7. Using Exercises 3–5 of Section 7, show that the characters of the representations Φ_n of SU_2 constructed in 7.4 constitute a complete orthogonal set in the space of continuous central functions on SU_2. Conclude from this that every irreducible representation of SU_2 is isomorphic to one of the representations Φ_n.

8. Use the theory of characters to derive the following relation for the representations of SU_2: if $m \geq n$, then

$$\Phi_m \Phi_n \simeq \Phi_{m-n} + \Phi_{m-n+2} + \ldots + \Phi_{m+n}.$$

9. Using Exercise 7, show that every irreducible representation of SO_3 is isomorphic to one of the representations Ψ_n constructed in 7.4.

10. Let R and S be representations of SU_2 such that $R|_{\mathbf{T}} \simeq S|_{\mathbf{T}}$. Show that $R \simeq S$. (See Exercise 3, Section 7.)

9. The Laplace Spherical Functions

9.1. We consider the following general problem. Suppose we are given a transitive action $s: G \to S(X)$ of the compact topological group G on the topological space X. We shall assume that s is continuous, i.e., the map

$$G \times X \to X, \qquad (g, x) \mapsto s(g)x,$$

is continuous. Under these circumstances, can we decompose the space of continuous functions on X into a topological direct sum of finite-dimensional G-invariant subspaces? If yes, how can the linear representations of G in these subspaces be described?

Let H be the isotropy subgroup of some point $o \in X$. It is a closed subgroup of G. Proceeding as in 5.5, we construct the bijective map

$$\bar{p}: G/H \to X, \qquad gH \mapsto go,$$

which commutes with the actions of the group G, where one assumes, as usual, that G acts on G/H by left translations.

The left coset space G/H is endowed with a topology as follows: a subset $B \subset G/H$ is said to be closed if and only if its preimage under the canonical projection $\pi: G \to G/H$ is closed.

Lemma. *$\bar{p}$ is a homeomorphism.*

The proof is analogous to that of the Lemma of 7.3. □

Thus, the space X can be identified with G/H. Under this identification s becomes the action l^H, and the continuous functions on X become the continuous functions on G/H or, equivalently, the continuous functions on G which are constant on the left cosets of H in G. Accordingly, the representation s_* of G in the space of continuous functions on X is isomorphic to the representation L^H of G in the space $C_2(G)^H$. The decomposition of L^H obtained in 8.4 permits us to answer the questions formulated above, provided that the irreducible representations of the group G are known.

Specifically, let $(e_1, \ldots, e_n)$ be an orthonormal basis of the space V of the irreducible representation T of G, chosen so that $(e_1, \ldots, e_m)$ is a basis of the subspace V^H. Let T_{ij} be the matrix elements of T in this basis. Then for each $j = 1, \ldots, m$ the functions $T_{1j}, \ldots, T_{nj}$ are constant on the left cosets of H in G. Regarded as functions on X, they form a basis for a minimal G-invariant subspace of functions in which a representation isomorphic to T' is realized. The functions constructed in this manner for all irreducible representations T of G form a complete orthogonal set in the space of continuous functions on X. They will be referred to as SPHERICAL FUNCTIONS to emphasize the analogy with Laplace's spherical functions, to which the remaining part of this section is devoted.

9.2. A particular case of the problem considered in the preceding subsection is that of *decomposing the space of continuous functions on the two-dimensional sphere*

$$S = \{ (x_1, x_2, x_3) \in \mathbf{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1 \}$$

into a topological direct sum of finite-dimensional subspaces invariant under the group SO_3 *of rotations of* $\mathbf{R}^3$. We solve this problem here without resorting to the foregoing general considerations.

Let $C_2(S)$ denote the Hermitian space of all continuous complex functions on S with the inner product defined as in 8.1. As the (usual) integration on S is invariant under SO_3, the inner product in $C_2(S)$ is also invariant under SO_3.

Fix the point $o = (0, 0, 1)$, the "north pole" on S. The isotropy subgroup of o, that is, the group of rotations around the x_3-axis, is isomorphic to SO_2, and by a slight abuse of notation we shall denote it by SO_2.

Lemma 1. *In every nonnull finite-dimensional* SO_3*-invariant subspace* $U \subset C_2(S)$ *there is a nonnull* SO_2*-invariant function.*

PROOF. First of all, let us show that U contains functions that do not vanish at the point o. Let $f \in U$ be an arbitrary nonnull function, and let $x \in S$ be such that $f(x) \neq 0$. There is a rotation $g \in \mathrm{SO}_3$ such that $gx = o$, and then $(g_* f)(o) = f(x) \neq 0$.

Now consider the subspace

$$U_0 = \{ f \in U \mid f(o) = 0 \}.$$

By the foregoing, it has codimension one. It is clearly SO_2-invariant. Hence, its orthogonal complement $U_0^\perp$ is also SO_2-invariant.

Now let $f_0 \in U_0^\perp$ be an arbitrary nonnull function. Rotations around the x_3-axis can affect f_0 only by a multiplicative constant. However, since the point o is fixed, that constant is in fact equal to 1. Hence, f_0 is the desired function. □

9.3. Let A denote the algebra of polynomials with complex coefficients in the coordinates x_1, x_2, x_3. Its elements will be regarded as functions on $\mathbf{R}^3$. It is plain that the algebra A is SO_3-invariant. Moreover, it splits into a direct sum of finite-dimensional invariant subspaces:

$$A = A_0 \oplus A_1 \oplus A_2 \oplus \dots,$$

where A_m designates the space of homogeneous polynomials of degree m. The monomials $x_1^{k_1} x_2^{k_2} x_3^{k_3}$ provide a basis for A. We endow A with a Hermitian inner product chosen so that this basis is orthogonal and

$$(x_1^{k_1} x_2^{k_2} x_3^{k_3}, x_1^{k_1} x_2^{k_2} x_3^{k_3}) = k_1! k_2! k_3!.$$

Lemma 2. *The operator of multiplication by x_i in A is the adjoint of the differentiation operator $\partial/\partial x_i$ with respect to the inner product $(\cdot\,,\cdot)$.*

PROOF. For the sake of definiteness, let $i = 1$. We have to show that $(\partial u/\partial x_1, v) = (u, x_1 v)$ for any two polynomials u, v. It suffices to check this for monomials u and v. Let $u = x_1^{k_1} x_2^{k_2} x_3^{k_3}$. If $v \neq x_1^{k_1-1} x_2^{k_2} x_3^{k_3}$, then both sides of the needed equality vanish. If now $v = x_1^{k_1-1} x_2^{k_2} x_3^{k_3}$, then we get $k_1! k_2! k_3!$ in both sides. □

Corollary. *The operator of multiplication by $r^2 = x_1^2 + x_2^2 + x_3^2$ is the adjoint to the Laplace operator*

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}.$$

□

Notice that $r^2 A_m \subset A_{m+2}$ and $\Delta A_m \subset A_{m-2}$.

The functions annihilated by the operator Δ are called HARMONIC. We let H denote the space of harmonic polynomials. Obviously $H = \sum_m H_m$, where $H_m = H \cap A_m$. Since the kernel of any linear operator coincides with the orthogonal complement to the range of its adjoint,

$$(1) \qquad A_m = H_m \oplus r^2 A_{m-2}, \qquad m = 0, 1, 2, \dots.$$

This yields the following direct sum decomposition of the space A_m:

$$(2) \qquad A_m = H_m \oplus r^2 H_{m-2} \oplus r^4 H_{m-4} \oplus \dots.$$

Since both the Laplace operator and the operator of multiplication by r^2 commute with rotations, the summands in this decomposition are SO_3-invariant. In point of fact, they are *minimal invariant subspaces,* as we next show.

9.4. Let ρ denote the operation of *restriction* of functions from $\mathbf{R}^3$ to the sphere S. Set

$$\mathbf{C}[S] = \rho(A).$$

It is plain that the map

$$\rho: A \to \mathbf{C}[S]$$

is linear and commutes with the actions of SO_3 on A and $\mathbf{C}[S]$.

It follows from Weierstrass's Theorem that $\mathbf{C}[S]$ is a dense subspace of $C_2(S)$ (cf. 8.3).

Lemma 3. Ker $\rho \cap A_m = 0$.

PROOF. In fact, any homogeneous polynomial that vanishes identically on S vanishes identically in $\mathbf{R}^3$, too. □

Theorem 1. *(2) is a decomposition of A_m into a direct sum of minimal* SO_3*-invariant subspaces.*

PROOF. By Lemma 3, ρ maps A_m isomorphically onto $\rho(A_m)$. By Lemma 1, the number of components in a direct sum decomposition of A_m into minimal invariant subspaces does not exceed $\dim A_m^{SO_2}$. We next show that $\dim A_m^{SO_2} = [m/2] + 1$ (Lemma 4 below; here [ ] stands for integer part). Since the number of components in the decomposition (2) is also equal to $[m/2] + 1$, we conclude that the latter are minimal invariant subspaces. □

Let us parametrize SO_2 by complex numbers of modulus one. Namely, for each $z = e^{it}$, $t \in \mathbf{R}$, we let $h(z)$ denote the rotation through angle t around the x_3-axis.

Lemma 4. *The space A_m admits a basis of joint eigenfunctions of the transformations $h(z) \in SO_2$. The corresponding eigenvalues have the form z^k, for $k = 0, \pm 1, \pm 2, \ldots, \pm m$. The multiplicity of the eigenvalue z^k is $[\frac{1}{2}(m - |k|)] + 1$. In particular,*

$$\dim A_m^{SO_2} = \left[\frac{m}{2}\right] + 1.$$

PROOF. Put

$$u = x_1 - ix_2, \qquad \bar{u} = x_1 + ix_2.$$

It is readily seen that

$$h(z)_* u = zu, \qquad h(z)_* \bar{u} = z^{-1}\bar{u}.$$

Any polynomial in A_m is uniquely expressible as a linear combination of monomials $u^p \bar{u}^q x_3^\ell$, where $p + q + \ell = m$. From the above formulas it follows that

$$h(z)_* u^p \bar{u}^q x_3^\ell = z^{p-q} u^p \bar{u}^q x_3^\ell,$$

i.e., the monomials $u^p \bar{u}^q x_3^\ell$ are joint eigenfunctions for all transformations in SO_2 with corresponding eigenvalues z^{p-q}. Since $|p - q| \leq p + q = m - \ell$, we have that $|p - q| \leq m$. For $p - q = k$ the exponent ℓ can assume the values $m - |k|, m - |k| - 2, m - |k| - 4, \ldots$. Hence, the number of monomials $u^p \bar{u}^q x_3^\ell$ with $p - q = k$ is $[\frac{1}{2}(m - |k|)] + 1$. □

9.5. We can now formulate the main result of this section.

Theorem 2. *The space $\mathbf{C}[S]$ decomposes into the orthogonal direct sum of the minimal SO_3-invariant subspaces $U_m = \rho(H_m)$, $m = 0, 1, 2, \ldots$. The subspace U_m has dimension $2m + 1$. It admits an orthogonal basis $(Y_{m,0}, Y_{m,\pm 1}, \ldots, Y_{m,\pm m})$ consisting of joint eigenfunctions of the transformations $h(z) \in SO_2$. The eigenvalue corresponding to $Y_{m,k}$ is z^k.*

The functions $Y_{m,k}$ are called the LAPLACE SPHERICAL FUNCTIONS.

PROOF. It follows from Lemma 3 and Theorem 1 that for each m

$$\rho(A_m) = \rho(H_m) \oplus \rho(H_{m-2}) \oplus \rho(H_{m-4}) \oplus \ldots,$$

and that $U_m = \rho(H_m)$ is a minimal SO_3-invariant subspace. Since

$$\mathbf{C}[S] = \rho(A) = \rho(A_0) + \rho(A_1) + \rho(A_2) + \ldots,$$

we have that

$$\mathbf{C}[S] = U_0 + U_1 + U_2 + \dots .$$

The representation of SO_3 in U_m is isomorphic to its representation in the space $H_m \subset A_m$. The representation of the group $SO_2 \subset SO_3$ in A_m is a sum of one-dimensional representations of the type $h(z) \mapsto z^k$ ($|k| \le m$), whose multiplicities were computed in Lemma 4. Consequently, the representation of SO_2 in H_m is also a sum of one-dimensional representations. It follows from decomposition (1) that the multiplicity of the one-dimensional representation $h(z) \mapsto z^k$ ($|k| \le m$) in the representation of SO_2 in the space H_m is equal to the difference of its multiplicities in the representations of SO_2 in the spaces A_m and A_{m-2}, which in turn is equal to 1:

$$[\frac{1}{2}(m - |k|)] - [\frac{1}{2}(m - 2 - |k|)] = 1.$$

Thus, in U_m there is a basis $(Y_{m,0}, Y_{m,\pm 1}, \dots, Y_{m,\pm m})$ such that

$$h(z)_* Y_{m,k} = z^k Y_{m,k}.$$

In particular, $\dim U_m = 2m + 1$.

Comparing dimensions we see that for $m \neq \ell$ the representations of SO_3 in U_m and U_ℓ are not isomorphic. Applying Theorem 9 of 4.7, we conclude that the subspaces U_m are mutually orthogonal, and consequently linearly independent. By the same theorem, applied now to the representation of SO_2 in U_m, the functions $Y_{m,0}, Y_{m,\pm 1}, \dots, Y_{m,\pm m}$ are pairwise orthogonal. This completes the proof of the present theorem. □

Corollary. *The functions $Y_{m,k}$ ($m = 0, 1, 2, \dots;\ k = 0, \pm 1, \dots, \pm m$) constitute a complete orthonormal set in $C_2(S)$.* □

9.6. Now let us find *explicit expressions for the Laplace functions.* We denote by ξ_1, ξ_2, ξ_3 the restrictions of the coordinate functions x_1, x_2, x_3 to the sphere S.

The space $U_m = \rho(H_m)$ contains a unique (up to a multiplicative constant) SO_2-invariant function, namely, $Y_{m,0}$. It is called the ZONAL SPHERICAL FUNCTION OF ORDER m. It is the restriction to S of a linear combination of polynomials of the form

$$u^p \bar{u}^p x_3^\ell = (u\bar{u})^p x_3^\ell = (x_1^2 + x_2^2)^p x_3^\ell \qquad (\text{with } 2p + \ell = m).$$

Since $\rho(x_1^2+x_2^2)=\rho(1-x_3^2)=1-\xi_3^2$, it follows that $Y_{m,0}$ is a polynomial of degree $\le m$ in ξ_3:

$$Y_{m,0}=P_m(\xi_3).$$

The linear independence of the functions $Y_{0,0}, Y_{1,0}, \ldots, Y_{m,0}$ implies that $P_0, P_1, \ldots, P_m$ constitute a basis in the space of polynomials of degree $\le m$.

Lemma. *$\int_{-1}^{1} P_m(t)\overline{P_\ell(t)}\,dt=0$ for $m\ne\ell$.*

PROOF. We calculate the inner product of the functions $Y_{m,0}$ and $Y_{\ell,0}$. To this end we note that the area of the infinitesimally thin belt on the sphere S specified by the inequalities $t\le\xi_3\le t+dt$ equals $2\pi\,dt$. Consequently, the integral of any function of the form $P(\xi_3)$ over S equals $2\pi\int_{-1}^{1}P(t)\,dt$. In particular,

$$(Y_{m,0},Y_{\ell,0})=2\pi\int_{-1}^{1}P_m(t)\overline{P_\ell(t)}\,dt$$

which, in view of the orthogonality of the functions $Y_{m,0}$ and $Y_{\ell,0}$, proves the lemma. □

We define an inner product in the space of polynomials of one variable by the rule

$$(P,Q)=\int_{-1}^{1}P(t)\overline{Q(t)}\,dt.$$

Then the lemma asserts that the polynomials $P_0, P_1, P_2, \ldots$ are mutually orthogonal. Since $(P_0, P_1, \ldots, P_{m-1})$ is a basis of the space of polynomials of degree $< m$, it follows that P_m is a polynomial of degree m that is orthogonal to all polynomials of degree $< m$. As such, it is uniquely determined up to a multiplicative constant. It is called the LEGENDRE POLYNOMIAL of degree m. We have thus established

Theorem 3. *$Y_{m,0}=P_m(\xi_3)$, where P_m is the Legendre polynomial of degree m.* □

Here are explicit expressions for a few of the first Legendre polynomials:

$$P_0(t)=1, \qquad P_1(t)=t,$$
$$P_2(t)=3t^2-1, \qquad P_3(t)=5t^3-3t,$$
$$P_4(t)=35t^4-30t^2+3, \qquad P_5(t)=63t^5-70t^3+15t.$$

As regards the remaining spherical functions, they can be expressed through Legendre polynomials as follows:

$$(3)\qquad \begin{aligned} Y_{m,k} &= (\xi_1 - i\xi_2)^k P_m^{(k)}(\xi_3), \qquad (k > 0), \\ Y_{m,-k} &= (\xi_1 + i\xi_2)^k P_m^{(k)}(\xi_3), \qquad (k > 0). \end{aligned}$$

We are going to see this in Section 11, where we will also show that the representation of SO_3 in the space U_m is isomorphic to the representation Ψ_{2m} constructed in 7.4.

Questions and Exercises

1. Suppose the compact group G acts transitively on a topological space X. Let H be the isotropy subgroup of the point $o \in X$. Show that in every finite-dimensional nonzero G-invariant subspace of continuous functions on X there exists a nonzero H-invariant function.

2. Under the assumptions of the preceding exercise, suppose that $T: G \to \mathrm{GL}(V)$ is a finite-dimensional irreducible representation of G. Prove that if V contains a nonzero H-invariant vector, then T is isomorphic to a representation of G in a space of continuous functions on X. (*Hint:* Establish first the existence of a nonzero H-invariant linear function $f \in V'$; then consider the map that assigns to each vector $v \in V$ the continuous function f_v on X defined by the formula $f_v(go) = f(g^{-1}v)$.)

Work out Exercises 3–6 without resorting to formulas (3).

3. Show that $Y_{m,k}(o) = 0$ if $k \neq 0$.

4. Find explicit expressions for the functions $Y_{m,k}$ for $m = 1, 2$.

5. Show that $Y_{m,m} = (\xi_1 - i\xi_2)^m$.

6. Show that $\overline{U}_m = U_m$. Deduce from this that $\overline{Y}_{m,k} = Y_{m,-k}$ (where the bar denotes complex conjugation).

7. Prove the following formula of Rodrigues:

$$P_m(t) = \frac{d^m}{dt^m}\Big[(t^2-1)^m\Big].$$