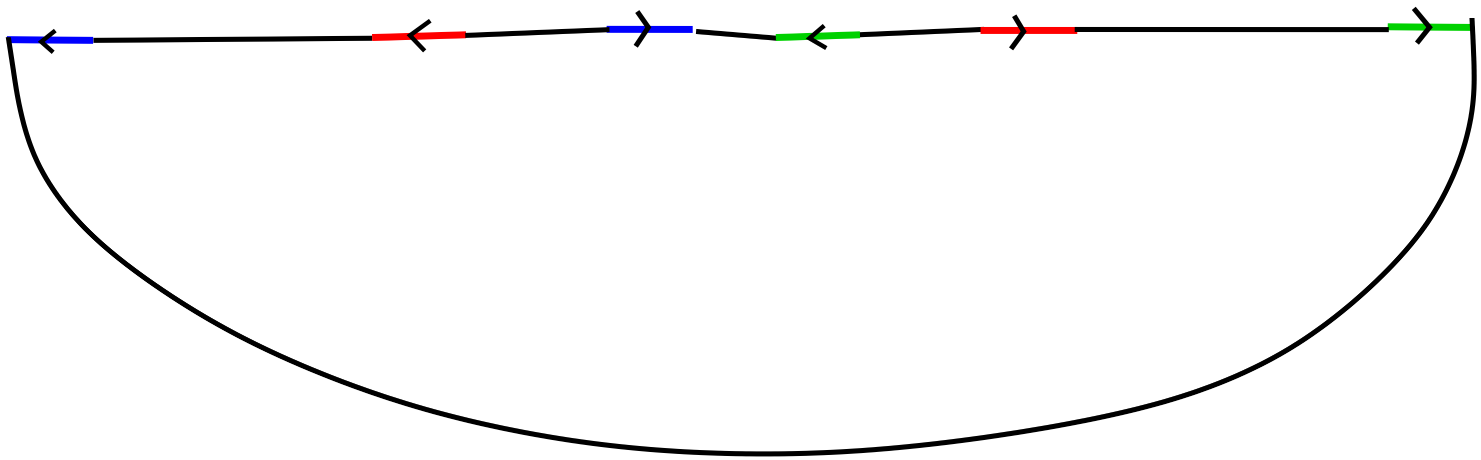


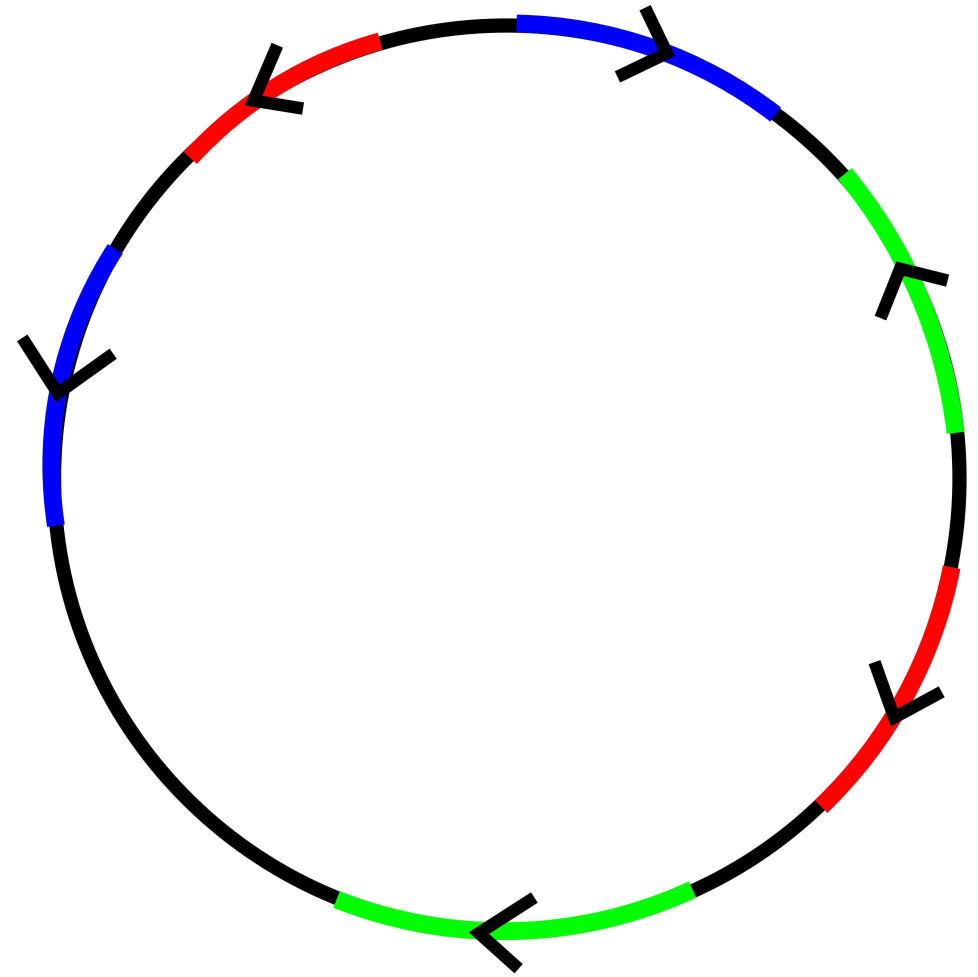
¡Aaah!

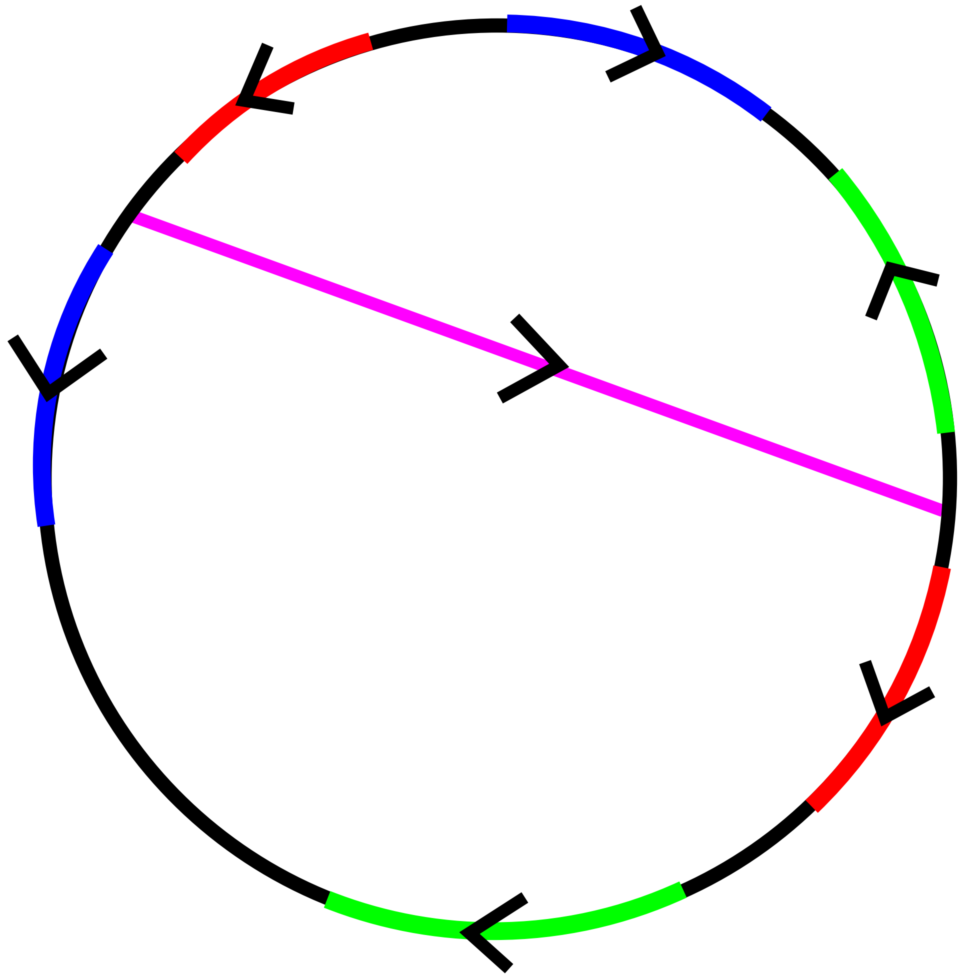


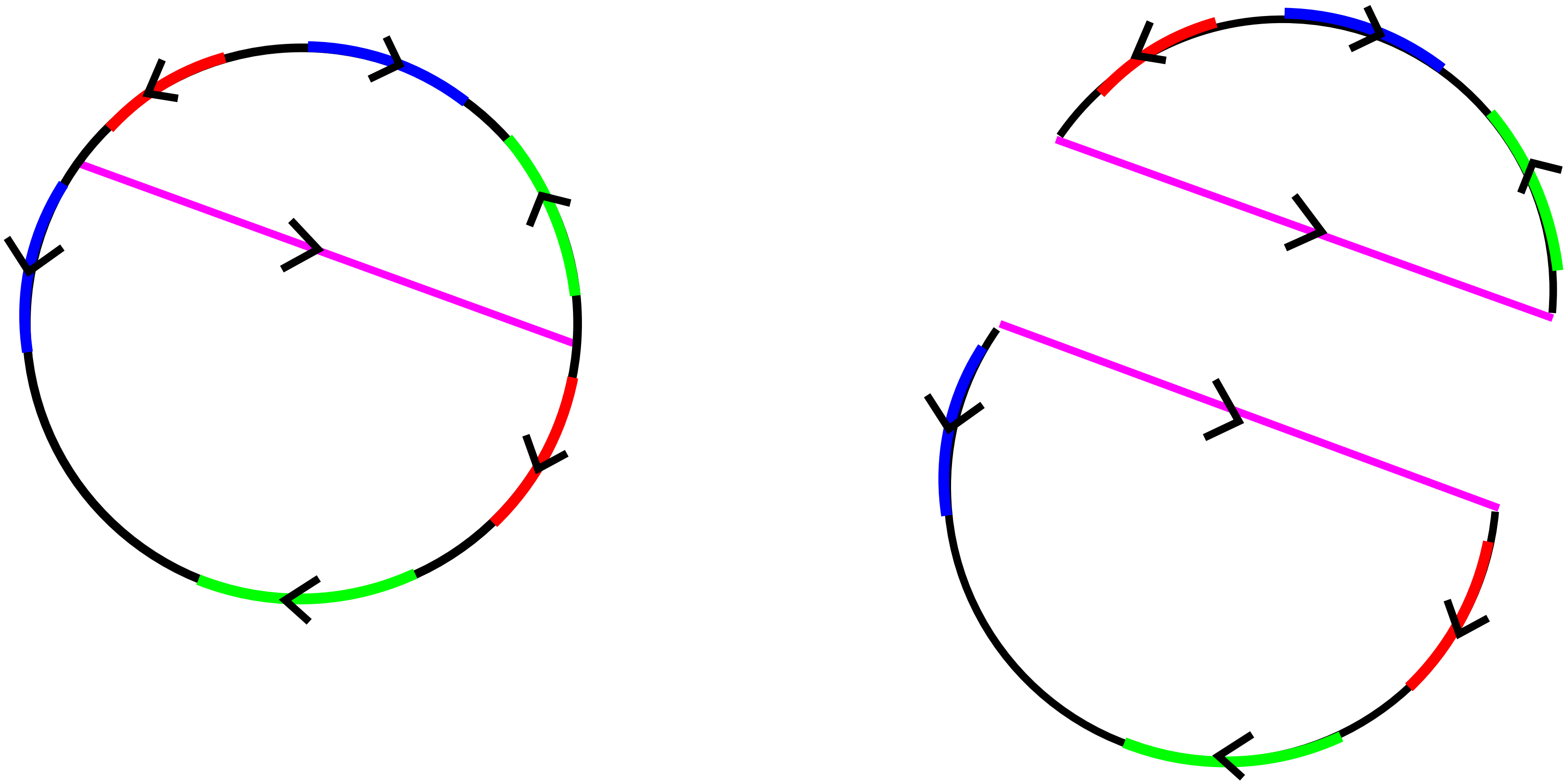


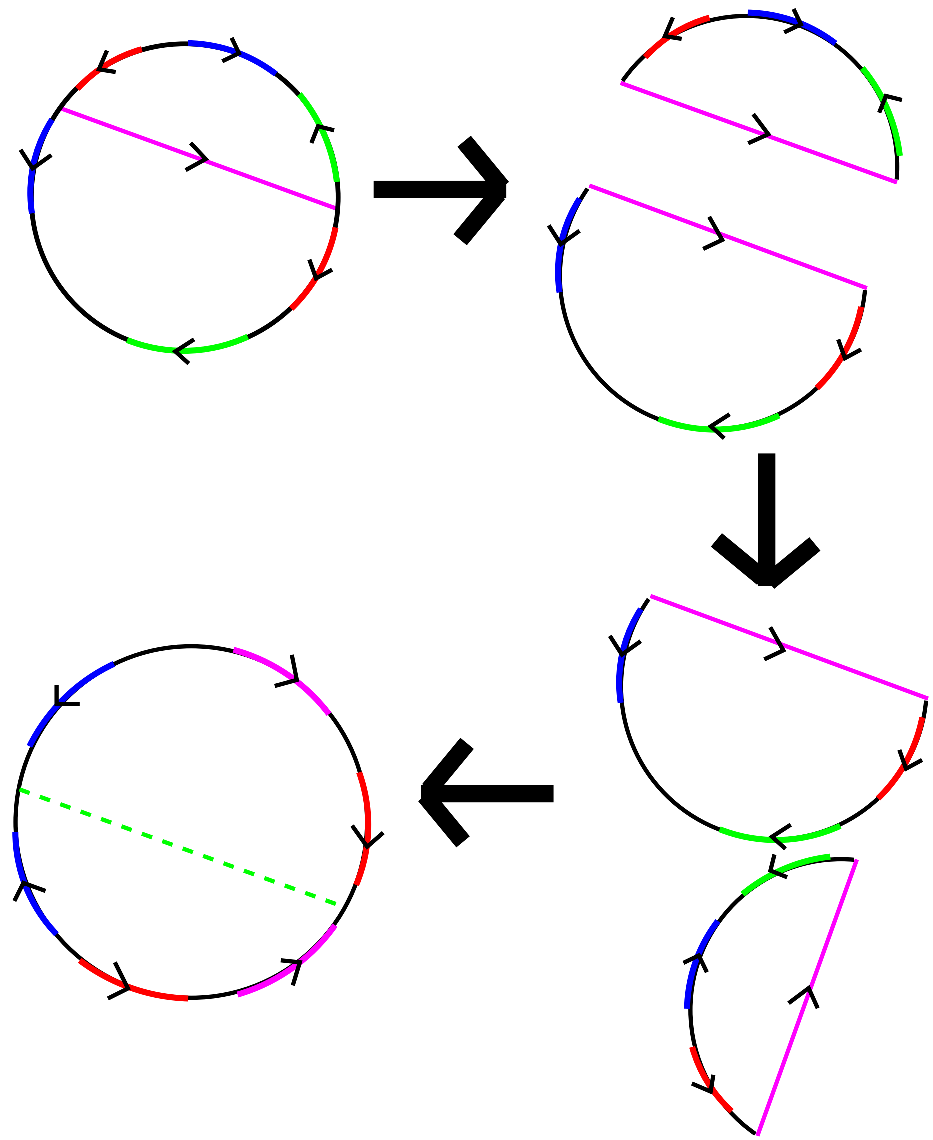


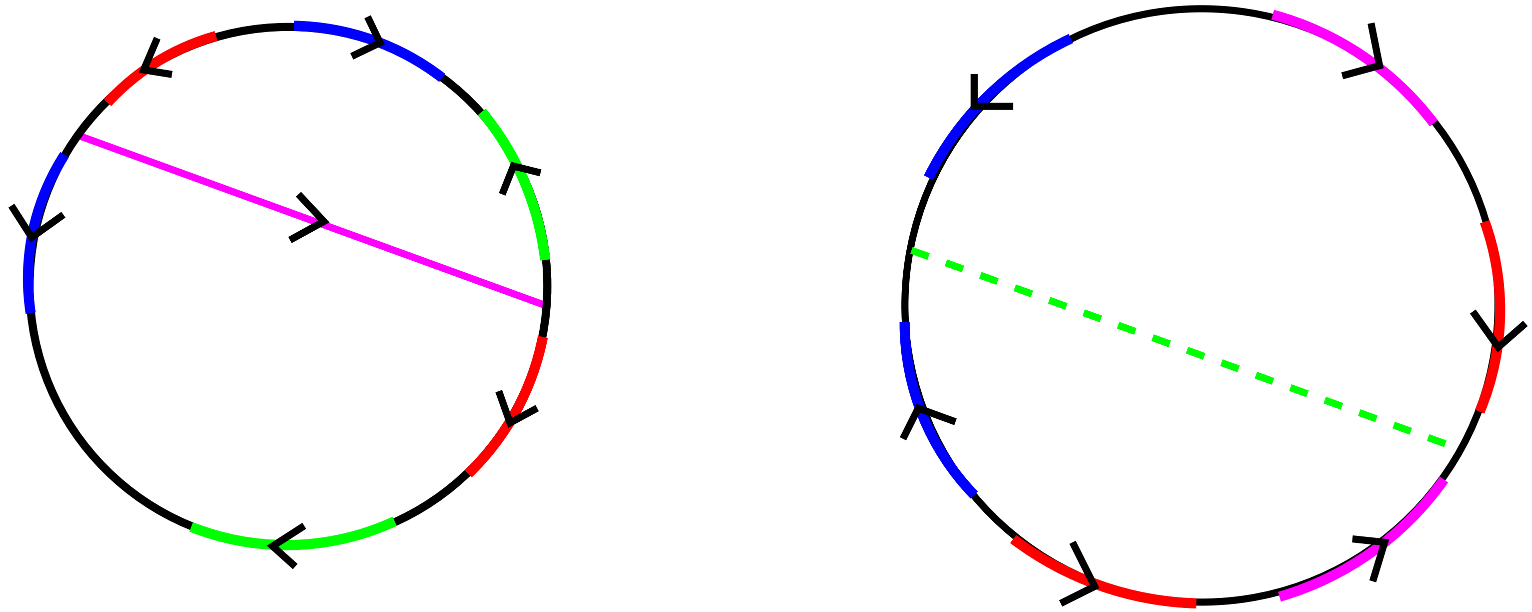


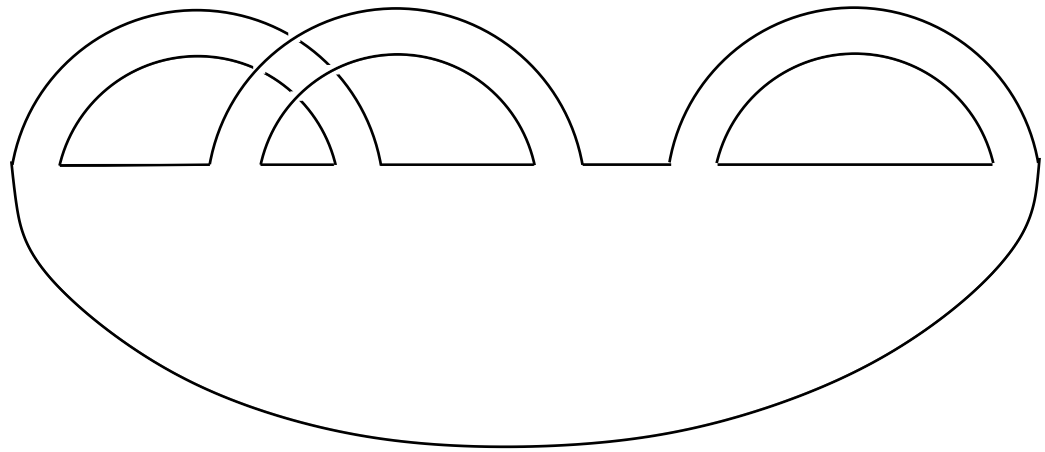


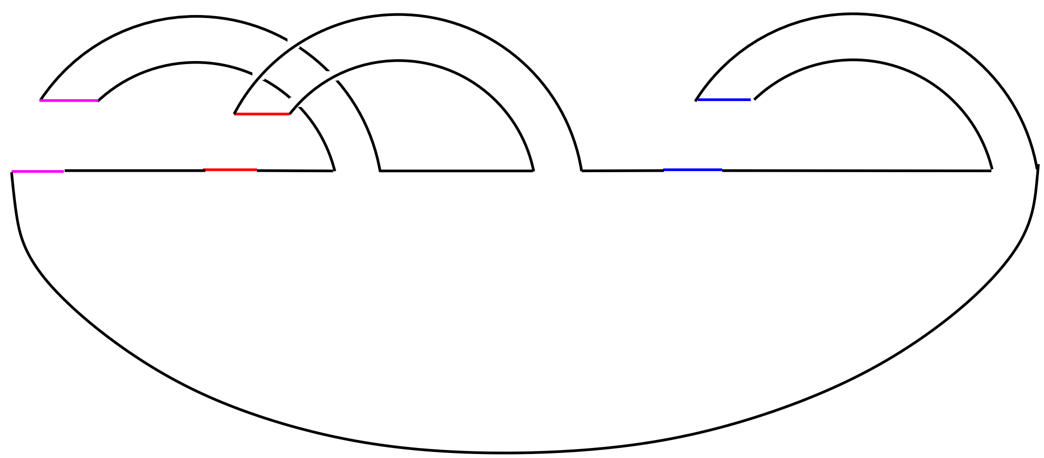


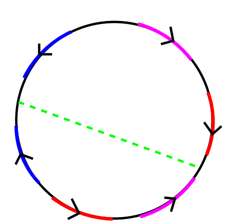


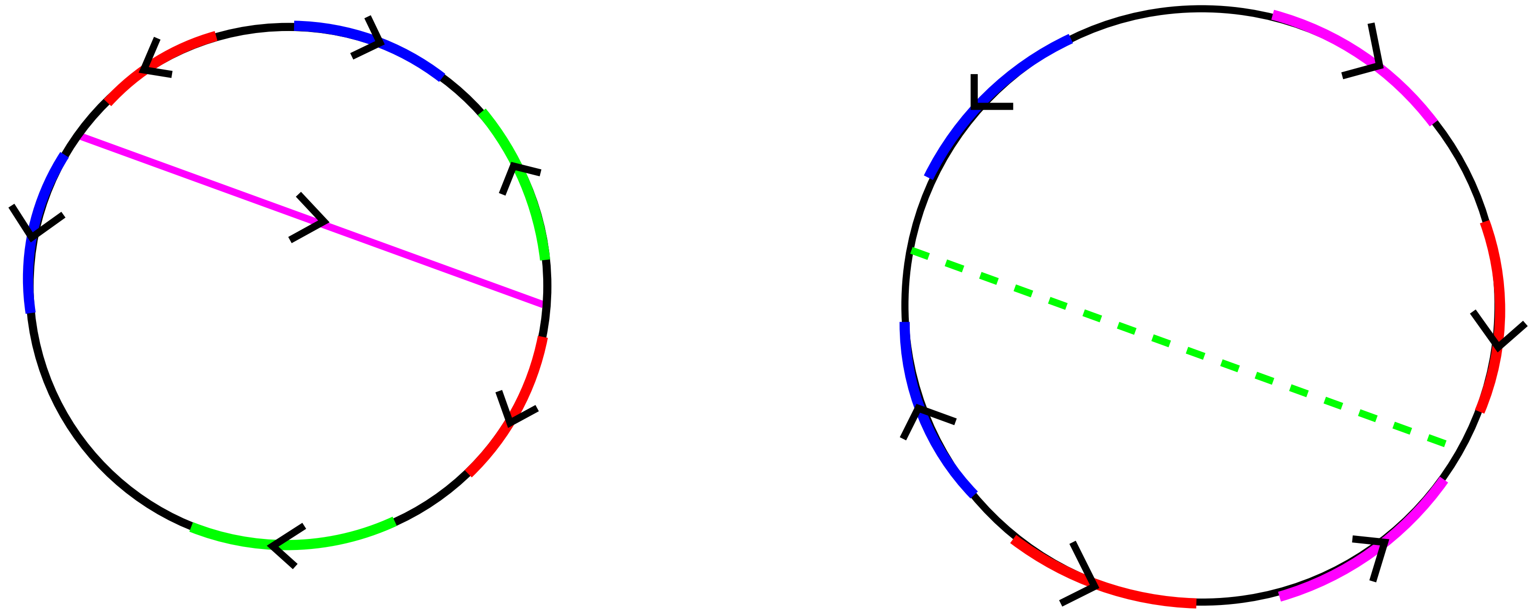


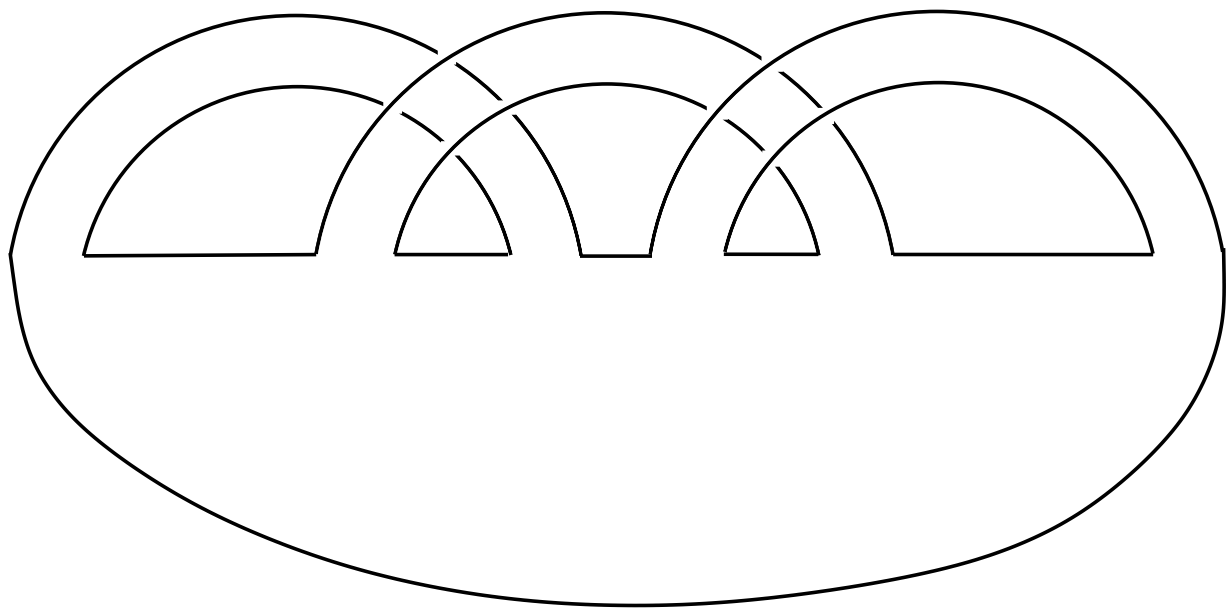


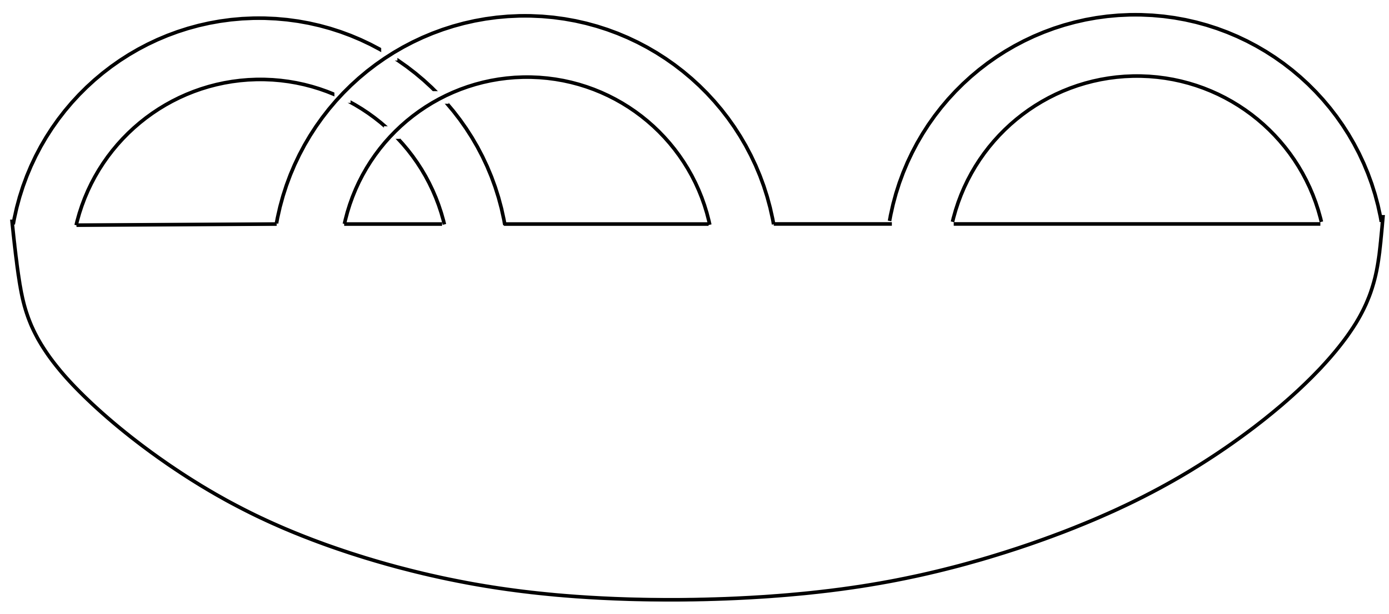




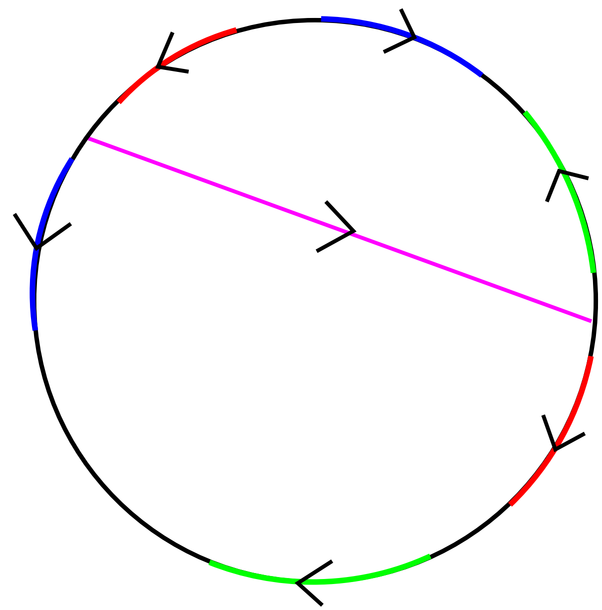


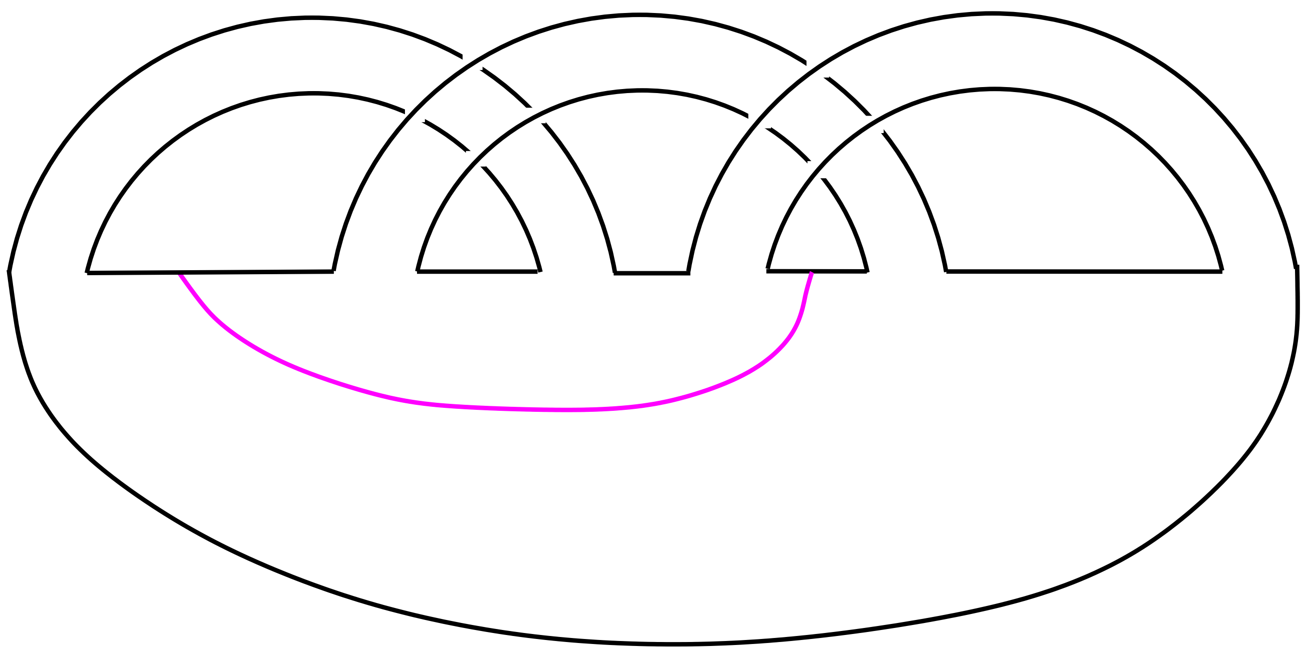


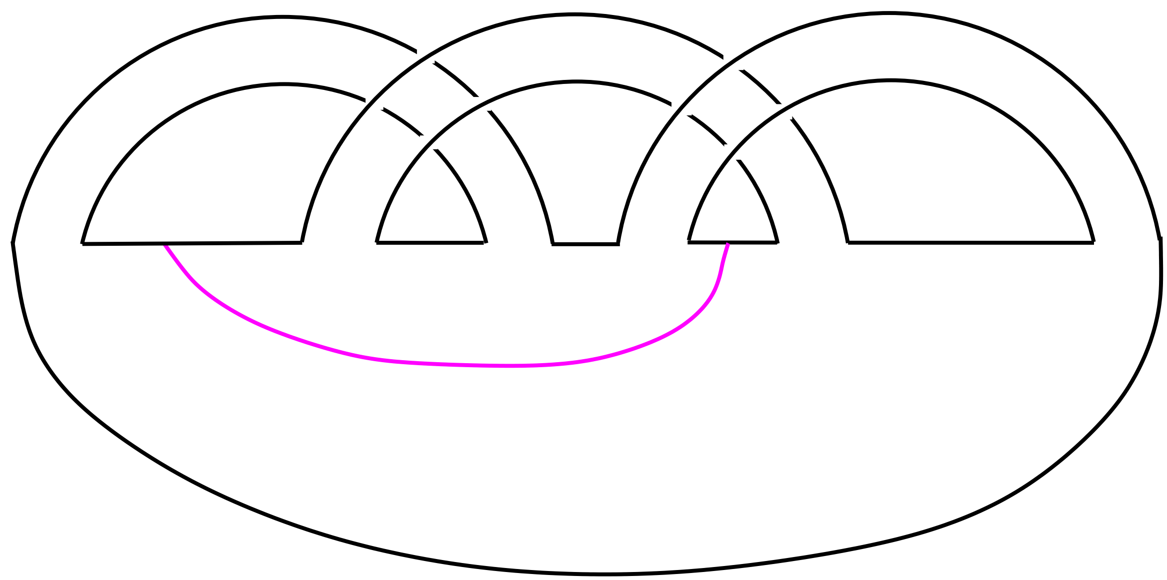


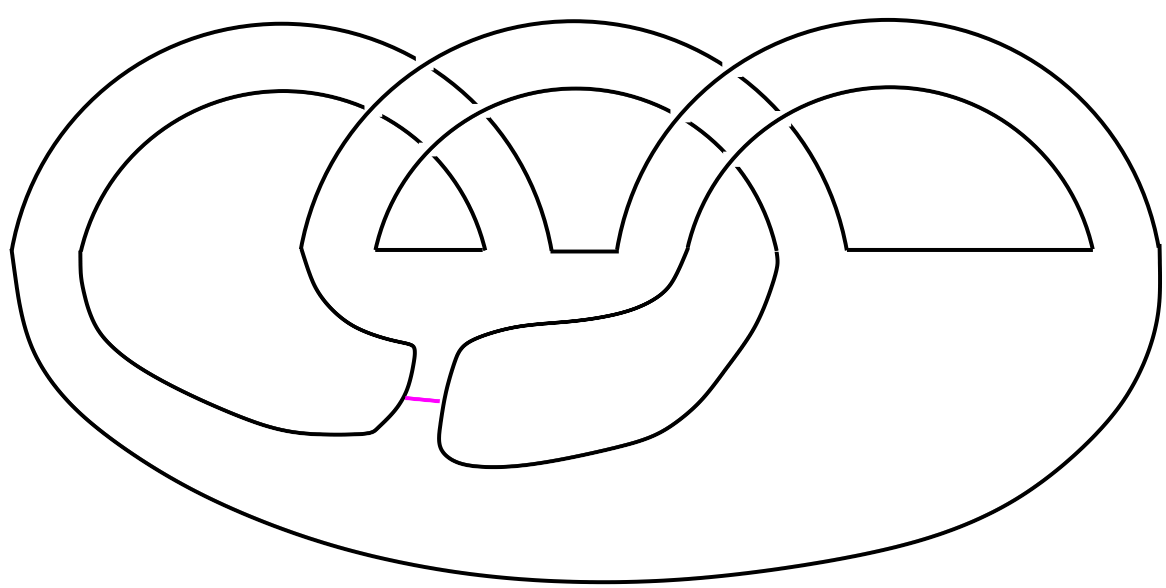


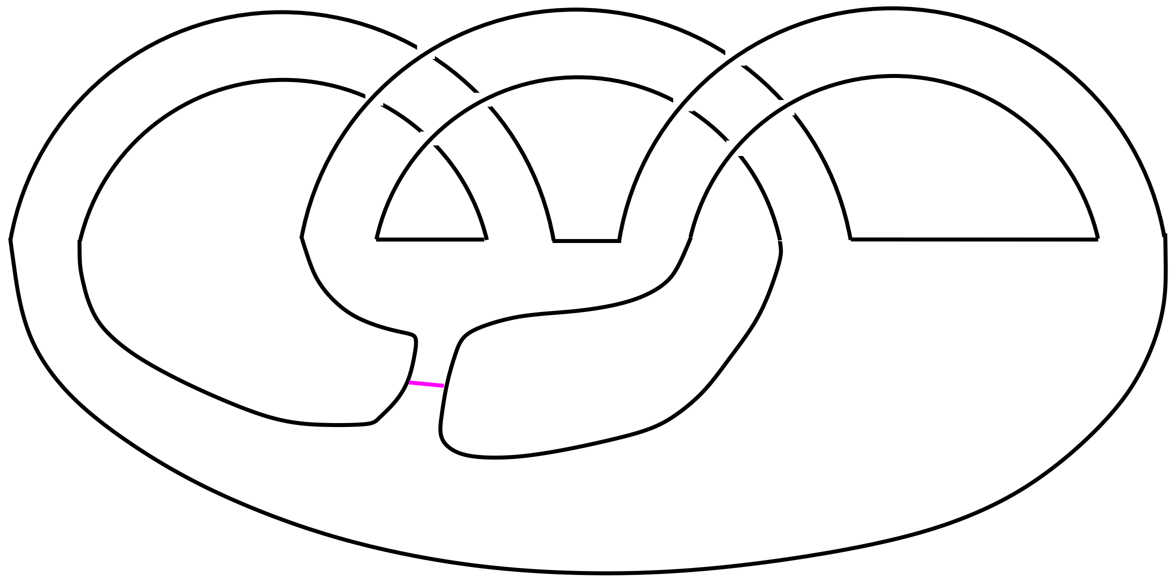
“Explicación”

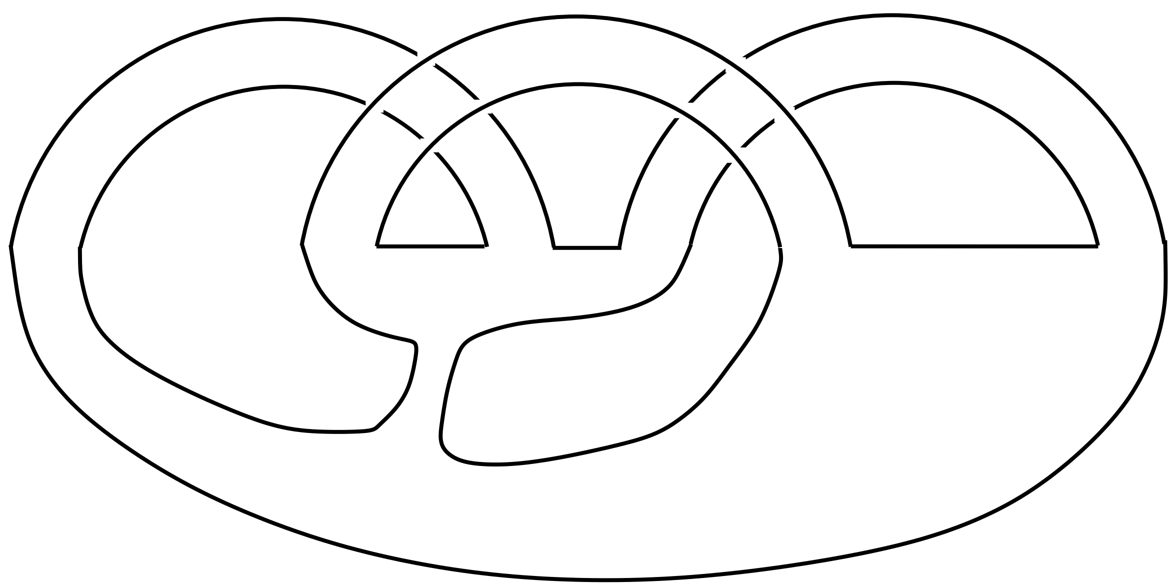


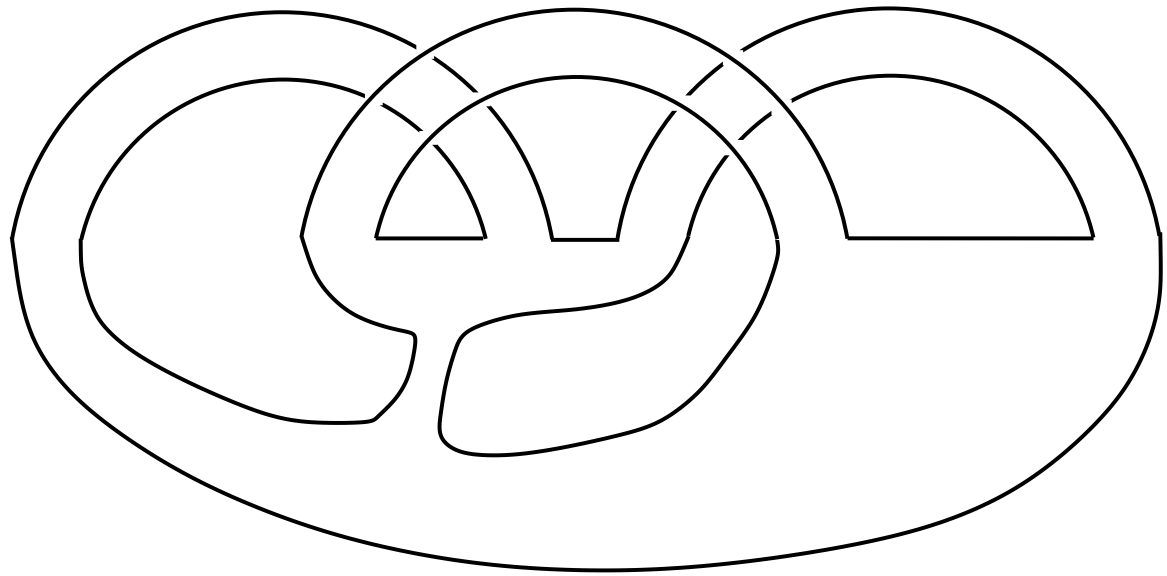


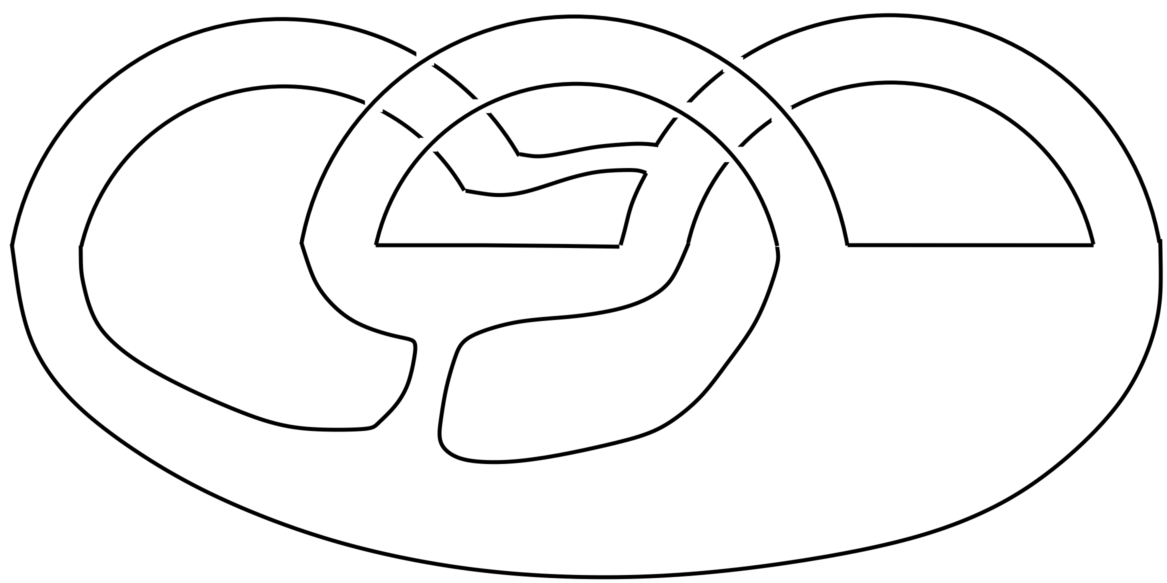


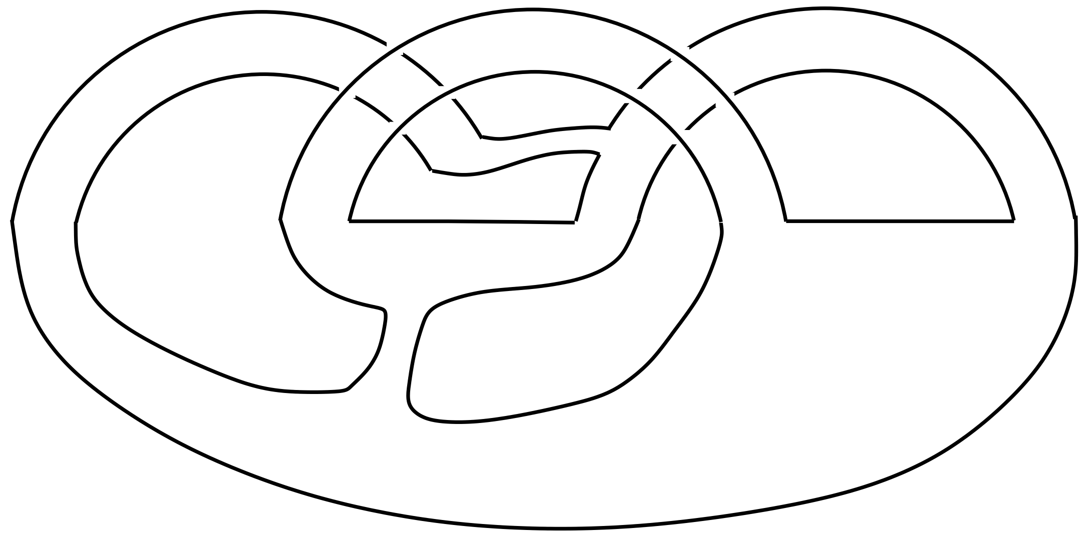


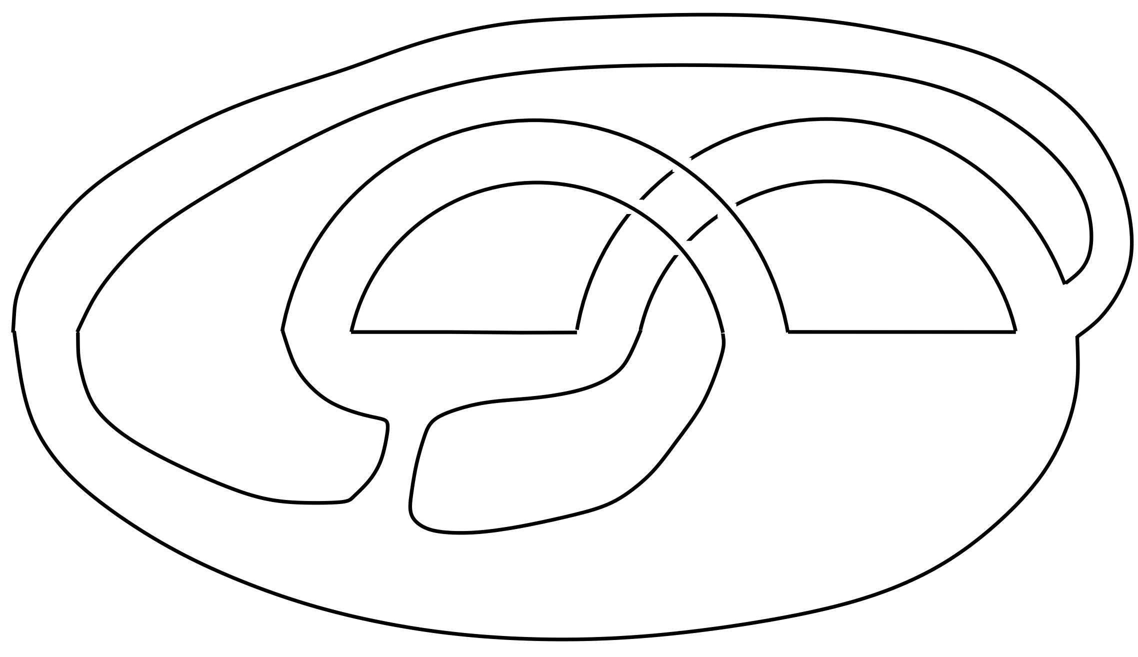


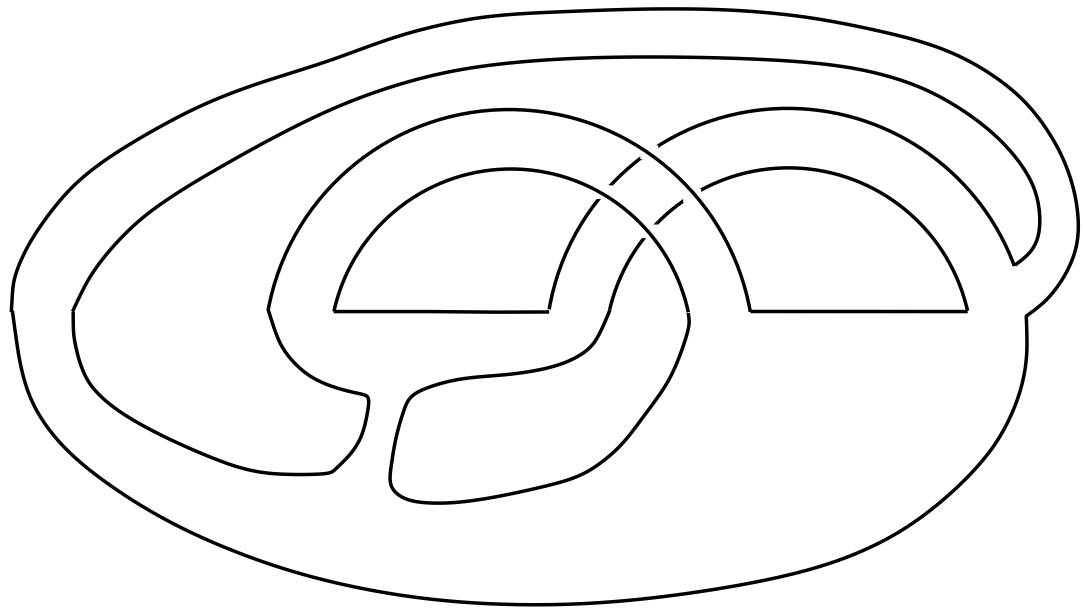


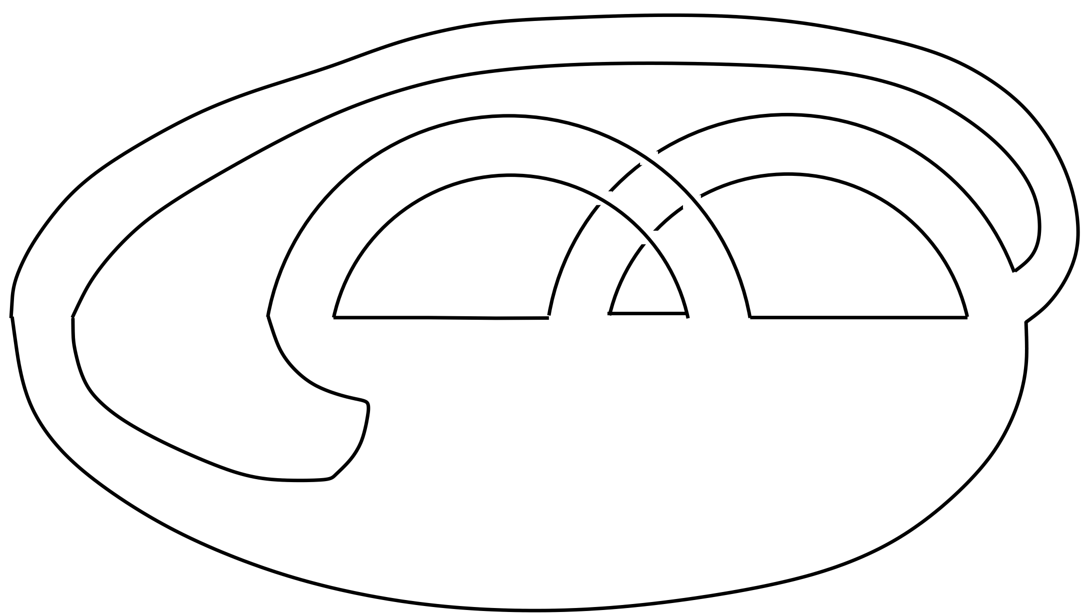


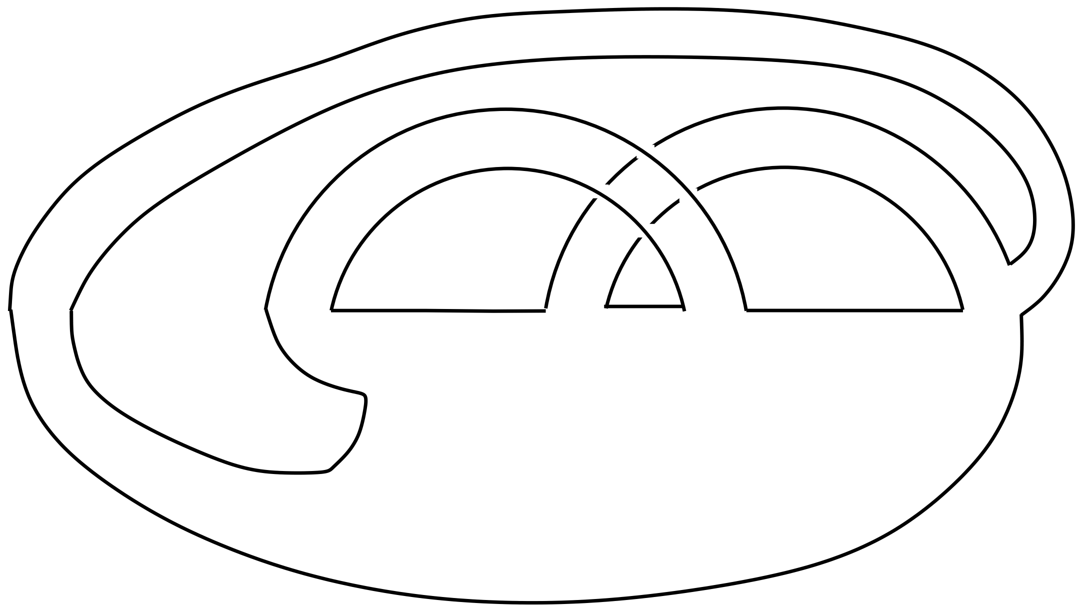


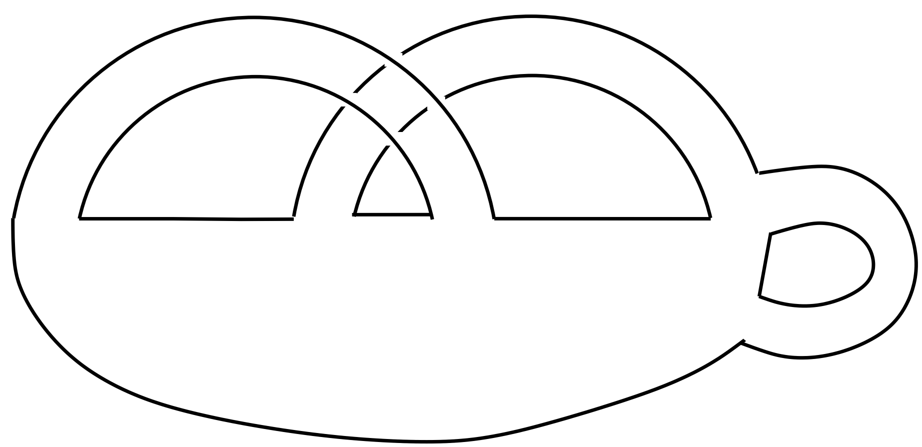












Algo está bien

Algo está bien

¿Por qué?

Conjuntos

Sea $\sim$ una relación de equivalencia en X.

Escribimos

$$[a] = \{y \in X | a \sim y\}$$

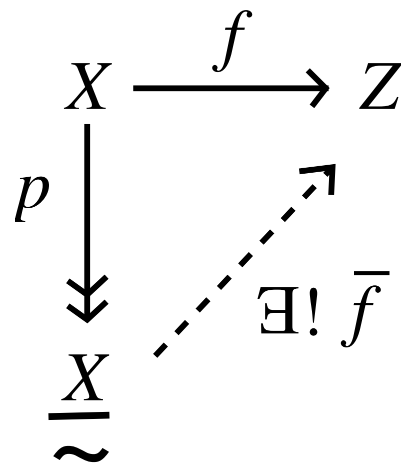
$$\frac{X}{\sim} = \{[a] | a \in X\}$$

$$p : X \to \frac{X}{\sim}$$
$$a \mapsto [a]$$

Si X es un espacio, le damos al cociente $X/\sim$ la topología más grande que hace continua a la proyección canónica $p : X \to X/\sim$.

(o sea, $U \subset \dfrac{X}{\sim}$ es abierto $\Leftrightarrow p^{-1}(U) \subset X$ es abierto)

Propiedad Universal de los Cocientes.



Para todo espacio Z y toda función continua $f : X \to Z$, si

$$(\forall a, b \in X, p(a) = p(b) \Rightarrow f(a) = f(b)),$$

entonces existe una única función continua $\bar{f} : \dfrac{X}{\sim} \to Z$ tal que $\bar{f} \circ p = f$.

En particular,

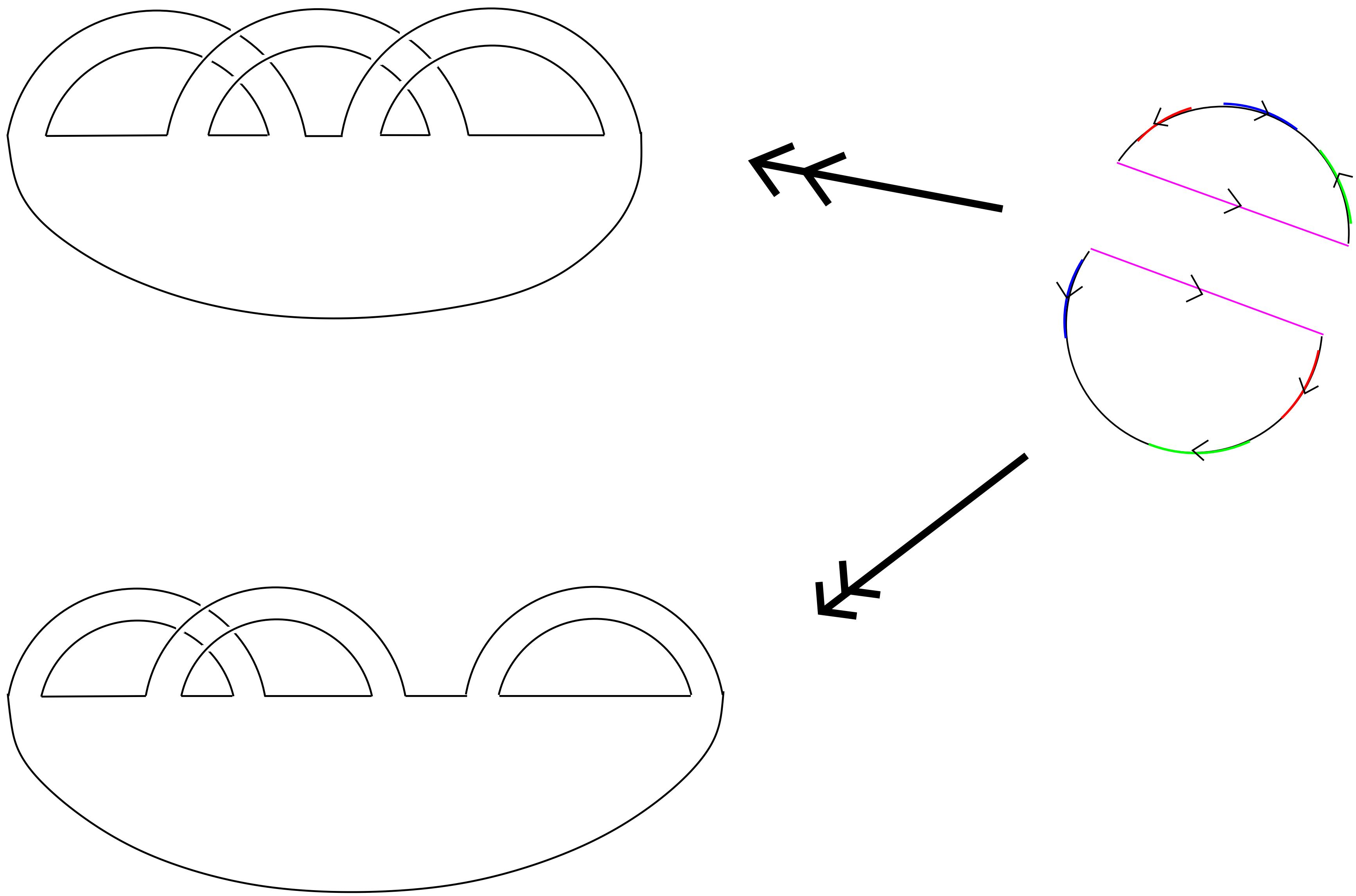
si R_1 y R_2 son relaciones de equivalencia en X con proyecciones canónicas $p : X \to X/R_1$ y $q : X \to X/R_2$ y se puede probar que

$$(\forall a, b \in X, p(a) = p(b) \Leftrightarrow q(a) = q(b)),$$

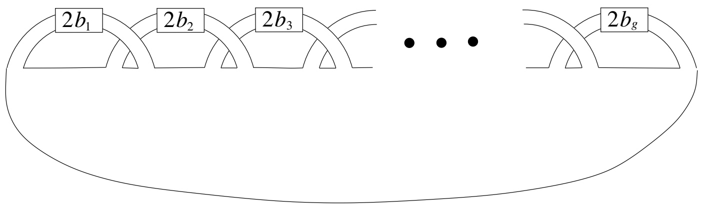
entonces la única función continua que existe

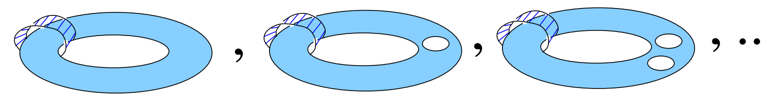
$$\bar{q} : X/R_1 \to X/R_2$$

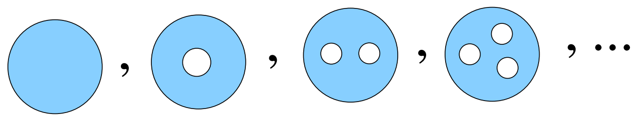
tal que $\bar{q} \circ p = q$, es un <u>homeomorfismo</u>.

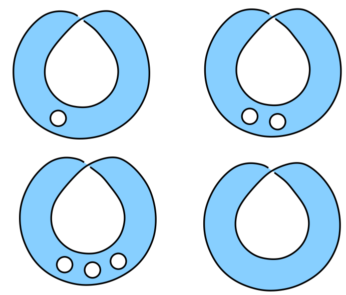


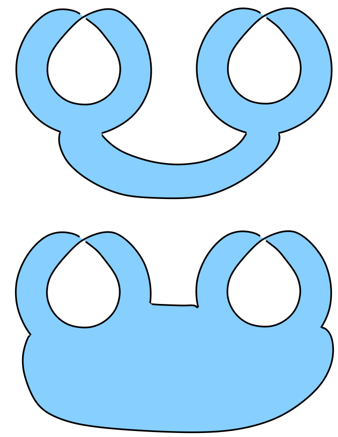
Las superficies con frontera son discos con bandas

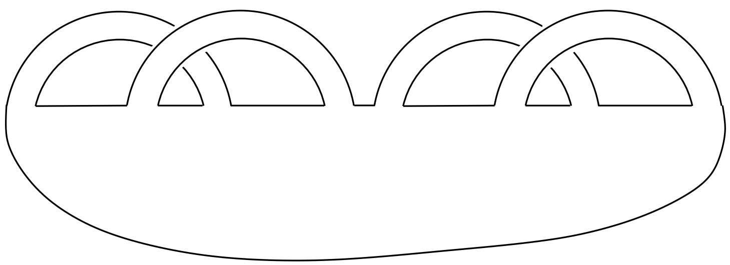

$2b_1$
$2b_2$
$2b_3$
$2b_g$

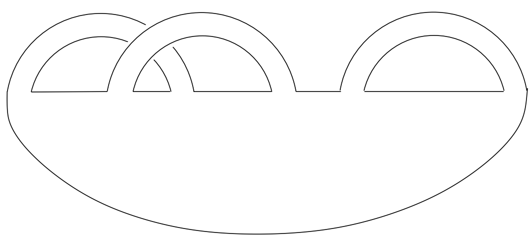


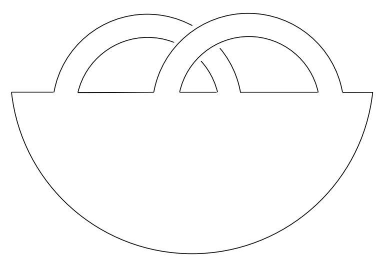


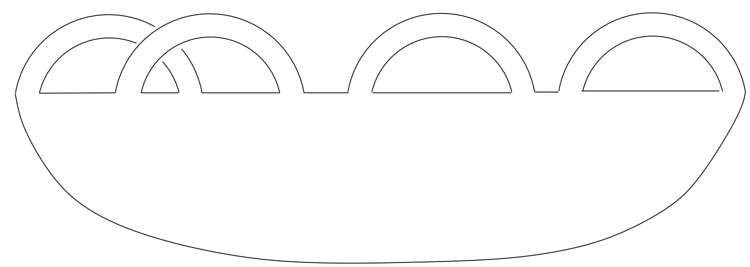


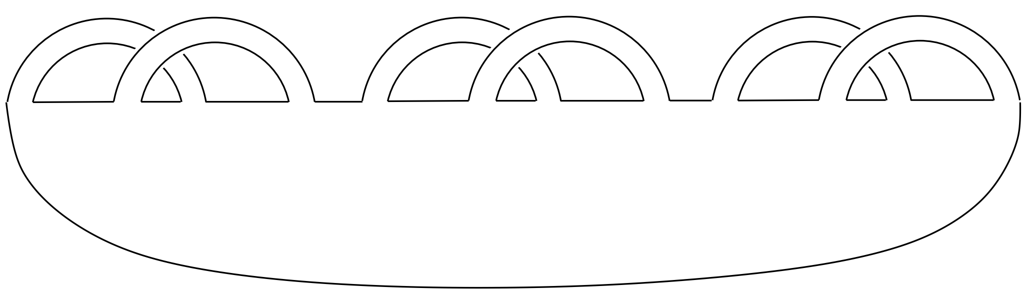












Las superficies con frontera son discos con bandas

Las superficies con frontera son discos con bandas

Para reconocer una superficie, podemos cortarla y...

Las superficies con frontera son discos con bandas

Para reconocer una superficie, podemos cortarla y...

¡No!

Alguien ya sistematizó este proceso y

todo puede ser más Fácil

(Sí)

Def. A, B espacios, $A \subset B$.

Decimos que A *desconecta* a B, si $B - A$ es disconexo.

Sea X una superficie compacta con frontera.

Decimos que X es <u>1-conexa</u> si todo arco propiamente encajado en X...

¿...?

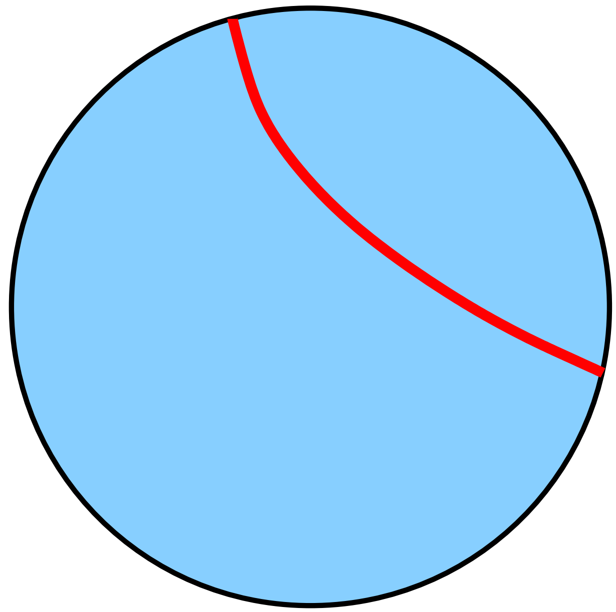
Sea X una superficie compacta con frontera.

Un arco $\alpha \subset X$ se dice que está *propiamente encajado* si $\partial\alpha \subset \partial X$ y $\mathring{\alpha} \subset \mathring{X}$.

Sea X una superficie compacta con frontera.

Decimos que X es <u>1-conexa</u> si todo arco propiamente encajado en X desconecta a X.

Ejemplo

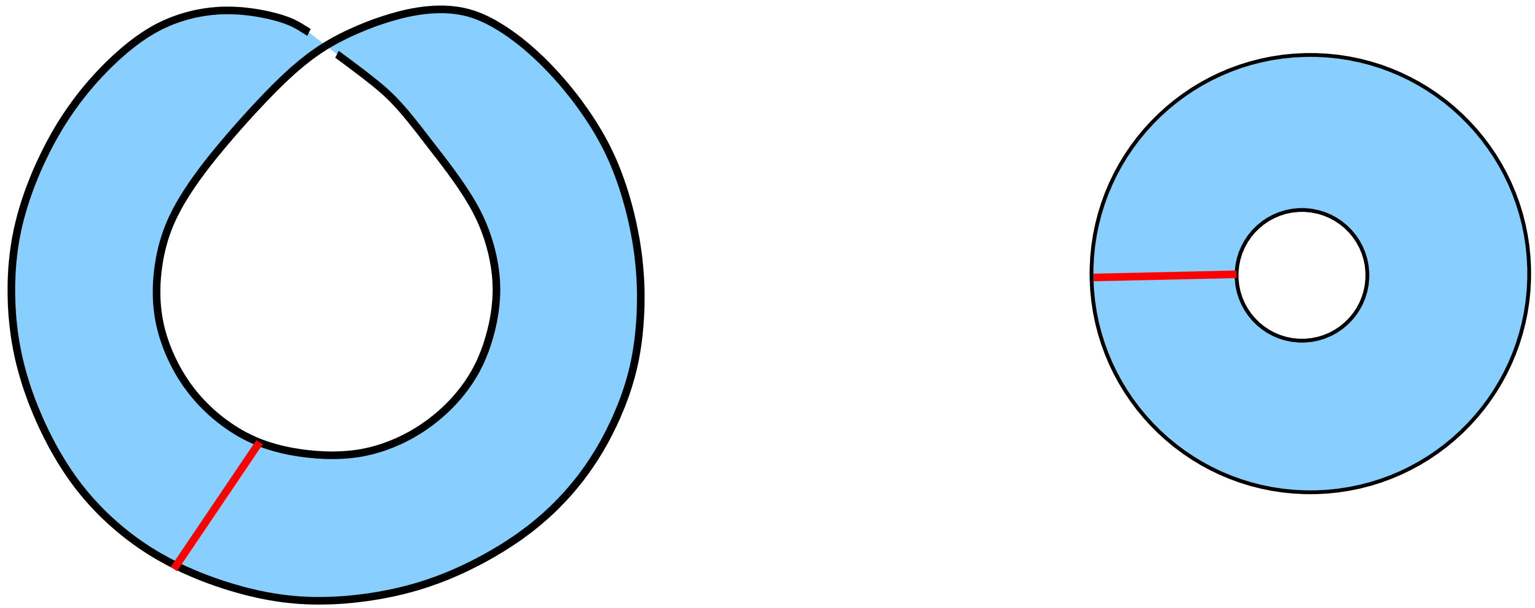


(se puede ver que la única superficie conexa, compacta y 1-conexa es el disco D^2)

Sea X una superficie compacta con frontera.

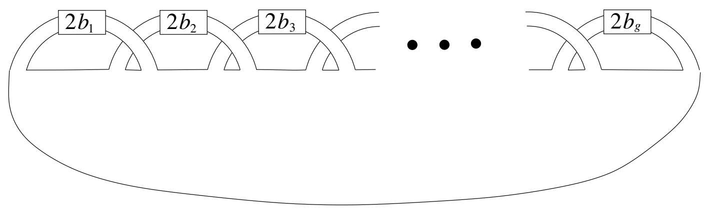
Decimos que X es <u>2-conexa</u> si X no es 1-conexa y todo par de arcos propiamente encajados en X desconecta a X.

Ejemplo



Sea X una superficie compacta con frontera.

Decimos que X es n–conexa
si X no es $(n-1)$–conexa y toda n–ada de arcos propiamente encajados en X desconecta a X.


$2b_1$
$2b_2$
$2b_3$
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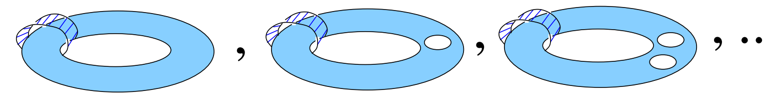

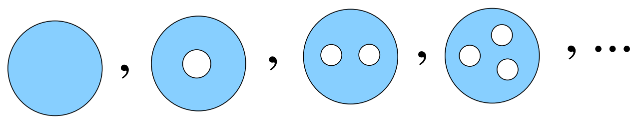

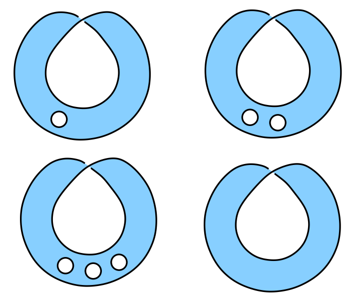

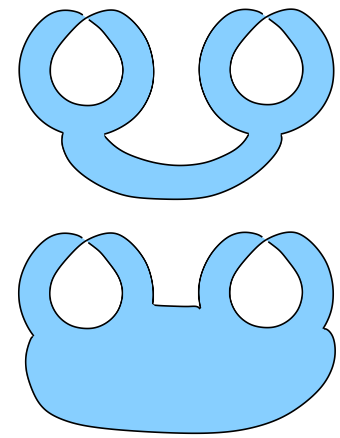

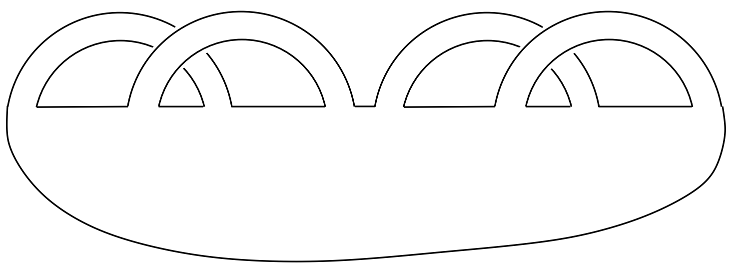

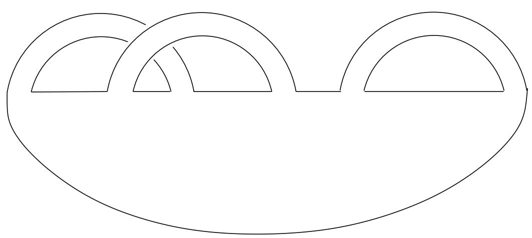

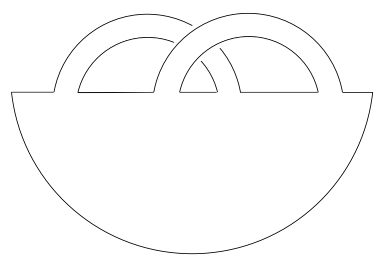

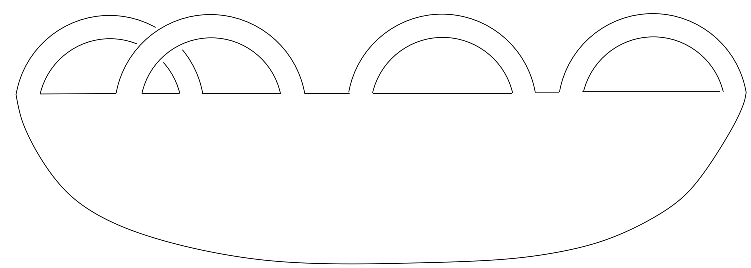

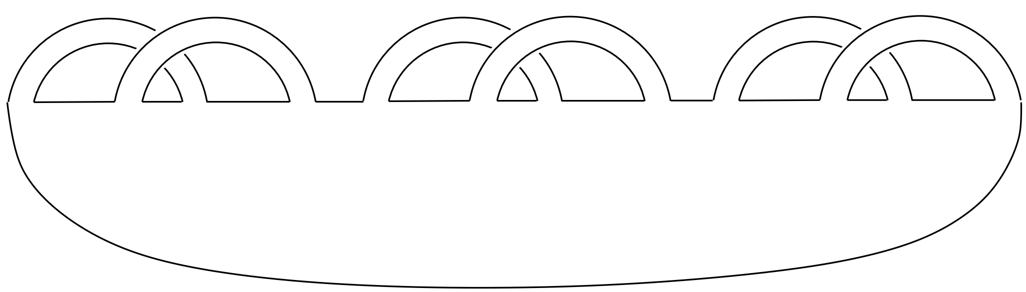

Observación:

Sean X y Y superficies compactas, conexas y con frontera.

Si $X \cong Y$,

entonces

- $|\partial X| = |\partial Y|$.
- X y Y son ambas orientables o ambas son no orientables.
- X es n-conexa $\Leftrightarrow$ Y es n-conexa.

Teorema de Clasificación de las Superficies

Sean X y Y superficies compactas, conexas y con frontera.

$X \cong Y$

$\Leftrightarrow$

- $|\partial X| = |\partial Y|$.
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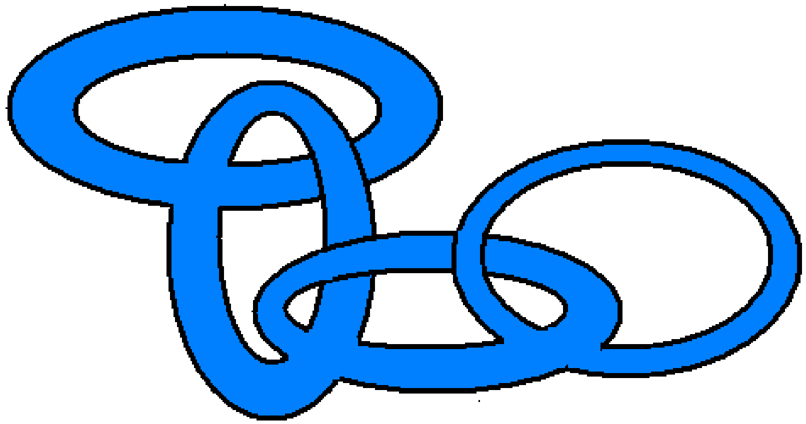
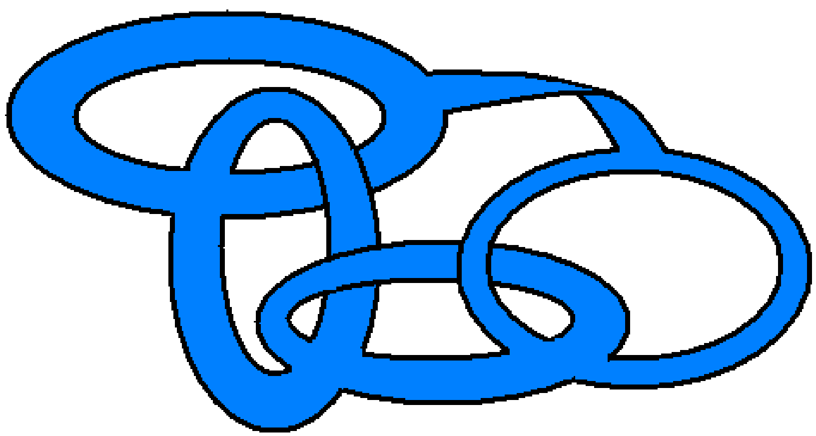
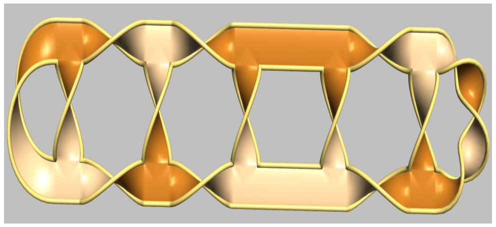
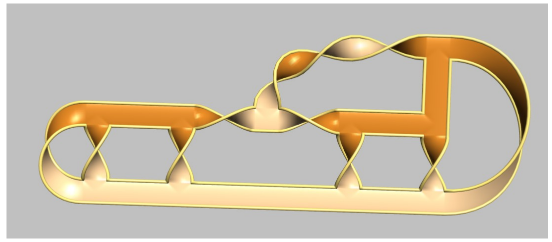
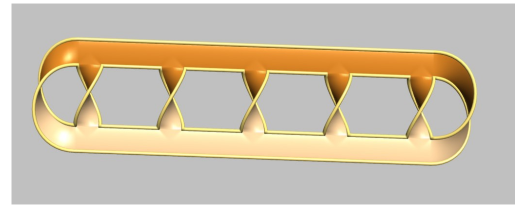
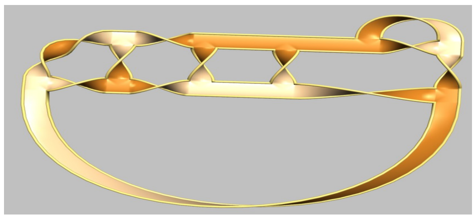
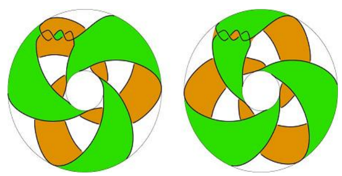
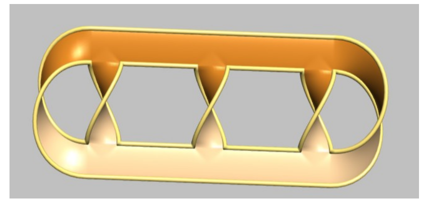
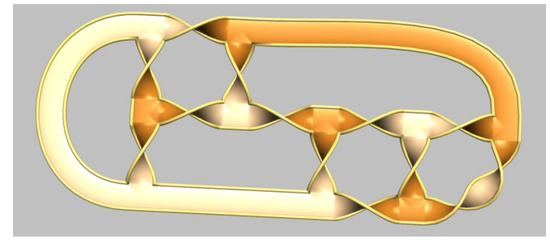
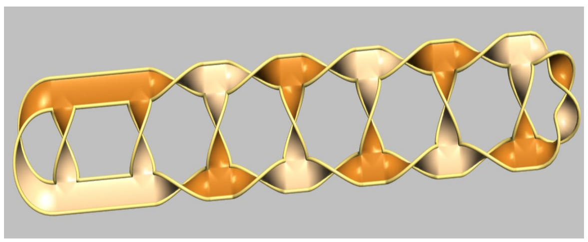
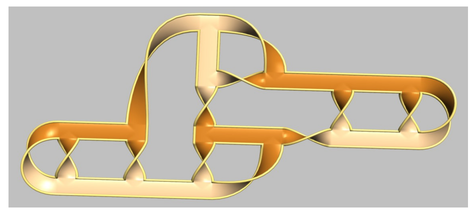
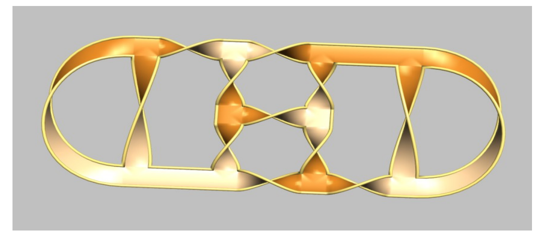
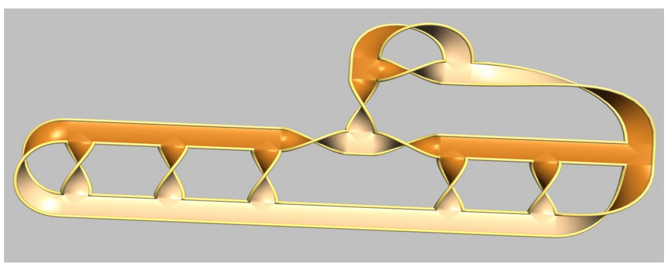
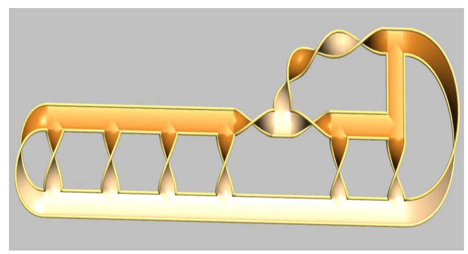
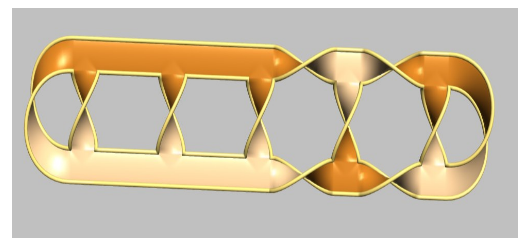
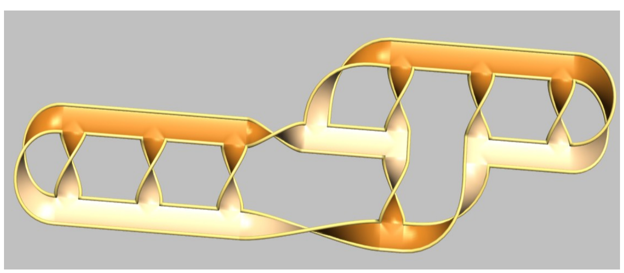
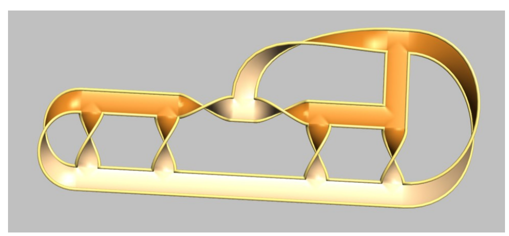
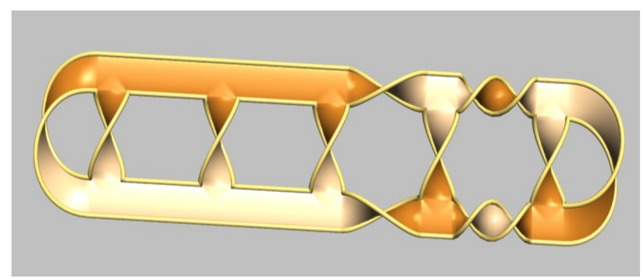
- X es n-conexa $\Leftrightarrow$ Y es n-conexa.





Nota

Se tiene el “invariante”:

$\chi(A) =$ número de vértices $-$ número de aristas $+$ número de caras $- \cdots$

La *característica de Euler* del espacio A.

Teorema.

$A \cong B \Rightarrow \chi(A) = \chi(B)$. $\square$

(por eso se llama un *invariante*)

Nota

Ejemplos:

$$\chi(\emptyset) = 0$$

$$\chi(S^1) = 0$$

$$\chi(B^n) = 1$$

Lema Fundamental.

$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$. $\square$

Nota

Sea X una superficie $(n+1)$–conexa.

Entonces X es un disco con n bandas,

$$X = D^2 \cup B_1 \cup \cdots \cup B_n.$$

$$\chi(D^2 \cup B_1) = \chi(D^2) + \chi(B_1) - \chi(D^2 \cap B_1) = 1 + 1 - 2 = 0$$

$\chi(D^2 \cup B_1 \cup B_2) = \chi(D^2 \cup B_1) + \chi(B_2) - \chi((D^2 \cup B_1) \cap B_2) =$
$0 + 1 - 2 = -1$

$\chi(D^2 \cup B_1 \cup B_2 \cup B_3) =$
$\chi(D^2 \cup B_1 \cup B_2) + \chi(B_3) - \chi((D^2 \cup B_1 \cup B_2) \cap B_3) = -1 + 1 - 2 =$
-2

$\vdots$

$\chi(X) = 1 - n.$

Teorema de Clasificación de las Superficies

Sean X y Y superficies compactas, conexas y con frontera.

$X \cong Y$

$\Leftrightarrow$

- $|\partial X| = |\partial Y|$.
- X y Y son ambas orientables o ambas son no orientables.
- $\chi(X) = \chi(Y)$.

Sea X una superficie $(n+1)$-conexa (X compacta con frontera).

Entonces existe un sistema de arcos, $\alpha_1, \ldots, \alpha_n \subset X$, propiamente encajados en X tales que $X - \alpha_1 \cup \cdots \cup \alpha_n$ es un disco.