Beltrami problem and Nijenhuis operators

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Two (pseudo)-Riemannian metrics are called geodesically equivalent if their geodesics coincide as unparameterized curves. In 1865, E. Beltrami explicitly posed the problem of describing all such metrics. This problem was solved by U. Dini in 1869 for the two-dimensional Riemannian case near almost every point, and in 1896, T. Levi-Civita extended the solution to all dimensions. The pseudo-Riemannian case, near almost every point and in any dimension, was resolved much later, by A. Bolsinov and V. Matveev in 2015 (with the two- and three-dimensional cases addressed earlier by A.Z. Petrov in 1949 and by Bolsinov-Matveev-Pucacco in 2009).

The new results of my talk, in collaboration with A. Bolsinov, provide a solution to the two-dimensional Beltrami problem at the remaining points, where the (1,1)-tensor connecting the geodesically equivalent metrics undergoes a change in its Segre characteristics.

One of the methods used in the proof arises from the relationship between geodesically equivalent metrics and quadratic integrals of the geodesic flow. Another group of methods stems from Nijenhuis geometry, a novel research direction that provides fresh insights into differential geometry and mathematical physics.

If time permits, I will comment on the multidimensional case, emphasizing the connection between geodesic equivalence, integrable geodesic flows, and Nijenhuis geometry.

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