Autonomous and non-autonomous restrictions of soliton hierarchies as a source of Stäckel and Painlevé-type systems

Krzysztof Marciniak
Department of Science and Technology, Campus Norrköping, Linköping University
601-74 Norrköping, Sweden
krzma@itn.liu.se

In [1]-[3] the authors presented a systematic way of deforming, using appropriate Killing vectors, Stäckel separable systems $\xi_{t_r} = Y_r(\xi) = \pi dH_r(\xi)$, $r = 1, \ldots, n$ on a $2n$-dimensional Poisson manifold (M, π) to non-autonomous integrable systems $\xi_{t_r} = Y_r(\xi, t) = \pi dH_r(\xi, t_1, \ldots, t_n)$ that are Frobenius integrable so that $\frac{\partial Y_r}{\partial t_s} - \frac{\partial Y_s}{\partial t_r} + [Y_s, Y_r] = 0$, $r, s = 1, \ldots, n$. Recently, in [7], we have developed an analogous theory for soliton hierarchies. We have demonstrated how to deform, using an appropriate master symmetry, an autonomous soliton hierarchy $u_{t_n} = K_n[u]$, $n = 1, 2, \ldots$ into a Frobenius integrable non-autonomous soliton hierarchy $u_{t_n} = K_n([u], x, t_1, t_2, \ldots)$, $n = 1, 2, \ldots$ so that the Frobenius integrability condition $\frac{\partial K_r}{\partial t_s} - \frac{\partial K_s}{\partial t_r} + [K_r, K_s] = 0$ holds.

Since the classical works of Bogoyavlensky, Novikov, Mokhov and others there has been a tremendous amount of research devoted to connections between soliton hierarchies $u_{t_n} = K_n[u]$, $n = 1, 2, \ldots$and their integrable finite-dimensional reductions; a substantial portion of this research focused on stationary flows. In [4]-[6] we have revisited this idea in a novel, systematic way by investigating not only a particular stationary flow $K_{n+1}[u] = 0$ of a soliton hierarchies but rather it's so called stationary system, that is the finite dimensional system defined by the stationary flow $K_{n+1}[u] = 0$ itself together with all the lower flows from the hierarchy:

$$u_{t_1} = K_1[u], \ldots, u_{t_n} = K_n[u], \qquad K_{n+1}[u] = 0$$

In result, we were able to show, by presenting explicit Miura-type maps between the jet variables $(u, u_x, u_{xx}, \ldots)$ and the Viéte variables (q, p), that in the case of particular soliton hierarchies (KdV, cKdV and AKNS) the related stationary systems can be represented as classical Stäckel separable systems.

In an ongoing research project we generalize the above concept of stationary systems to what we call *non-autonomous restrictions of soliton hierarchies*. These restrictions are defined through invariant time-dependent constraints that are appropriate deformations of the stationary flow $K_{n+1}[u] = 0$ by times-dependent linear combinations of a master symmetry and lower flows, the idea which is based on results in [7]. It turns out that this class of time-dependent restrictions of soliton hierarchies is represented by non-autonomous Hamiltonian finite-dimensional dynamical systems of Painlevé type (in the sense that they possess an isomonodromic rather than isospectral representation). The original Painlevé equations are non-autonomous nonlinear ODE's that, at the beginning of the 20th century, allowed for defining new special functions. Our hope it that our approach will yield a *systematic* way of producing new Painlevé-type systems.

In this talk I will present the above ideas, with focus on autonomous and non-autonomous restrictions of soliton hierarchies. This talk is a result of a joint work with **Maciej Błaszak** and **Błażej M. Szablikowski**, Poznań University, Poland.

References

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