Chapter 7

Identifying second degree equations

7.1 The eigenvalue method

In this section we apply eigenvalue methods to determine the geometrical nature of the second degree equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \tag{7.1}$$

where not all of a, h, b are zero.

Let $A = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$ be the matrix of the quadratic form $ax^2 + 2hxy + by^2$. We saw in section 6.1, equation 6.2 that A has real eigenvalues λ_1 and λ_2, given by

$$\lambda_1 = \frac{a + b - \sqrt{(a-b)^2 + 4h^2}}{2},\ \lambda_2 = \frac{a + b + \sqrt{(a-b)^2 + 4h^2}}{2}.$$

We show that it is always possible to rotate the x, y axes to x_1, x_2 axes whose positive directions are determined by eigenvectors X_1 and X_2 corresponding to $\lambda_1$and λ_2 in such a way that relative to the x_1, y_1 axes, equation 7.1 takes the form

$$a'x^2 + b'y^2 + 2g'x + 2f'y + c = 0. \tag{7.2}$$

Then by completing the square and suitably translating the x_1, y_1 axes, to new x_2, y_2 axes, equation 7.2 can be reduced to one of several standard forms, each of which is easy to sketch. We need some preliminary definitions.

DEFINITION 7.1.1 (Orthogonal matrix) An $n \times n$ real matrix P is called *orthogonal* if

$$P^t P = I_n.$$

It follows that if P is orthogonal, then $\det P = \pm 1$. For

$$\det(P^t P) = \det P^t \det P = (\det P)^2,$$

so $(\det P)^2 = \det I_n = 1$. Hence $\det P = \pm 1$.

If P is an orthogonal matrix with $\det P = 1$, then P is called a *proper* orthogonal matrix.

THEOREM 7.1.1 If P is a 2×2 orthogonal matrix with $\det P = 1$, then

$$P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

for some θ.

REMARK 7.1.1 Hence, by the discusssion at the beginning of Chapter 6, if P is a proper orthogonal matrix, the coordinate transformation

$$\begin{bmatrix} x \\ y \end{bmatrix} = P \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

represents a rotation of the axes, with new x_1 and y_1 axes given by the repective columns of P.

Proof. Suppose that $P^t P = I_2$, where $\Delta = \det P = 1$. Let

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then the equation

$$P^t = P^{-1} = \frac{1}{\Delta} \operatorname{adj} P$$

gives

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Hence $a = d$, $b = -c$ and so

$$P = \begin{bmatrix} a & -c \\ c & a \end{bmatrix},$$

where $a^2 + c^2 = 1$. But then the point (a, c) lies on the unit circle, so $a = \cos\theta$ and $c = \sin\theta$, where θ is uniquely determined up to multiples of 2π.

DEFINITION 7.1.2 (Dot product). If $X = \begin{bmatrix} a \\ b \end{bmatrix}$ and $Y = \begin{bmatrix} c \\ d \end{bmatrix}$, then $X \cdot Y$, the *dot product* of X and Y, is defined by

$$X \cdot Y = ac + bd.$$

The dot product has the following properties:

(i) $X \cdot (Y + Z) = X \cdot Y + X \cdot Z$;

(ii) $X \cdot Y = Y \cdot X$;

(iii) $(tX) \cdot Y = t(X \cdot Y)$;

(iv) $X \cdot X = a^2 + b^2$ if $X = \begin{bmatrix} a \\ b \end{bmatrix}$;

(v) $X \cdot Y = X^t Y$.

The *length* of X is defined by

$$||X|| = \sqrt{a^2 + b^2} = (X \cdot X)^{1/2}.$$

We see that $||X||$ is the distance between the origin $O = (0,\, 0)$ and the point $(a,\, b)$.

THEOREM 7.1.2 (Geometrical interpretation of the dot product) Let $A = (x_1,\, y_1)$ and $B = (x_2,\, y_2)$ be points, each distinct from the origin $O = (0,\, 0)$. Then if $X = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $Y = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$, we have

$$X \cdot Y = OA \cdot OB \cos\theta,$$

where θ is the angle between the rays OA and OB.

Proof. By the cosine law applied to triangle OAB, we have

$$AB^2 = OA^2 + OB^2 - 2OA \cdot OB \cos\theta. \tag{7.3}$$

Now $AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$, $OA^2 = x_1^2 + y_1^2$, $OB^2 = x_2^2 + y_2^2$.

Substituting in equation 7.3 then gives

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = (x_1^2 + y_1^2) + (x_2^2 + y_2^2) - 2OA \cdot OB \cos\theta,$$

which simplifies to give

$$OA \cdot OB \cos\theta = x_1x_2 + y_1y_2 = X \cdot Y.$$

It follows from theorem 7.1.2 that if $A = (x_1, y_1)$ and $B = (x_2, y_2)$ are points distinct from $O = (0, 0)$ and $X = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $Y = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$, then $X \cdot Y = 0$ means that the rays OA and OB are perpendicular. This is the reason for the following definition:

DEFINITION 7.1.3 (Orthogonal vectors) Vectors X and Y are called orthogonal if

$$X \cdot Y = 0.$$

There is also a connection with orthogonal matrices:

THEOREM 7.1.3 Let P be a 2×2 real matrix. Then P is an orthogonal matrix if and only if the columns of P are orthogonal and have unit length.

Proof. P is orthogonal if and only if $P^tP = I_2$. Now if $P = [X_1|X_2]$, the matrix P^tP is an important matrix called the *Gram* matrix of the column vectors X_1 and X_2. It is easy to prove that

$$P^tP = [X_i \cdot X_j] = \begin{bmatrix} X_1 \cdot X_1 & X_1 \cdot X_2 \\ X_2 \cdot X_1 & X_2 \cdot X_2 \end{bmatrix}.$$

Hence the equation $P^tP = I_2$ is equivalent to

$$\begin{bmatrix} X_1 \cdot X_1 & X_1 \cdot X_2 \\ X_2 \cdot X_1 & X_2 \cdot X_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

or, equating corresponding elements of both sides:

$$X_1 \cdot X_1 = 1,\ X_1 \cdot X_2 = 0,\ X_2 \cdot X_2 = 1,$$

which says that the columns of P are orthogonal and of unit length.

The next theorem describes a fundamental property of real symmetric matrices and the proof generalizes to symmetric matrices of any size.

THEOREM 7.1.4 If X_1 and X_2 are eigenvectors corresponding to distinct eigenvalues λ_1 and λ_2 of a real symmetric matrix A, then X_1 and X_2 are orthogonal vectors.

Proof. Suppose

$$AX_1 = \lambda_1 X_1,\ AX_2 = \lambda_2 X_2, \tag{7.4}$$

where X_1 and X_2 are non–zero column vectors, $A^t = A$ and $\lambda_1 \neq \lambda_2$.

We have to prove that $X_1^t X_2 = 0$. From equation 7.4,

$$X_2^t A X_1 = \lambda_1 X_2^t X_1 \tag{7.5}$$

and

$$X_1^t A X_2 = \lambda_2 X_1^t X_2. \tag{7.6}$$

From equation 7.5, taking transposes,

$$(X_2^t A X_1)^t = (\lambda_1 X_2^t X_1)^t$$

so

$$X_1^t A^t X_2 = \lambda_1 X_1^t X_2.$$

Hence

$$X_1^t A X_2 = \lambda_1 X_1^t X_2. \tag{7.7}$$

Finally, subtracting equation 7.6 from equation 7.7, we have

$$(\lambda_1 - \lambda_2) X_1^t X_2 = 0$$

and hence, since $\lambda_1 \neq \lambda_2$,

$$X_1^t X_2 = 0.$$

THEOREM 7.1.5 Let A be a real 2×2 symmetric matrix with distinct eigenvalues λ_1 and λ_2. Then a proper orthogonal 2×2 matrix P exists such that

$$P^t A P = \operatorname{diag}(\lambda_1,\ \lambda_2).$$

Also the rotation of axes

$$\begin{bmatrix} x \\ y \end{bmatrix} = P \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

"diagonalizes" the quadratic form corresponding to A:

$$X^t A X = \lambda_1 x_1^2 + \lambda_2 y_1^2.$$

Proof. Let X_1 and X_2 be eigenvectors corresponding to λ_1 and λ_2. Then by theorem 7.1.4, X_1 and X_2 are orthogonal. By dividing X_1 and X_2 by their lengths (i.e. *normalizing* X_1 and X_2) if necessary, we can assume that X_1 and X_2 have unit length. Then by theorem 7.1.1, $P = [X_1|X_2]$ is an orthogonal matrix. By replacing X_1 by $-X_1$, if necessary, we can assume that $\det P = 1$. Then by theorem 6.2.1, we have

$$P^tAP = P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Also under the rotation $X = PY$,

$$\begin{aligned} X^tAX &= (PY)^tA(PY) = Y^t(P^tAP)Y = Y^t \operatorname{diag}(\lambda_1,\, \lambda_2)Y \\ &= \lambda_1 x_1^2 + \lambda_2 y_1^2. \end{aligned}$$

EXAMPLE 7.1.1 Let A be the symmetric matrix

$$A = \begin{bmatrix} 12 & -6 \\ -6 & 7 \end{bmatrix}.$$

Find a proper orthogonal matrix P such that P^tAP is diagonal.

Solution. The characteristic equation of A is $\lambda^2 - 19\lambda + 48 = 0$, or

$$(\lambda - 16)(\lambda - 3) = 0.$$

Hence A has distinct eigenvalues $\lambda_1 = 16$ and $\lambda_2 = 3$. We find corresponding eigenvectors

$$X_1 = \begin{bmatrix} -3 \\ 2 \end{bmatrix} \text{ and } X_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Now $||X_1|| = ||X_2|| = \sqrt{13}$. So we take

$$X_1 = \frac{1}{\sqrt{13}}\begin{bmatrix} -3 \\ 2 \end{bmatrix} \text{ and } X_2 = \frac{1}{\sqrt{13}}\begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Then if $P = [X_1|X_2]$, the proof of theorem 7.1.5 shows that

$$P^tAP = \begin{bmatrix} 16 & 0 \\ 0 & 3 \end{bmatrix}.$$

However $\det P = -1$, so replacing X_1 by $-X_1$ will give $\det P = 1$.

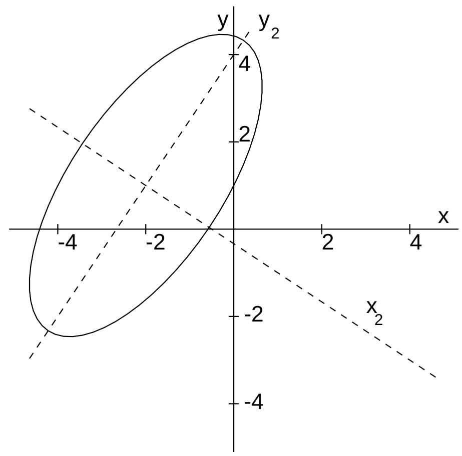

Figure 7.1: $12x^2 - 12xy + 7y^2 + 60x - 38y + 31 = 0$.

REMARK 7.1.2 (A shortcut) Once we have determined one eigenvector $X_1 = \begin{bmatrix} a \\ b \end{bmatrix}$, the other can be taken to be $\begin{bmatrix} -b \\ a \end{bmatrix}$, as these these vectors are always orthogonal. Also $P = [X_1|X_2]$ will have $\det P = a^2 + b^2 > 0$.

We now apply the above ideas to determine the geometric nature of second degree equations in x and y.

EXAMPLE 7.1.2 Sketch the curve determined by the equation

$$12x^2 - 12xy + 7y^2 + 60x - 38y + 31 = 0.$$

Solution. With P taken to be the proper orthogonal matrix defined in the previous example by

$$P = \begin{bmatrix} 3/\sqrt{13} & 2/\sqrt{13} \\ -2/\sqrt{13} & 3/\sqrt{13} \end{bmatrix},$$

then as theorem 7.1.1 predicts, P is a rotation matrix and the transformation

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = PY = P \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

or more explicitly

$$x = \frac{3x_1 + 2y_1}{\sqrt{13}},\ y = \frac{-2x_1 + 3y_1}{\sqrt{13}}, \tag{7.8}$$

will rotate the x, y axes to positions given by the respective columns of P. (More generally, we can always arrange for the x_1 axis to point either into the first or fourth quadrant.)

Now $A = \begin{bmatrix} 12 & -6 \\ -6 & 7 \end{bmatrix}$ is the matrix of the quadratic form

$$12x^2 - 12xy + 7y^2,$$

so we have, by Theorem 7.1.5

$$12x^2 - 12xy + 7y^2 = 16x_1^2 + 3y_1^2.$$

Then under the rotation $X = PY$, our original quadratic equation becomes

$$16x_1^2 + 3y_1^2 + \frac{60}{\sqrt{13}}(3x_1 + 2y_1) - \frac{38}{\sqrt{13}}(-2x_1 + 3y_1) + 31 = 0,$$

or

$$16x_1^2 + 3y_1^2 + \frac{256}{\sqrt{13}}x_1 + \frac{6}{\sqrt{13}}y_1 + 31 = 0.$$

Now complete the square in x_1 and y_1:

$$16\left(x_1^2 + \frac{16}{\sqrt{13}}x_1\right) + 3\left(y_1^2 + \frac{2}{\sqrt{13}}y_1\right) + 31 = 0,$$

$$\begin{aligned} 16\left(x_1 + \frac{8}{\sqrt{13}}\right)^2 + 3\left(y_1 + \frac{1}{\sqrt{13}}\right)^2 &= 16\left(\frac{8}{\sqrt{13}}\right)^2 + 3\left(\frac{1}{\sqrt{13}}\right)^2 - 31 \\ &= 48. \end{aligned} \tag{7.9}$$

Then if we perform a translation of axes to the new origin $(x_1, y_1) = (-\frac{8}{\sqrt{13}}, -\frac{1}{\sqrt{13}})$:

$$x_2 = x_1 + \frac{8}{\sqrt{13}},\ y_2 = y_1 + \frac{1}{\sqrt{13}},$$

equation 7.9 reduces to

$$16x_2^2 + 3y_2^2 = 48,$$

or

$$\frac{x_2^2}{3} + \frac{y_2^2}{16} = 1.$$

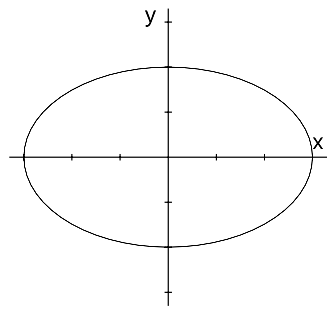

Figure 7.2: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $0<b<a$: an ellipse.

This equation is now in one of the standard forms listed below as Figure 7.2 and is that of a whose centre is at $(x_2,\, y_2) = (0,\, 0)$ and whose axes of symmetry lie along the x_2, y_2 axes. In terms of the original x, y coordinates, we find that the centre is $(x,\, y) = (-2,\, 1)$. Also $Y = P^tX$, so equations 7.8 can be solved to give

$$x_1 = \frac{3x_1 - 2y_1}{\sqrt{13}},\ y_1 = \frac{2x_1 + 3y_1}{\sqrt{13}}.$$

Hence the y_2–axis is given by

$$\begin{aligned} 0 = x_2 &= x_1 + \frac{8}{\sqrt{13}} \\ &= \frac{3x - 2y}{\sqrt{13}} + \frac{8}{\sqrt{13}}, \end{aligned}$$

or $3x - 2y + 8 = 0$. Similarly the x_2 axis is given by $2x + 3y + 1 = 0$.

This ellipse is sketched in Figure 7.1.

Figures 7.2, 7.3, 7.4 and 7.5 are a collection of standard second degree equations: Figure 7.2 is an ellipse; Figures 7.3 are hyperbolas (in both these examples, the asymptotes are the lines $y = \pm\frac{b}{a}x$); Figures 7.4 and 7.5 represent parabolas.

EXAMPLE 7.1.3 Sketch $y^2 - 4x - 10y - 7 = 0$.

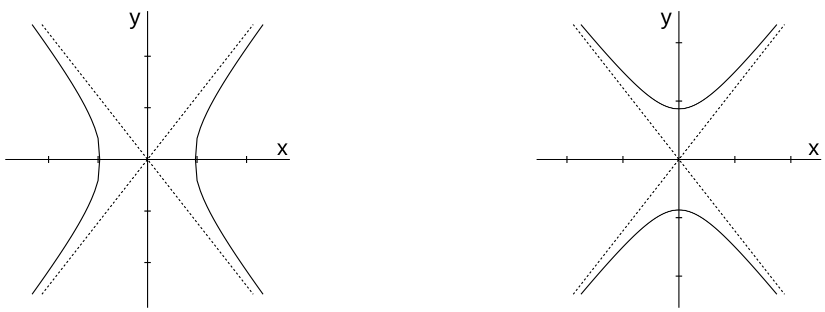

Figure 7.3: (i) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$; (ii) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = -1$, $0 < b$, $0 < a$.

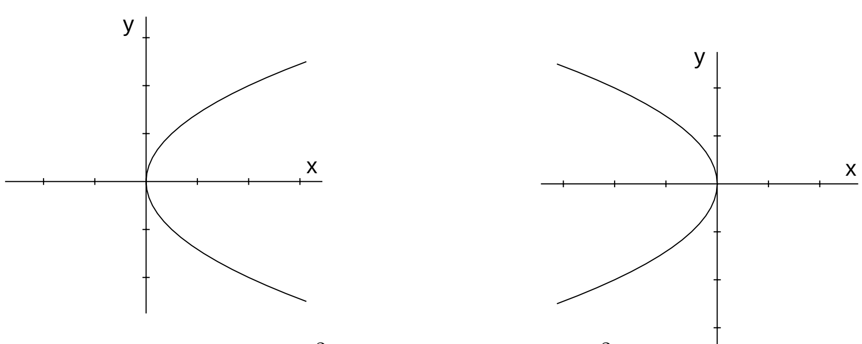

Figure 7.4: (i) $y^2 = 4ax$, $a > 0$; (ii) $y^2 = 4ax$, $a < 0$.

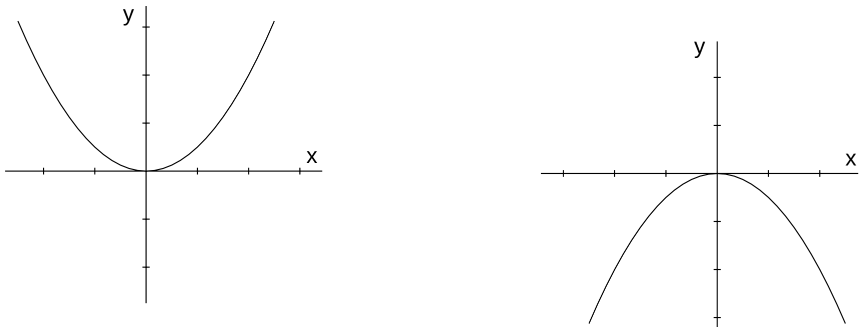

Figure 7.5: (iii) $x^2 = 4ay$, $a > 0$; (iv) $x^2 = 4ay$, $a < 0$.

Solution. Complete the square:

$$\begin{aligned} y^2 - 10y + 25 - 4x - 32 &= 0 \\ (y-5)^2 = 4x + 32 &= 4(x+8), \end{aligned}$$

or $y_1^2 = 4x_1$, under the translation of axes $x_1 = x + 8$, $y_1 = y - 5$. Hence we get a parabola with vertex at the new origin $(x_1, y_1) = (0, 0)$, i.e. $(x, y) = (-8, 5)$.

The parabola is sketched in Figure 7.6.

EXAMPLE 7.1.4 Sketch the curve $x^2 - 4xy + 4y^2 + 5y - 9 = 0$.

Solution. We have $x^2 - 4xy + 4y^2 = X^tAX$, where

$$A = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}.$$

The characteristic equation of A is $\lambda^2 - 5\lambda = 0$, so A has distinct eigenvalues $\lambda_1 = 5$ and $\lambda_2 = 0$. We find corresponding unit length eigenvectors

$$X_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \end{bmatrix},\ X_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Then $P = [X_1|X_2]$ is a proper orthogonal matrix and under the rotation of axes $X = PY$, or

$$\begin{aligned} x &= \frac{x_1 + 2y_1}{\sqrt{5}} \\ y &= \frac{-2x_1 + y_1}{\sqrt{5}}, \end{aligned}$$

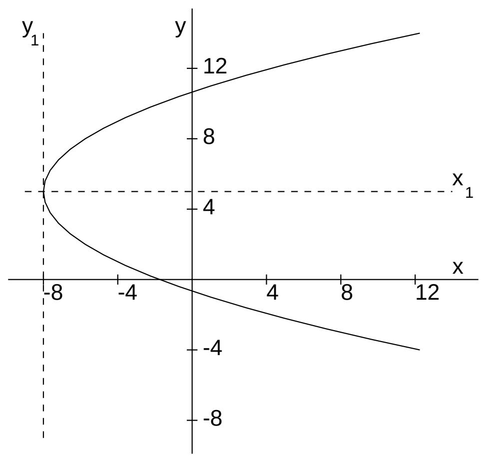

Figure 7.6: $y^2 - 4x - 10y - 7 = 0$.

we have

$$x^2 - 4xy + 4y^2 = \lambda_1 x_1^2 + \lambda_2 y_1^2 = 5x_1^2.$$

The original quadratic equation becomes

$$\begin{aligned} 5x_1^2 + \frac{\sqrt{5}}{\sqrt{5}}(-2x_1 + y_1) - 9 &= 0 \\ 5(x_1^2 - \frac{2}{\sqrt{5}}x_1) + \sqrt{5}y_1 - 9 &= 0 \\ 5(x_1 - \frac{1}{\sqrt{5}})^2 = 10 - \sqrt{5}y_1 &= \sqrt{5}(y_1 - 2\sqrt{5}), \end{aligned}$$

or $5x_2^2 = -\frac{1}{\sqrt{5}}y_2$, where the x_1, y_1 axes have been translated to x_2, y_2 axes using the transformation

$$x_2 = x_1 - \frac{1}{\sqrt{5}}, \quad y_2 = y_1 - 2\sqrt{5}.$$

Hence the vertex of the parabola is at $(x_2, y_2) = (0, 0)$, i.e. $(x_1, y_1) = (\frac{1}{\sqrt{5}}, 2\sqrt{5})$, or $(x, y) = (\frac{21}{5}, \frac{8}{5})$. The axis of symmetry of the parabola is the line $x_2 = 0$, i.e. $x_1 = 1/\sqrt{5}$. Using the rotation equations in the form

$$x_1 = \frac{x - 2y}{\sqrt{5}}$$

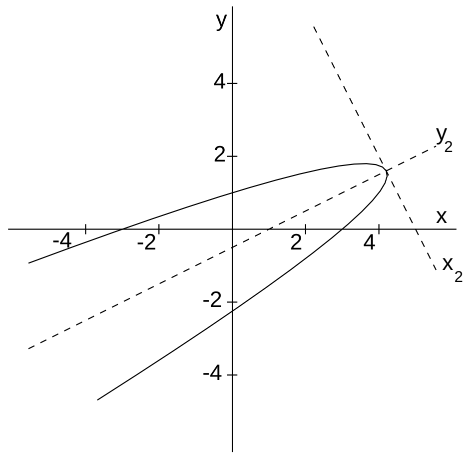

Figure 7.7: $x^2 - 4xy + 4y^2 + 5y - 9 = 0$.

$$y_1 = \frac{2x + y}{\sqrt{5}},$$

we have

$$\frac{x - 2y}{\sqrt{5}} = \frac{1}{\sqrt{5}}, \quad \text{or} \quad x - 2y = 1.$$

The parabola is sketched in Figure 7.7.

7.2 A classification algorithm

There are several possible degenerate cases that can arise from the general second degree equation. For example $x^2 + y^2 = 0$ represents the point $(0,\ 0)$; $x^2 + y^2 = -1$ defines the empty set, as does $x^2 = -1$ or $y^2 = -1$; $x^2 = 0$ defines the line $x = 0$; $(x + y)^2 = 0$ defines the line $x + y = 0$; $x^2 - y^2 = 0$ defines the lines $x - y = 0$, $x + y = 0$; $x^2 = 1$ defines the parallel lines $x = \pm 1$; $(x + y)^2 = 1$ likewise defines two parallel lines $x + y = \pm 1$.

We state without proof a complete classification [1] of the various cases

[1] This classification forms the basis of a computer program which was used to produce the diagrams in this chapter. I am grateful to Peter Adams for his programming assistance.

that can possibly arise for the general second degree equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \tag{7.10}$$

It turns out to be more convenient to first perform a suitable translation of axes, before rotating the axes. Let

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}, \quad C = ab - h^2,\ A = bc - f^2,\ B = ca - g^2.$$

If $C \neq 0$, let

$$\alpha = \frac{-\begin{vmatrix} g & h \\ f & b \end{vmatrix}}{C}, \qquad \beta = \frac{-\begin{vmatrix} a & g \\ h & f \end{vmatrix}}{C}. \tag{7.11}$$

CASE 1. $\Delta = 0$.

(1.1) $C \neq 0$. Translate axes to the new origin (α, β), where α and β are given by equations 7.11:

$$x = x_1 + \alpha, \quad y = y_1 + \beta.$$

Then equation 7.10 reduces to

$$ax_1^2 + 2hx_1y_1 + by_1^2 = 0.$$

(a) $C > 0$: **Single point** $(x, y) = (\alpha, \beta)$.

(b) $C < 0$: **Two non–parallel lines** intersecting in $(x, y) = (\alpha, \beta)$. The lines are

$$\begin{array}{ccc} \dfrac{y-\beta}{x-\alpha} & = & \dfrac{-h \pm \sqrt{-C}}{b} \quad \text{if } b \neq 0, \\ x = \alpha & \text{and} & \dfrac{y-\beta}{x-\alpha} = -\dfrac{a}{2h}, \quad \text{if } b = 0. \end{array}$$

(1.2) $C = 0$.

(a) $h = 0$.

(i) $a = g = 0$.

(A) $A > 0$: **Empty set**.

(B) $A = 0$: **Single line** $y = -f/b$.

(C) $A < 0$: **Two parallel lines**

$$y = \frac{-f \pm \sqrt{-A}}{b}$$

(ii) $b = f = 0$.

(A) $B > 0$: **Empty set**.

(B) $B = 0$: **Single line** $x = -g/a$.

(C) $B < 0$: **Two parallel lines**

$$x = \frac{-g \pm \sqrt{-B}}{a}$$

(b) $h \neq 0$.

(i) $B > 0$: **Empty set**.

(ii) $B = 0$: **Single line** $ax + hy = -g$.

(iii) $B < 0$: **Two parallel lines**

$$ax + hy = -g \pm \sqrt{-B}.$$

CASE 2. $\Delta \neq 0$.

(2.1) $C \neq 0$. Translate axes to the new origin (α, β), where α and β are given by equations 7.11:

$$x = x_1 + \alpha, \quad y = y_1 + \beta.$$

Equation 7.10 becomes

$$ax_1^2 + 2hx_1y_1 + by_1^2 = -\frac{\Delta}{C}. \tag{7.12}$$

CASE 2.1(i) $h = 0$. Equation 7.12 becomes $ax_1^2 + by_1^2 = \frac{-\Delta}{C}$.

(a) $C < 0$: **Hyperbola**.

(b) $C > 0$ and $a\Delta > 0$: **Empty set**.

(c) $C > 0$ and $a\Delta < 0$.

(i) $a = b$: **Circle**, centre (α, β), radius $\sqrt{\frac{g^2+f^2-ac}{a}}$.

(ii) $a \neq b$: **Ellipse**.

CASE 2.1(ii) $h \neq 0$.

Rotate the (x_1, y_1) axes with the new positive x_2–axis in the direction of

$$[(b-a+R)/2, -h],$$

where $R = \sqrt{(a-b)^2 + 4h^2}$.

Then equation 7.12 becomes

$$\lambda_1 x_2^2 + \lambda_2 y_2^2 = -\frac{\Delta}{C}. \tag{7.13}$$

where

$$\lambda_1 = (a+b-R)/2, \ \lambda_2 = (a+b+R)/2,$$

Here $\lambda_1\lambda_2 = C$.

(a) $C < 0$: **Hyperbola**.
Here $\lambda_2 > 0 > \lambda_1$ and equation 7.13 becomes

$$\frac{x_2^2}{u^2} - \frac{y_2^2}{v^2} = \frac{-\Delta}{|\Delta|},$$

where

$$u = \sqrt{\frac{|\Delta|}{C\lambda_1}}, \ v = \sqrt{\frac{|\Delta|}{-C\lambda_2}}.$$

(b) $C > 0$ and $a\Delta > 0$: **Empty set**.

(c) $C > 0$ and $a\Delta < 0$: **Ellipse**.
Here $\lambda_1, \lambda_2, a, b$ have the same sign and $\lambda_1 \neq \lambda_2$ and equation 7.13 becomes

$$\frac{x_2^2}{u^2} + \frac{y_2^2}{v^2} = 1,$$

where

$$u = \sqrt{\frac{\Delta}{-C\lambda_1}}, \ v = \sqrt{\frac{\Delta}{-C\lambda_2}}.$$

(2.1) $C = 0$.

(a) $h = 0$.

(i) $a = 0$: Then $b \neq 0$ and $g \neq 0$. **Parabola** with vertex

$$\left(\frac{-A}{2gb}, -\frac{f}{b}\right).$$

Translate axes to $(x_1,\, y_1)$ axes:

$$y_1^2 = -\frac{2g}{b}x_1.$$

(ii) $b = 0$: Then $a \neq 0$ and $f \neq 0$. **Parabola** with vertex

$$\left(-\frac{g}{a},\, \frac{-B}{2fa}\right).$$

Translate axes to $(x_1,\, y_1)$ axes:

$$x_1^2 = -\frac{2f}{a}y_1.$$

(b) $h \neq 0$: **Parabola**. Let

$$k = \frac{ga + bf}{a + b}.$$

The vertex of the parabola is

$$\left(\frac{(2akf - hk^2 - hac)}{d},\, \frac{a(k^2 + ac - 2kg)}{d}\right).$$

Now translate to the vertex as the new origin, then rotate to $(x_2,\, y_2)$ axes with the positive x_2–axis along $[sa,\, -sh]$, where $s = \operatorname{sign}(a)$.

(The positive x_2–axis points into the first or fourth quadrant.) Then the parabola has equation

$$x_2^2 = \frac{-2st}{\sqrt{a^2 + h^2}}y_2,$$

where $t = (af - gh)/(a + b)$.

REMARK 7.2.1 If $\Delta = 0$, it is not necessary to rotate the axes. Instead it is always possible to translate the axes suitably so that the coefficients of the terms of the first degree vanish.

EXAMPLE 7.2.1 Identify the curve

$$2x^2 + xy - y^2 + 6y - 8 = 0. \qquad (7.14)$$

Solution. Here

$$\Delta = \begin{vmatrix} 2 & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 & 3 \\ 0 & 3 & -8 \end{vmatrix} = 0.$$

Let $x = x_1 + \alpha$, $y = y_1 + \beta$ and substitute in equation 7.14 to get

$$2(x_1 + \alpha)^2 + (x_1 + \alpha)(y_1 + \beta) - (y_1 + \beta)^2 + 4(y_1 + \beta) - 8 = 0. \tag{7.15}$$

Then equating the coefficients of x_1 and y_1 to 0 gives

$$\begin{aligned} 4\alpha + \beta &= 0 \\ \alpha + 2\beta + 4 &= 0, \end{aligned}$$

which has the unique solution $\alpha = -\frac{2}{3}$, $\beta = \frac{8}{3}$. Then equation 7.15 simplifies to

$$2x_1^2 + x_1y_1 - y_1^2 = 0 = (2x_1 - y_1)(x_1 + y_1),$$

so relative to the x_1, y_1 coordinates, equation 7.14 describes two lines: $2x_1 - y_1 = 0$ or $x_1 + y_1 = 0$. In terms of the original x, y coordinates, these lines become $2(x + \frac{2}{3}) - (y - \frac{8}{3}) = 0$ and $(x + \frac{2}{3}) + (y - \frac{8}{3}) = 0$, i.e. $2x - y + 4 = 0$ and $x + y - 2 = 0$, which intersect in the point

$$(x,\, y) = (\alpha,\, \beta) = (-\frac{2}{3},\, \frac{8}{3}).$$

EXAMPLE 7.2.2 Identify the curve

$$x^2 + 2xy + y^2 + +2x + 2y + 1 = 0. \tag{7.16}$$

Solution. Here

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

Let $x = x_1 + \alpha$, $y = y_1 + \beta$ and substitute in equation 7.16 to get

$$(x_1+\alpha)^2+2(x_1+\alpha)(y_1+\beta)+(y_1+\beta)^2+2(x_1+\alpha)+2(y_1+\beta)+1 = 0. \tag{7.17}$$

Then equating the coefficients of x_1 and y_1 to 0 gives the same equation

$$2\alpha + 2\beta + 2 = 0.$$

Take $\alpha = 0$, $\beta = -1$. Then equation 7.17 simplifies to

$$x_1^2 + 2x_1y_1 + y_1^2 = 0 = (x_1 + y_1)^2,$$

and in terms of x, y coordinates, equation 7.16 becomes

$$(x + y + 1)^2 = 0, \text{ or } x + y + 1 = 0.$$

7.3 PROBLEMS

1. Sketch the curves

   (i) $x^2 - 8x + 8y + 8 = 0$;

   (ii) $y^2 - 12x + 2y + 25 = 0$.

2. Sketch the hyperbola
$$4xy - 3y^2 = 8$$
   and find the equations of the asymptotes.

   [Answer: $y = 0$ and $y = \frac{4}{3}x$.]

3. Sketch the ellipse
$$8x^2 - 4xy + 5y^2 = 36$$
   and find the equations of the axes of symmetry.

   [Answer: $y = 2x$ and $x = -2y$.]

4. Sketch the conics defined by the following equations. Find the centre when the conic is an ellipse or hyperbola, asymptotes if an hyperbola, the vertex and axis of symmetry if a parabola:

   (i) $4x^2 - 9y^2 - 24x - 36y - 36 = 0$;

   (ii) $5x^2 - 4xy + 8y^2 + 4\sqrt{5}x - 16\sqrt{5}y + 4 = 0$;

   (iii) $4x^2 + y^2 - 4xy - 10y - 19 = 0$;

   (iv) $77x^2 + 78xy - 27y^2 + 70x - 30y + 29 = 0$.

   [Answers: (i) hyperbola, centre $(3, -2)$, asymptotes $2x - 3y - 12 = 0$, $2x + 3y = 0$;
   (ii) ellipse, centre $(0, \sqrt{5})$;
   (iii) parabola, vertex $(-\frac{7}{5}, -\frac{9}{5})$, axis of symmetry $2x - y + 1 = 0$;
   (iv) hyperbola, centre $(-\frac{1}{10}, \frac{7}{10})$, asymptotes $7x + 9y + 7 = 0$ and $11x - 3y - 1 = 0$.]

5. Identify the lines determined by the equations:

   (i) $2x^2 + y^2 + 3xy - 5x - 4y + 3 = 0$;

(ii) $9x^2 + y^2 - 6xy + 6x - 2y + 1 = 0$;

(iii) $x^2 + 4xy + 4y^2 - x - 2y - 2 = 0$.

[Answers: (i) $2x + y - 3 = 0$ and $x + y - 1 = 0$; (ii) $3x - y + 1 = 0$; (iii) $x + 2y + 1 = 0$ and $x + 2y - 2 = 0$.]