Top dif. 30/9/2020

Def: Pt. crítico de $f: M \to \mathbb{R}$, $x \in M$, $df(x) = 0$.

Def: Función de Morse es cuando todos los p.c. so "no degenerados".

E.g. $f: \mathbb{R} \to \mathbb{R}$, $f'(0) = 0$, $f''(0) = \begin{cases} >0 & \text{mín} \\ <0 & \text{máx} \\ 0 & \text{"degenerado"} \end{cases}$] no degenerado

$f(x) = ax^2 + bx^3 + \dots$

$f'(0) = 0$

$\boxed{\begin{array}{cc} \cup & \cap \\ a>0 & a<0 \\ \frac{1}{2}f''(0) & \end{array}}$? $a=0$

Lemma de Morse:

si $f'(0) = 0$, $f''(0) \neq 0$

ent. la "situación es así"

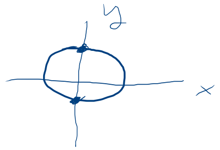
O sea, hay una coordenada $\varphi: \overset{x}{(-\varepsilon, \varepsilon)} \to \overset{u}{(-\delta, \delta)}$ (difeo)

t.q. $f(u) = (f \circ \varphi^{-1})(u) = \begin{cases} u^2 & f''(0) > 0 \\ -u^2 & f''(0) < 0 \end{cases}$

Eg: $f(x)$, $f'(x) = \Pi(x - x_i)$, sin raíces mult. $x_i \neq x_j$, $i \neq j$.

Eg. $M = S^1 = \{x^2 + y^2 = 1\}$

$f = y|_{S^1}$.



p.c. $(0, \pm 1)$

$df = \sum \frac{\partial f}{\partial x_i} dx_i$

¿Porqué son los únicos?

(A) $df = 0 \iff \frac{\partial f}{\partial \theta} = 0$, $x = \cos\theta$, $y = \sin\theta$

$\iff \cos\theta = 0$, $\theta = \frac{\pi}{2} + n\pi$, $n \in \mathbb{Z}$.

$\Rightarrow (x, y) = (0, \pm 1)$,

(B) $\underbrace{x^2 + y^2}_{F} = 1$ $S^1 = F^{-1}(1)$, $f|_{F^{-1}(1)}$

$2x\,dx + 2y\,dy = 0$

$dy = 0 \Rightarrow 2x\,dx = 0 \underset{dx \neq 0}{\Longrightarrow} x = 0 \Rightarrow (x, y) = (0, \pm 1)$.

(C) $y: \mathbb{R}^2 \to \mathbb{R}$. $f = y|_{S^1}$. $df(m) = dy|_{T_m S^1}$.

$\Rightarrow df(m) = \lambda\, dF(m)$

$T_m S^1 = \ker[dF(m)]$

dy, $dF = x\,dx + y\,dy$.

buscamos puntos $(x, y) \in S^1$ t.q. $dy = \lambda(x\,dx + y\,dy)$

$\Rightarrow (x, y) = (0, \pm 1)$

$f : S^n \to \mathbb{R}$, $\quad f = x_{n+1}\big|_{S^n}$; p.c. $(0,\dots,0,\pm 1)$

$$S^n = \left\{ \sum_{1}^{n+1} x_i^2 = 1 \right\}$$

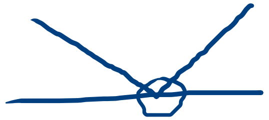
$$x_{n+1} = \pm\sqrt{1-(x_1^2+\dots+x_n^2)}$$

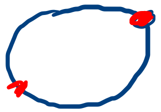
<u>no deg:</u> $d^2 f = \left(\frac{\partial^2 f}{\partial x_i x_j}\right)$ es nosingular

$$\Rightarrow f = f(0) + \sum_{1}^{k} x_i^2 - \sum_{k+1}^{n} u_i^2 ,$$

$n-k =$ ind. f del p.c.

E.g. $M \subset \mathbb{R}^n$. $p \in \mathbb{R}^n - M$. $f = \mathrm{dist}(\cdot, p)\big|_M$





Morse-Bott '50

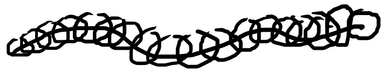
top. de var. de bandera.

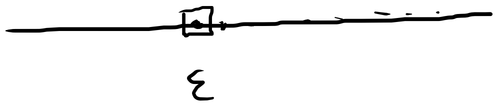
3 mnts Receso $\Rightarrow$ 1:48.

3. Suppose that Z is a submanifold of X with $\dim Z < \dim X$. Prove that Z has measure zero in X (without using Sard!).

medida 0? $\quad Z \subset X^n$ es de "medida 0"

si $\forall z \in Z$, $z \in U \underset{ab}{\subset} X$, $\varphi : U \to V \underset{ab}{\subset} \mathbb{R}^n$, $\varphi(U \cap Z)$ es "de medida 0" en $\mathbb{R}^n$.





$\mathbb{R} \subset \mathbb{R}^2$
eje de x

Teo Sard: $f : M \to N$ el conj de valores no reg. es de med. 0 ("casi" todos los valores son regulares).

$i : Z \to X$, $x \in X$ es valor irregular ssi $x \in Z \overset{}{\underset{\text{Sard}}{\Longrightarrow}} \operatorname{med}(Z) = 0$.

Top. Dif. 2 oct.

13. Show that the determinant function on $M(n)$ is Morse if $n = 2$, but not if $n > 2$.

mat $n \times n$.

$f : \mathbb{R}^{n^2} \longrightarrow \mathbb{R}$ det

si $df(A) = 0$ $\Rightarrow d^2 f(A) = \left(\frac{\partial^2 f}{\partial A_{ij} A_{kl}}\right)$ es H_f no deg.

$\underline{n=2}$: $f\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = x_{11}x_{22} - x_{12}x_{21}$

$f(\lambda A) = \lambda^n f(A)$

Ej $\Rightarrow$ los puntos crit. de det son las mat singulares.

$\begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}$ $d^2 f(A) = (?)$ $\boxed{HA}$ det $= 0$

$\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$ det $= 1$

$x_{11}, x_{12}, x_{21}, x_{22}$

$[H_f(x)]_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}(x)$ $f : \mathbb{R}^n \to \mathbb{R}$

Serie de Taylor en x:

$f(x+h) = f(x) + \sum a_i h_i + \boxed{\sum_{ij} b_{ij} h_i h_j}$ $h = (h_1, h_2, \ldots)$ "pequeño"

" " $+ \sum_{i,j,k} c_{ijk} h_i h_j h_k + \ldots$ $b_{ij} = b_{ji}$ $B = (b_{ij})$

$a_i = \frac{\partial f}{\partial x_i}(x)$, $b_{ij} = \frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}(x)$, etc.

$b_{ij} = b_{ji}$ $x_1^2, x_1 x_2, \ldots$ $x_2 x_1$

$\underline{x=0}$ ~~$f(h_1, h$~~ $f(x) = x_1 x_2 + (x_1)^3$

$H_f(0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\left(\frac{1}{2}\right)$?

$f = \sum_{ij} b_{ij} x_i x_j$

$H_f(0) = \begin{pmatrix} 2b_{11} & 2b_{12} \\ & \end{pmatrix} = 2B$ $\frac{\partial f}{\partial x_1 \partial x_2}$

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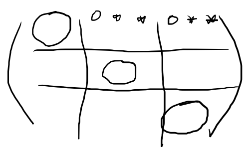
sea $A \in \mathrm{Mat}_{3\times 3}(\mathbb{R})$

$\det(A) = 0. \Rightarrow df_A = 0$, $f = \det$. (E.g. $A = 0$)

$\underline{\text{P.D.}}$ $H_f(A)$ es singular.

$A = \begin{pmatrix} x_{11} & x_{12} & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$, $f(A) = x_{11} x_{22} x_{33} \pm$ muchos otros,

$H_f(A) = \begin{pmatrix} 0 & 0 & 0 & 0 & x_{33} & x_{32} & 0 & -x_{23} & x_{22} \\ & & & & & & & & \end{pmatrix}$



$x_{11} \underbrace{x_{22}}_{5} \underbrace{x_{33}}_{9} - x_{11} \underbrace{x_{23}}_{6} \underbrace{x_{32}}_{8}$

$H_f(A) = ?$ $\qquad f(A+tH)$

$$f(x+h) = f(x) + \sum a_i h_i + \sum b_{ij} h_i h_j + \cdots$$

$$\underbrace{f(x+th)} = f(x) + t\left(\sum a_i h_i\right) + t^2 \underbrace{\left(\sum b_{ij} h_i h_j\right)} + \cdots$$

$$\left(\frac{\partial}{\partial t}\right)^2\Big|_{t=0} f(x+th)$$

$$\det(A+tH) = \det(A) + t(?)$$

Obs: hemos demostrado que para $n \geq 3$,

$A = 0$ es un punto criticado degenerado de det,

porque $H_{\det}(0) = 0$.

O sea, si $f(x_1, \ldots, x_N)$ es pol homog. de grado ≥ 3

ent. $x = 0$ es un p.c. deg. de f ya que $H_f(0) = 0$

la mat. 0

Dem: → homog. de grado $N-2$

Pregunta: Que tipo de cosa es $\boxed{\det^{-1}(0)}$?

~~Va~~ Resp: es una variedad "estratificada"