INTEGRABLE SUB-RIEMANNIAN GEODESIC FLOW IN THE ENGEL TYPE GROUP.

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The Engel-type group, denoted as $\mathrm{Eng}(n)$, is a $(2n+2)$-dimensional Carnot group equipped with a non-integrable distribution of rank $(n+1)$. Every Carnot group $\mathbb{G}$ admits the structure of a subRiemannian manifold. Given a Carnot group with a subRiemannian structure, there exists a Hamiltonian function $H_{sR} : T^*\mathbb{G} \to \mathbb{R}$ such that its solutions, when projected to $\mathbb{G}$, correspond to geodesics. When $n = 1$, $\mathrm{Eng}(n)$ reduces to the classical Engel group, whose sub-Riemannian geodesic flow is integrable when endowed with left-invariant subRiemannian metric. In joint work with Enrico Le Donne and Nicola Paddeu, we extend this setting by endowing $\mathrm{Eng}(n)$ with a left-invariant sub-Riemannian metric. We define $\mathbb{A}$ as the maximal abelian subgroup of $\mathrm{Eng}(n)$. The action of $\mathbb{A}$ on $\mathrm{Eng}(n)$ gives rise to $(n+2)$ integrals of motion. Together with the Hamiltonian function H_{sR}, this yields $(n+3)$ constants of motion in involution. Next, we define a Hamiltonian group action of the special orthogonal group $\mathrm{SO}(n)$ on $T^*\,\mathrm{Eng}(n)$ that preserves the Hamiltonian H_{sR}. This symmetry leads to additional, non-commutative constants of motion, and these integrals are quadratic in the momenta. Notably, the Hamiltonian group action of $\mathrm{SO}(n)$ on $T^*\,\mathrm{Eng}(n)$ does not arise from a co-lift of an action of $\mathrm{SO}(n)$ on $\mathrm{Eng}(n)$. By exploiting these symmetries, we derive $(n-1)$ new integrals of motion in involution, thereby ensuring that the sub-Riemannian geodesic flow is Liouville integrable. This construction thus provides:

- A new example of an integrable sub-Riemannian geodesic flow on a Carnot group with arbitrary rank distribution.
- A novel Hamiltonian group action that does not correspond to the co-lift of a group action.

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