8. What is the tangent space to the paraboloid defined by $x^2 + y^2 - z^2 = a$ at $(\sqrt{a}, 0, 0)$, where $(a > 0)$?

<u>Prop</u>: Sea $f: \mathbb{R}^3 \to \mathbb{R}$ (e.g. $f(x,y,z) = x^2 + y^2 - z^2$) suave, y $a \in \mathbb{R}$ un valor regular (i.e. $df = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \neq 0 \ \forall (x,y,z) \in f^{-1}(a)$).

Ent.

M ① $f^{-1}(a) \subset \mathbb{R}^3$ es una var. de dim $= 2$.

② $\forall \underset{m}{(x,y,z)} \in f^{-1}(a)$, $T_{(x,y,z)} M = [d f(x,y,z)]^{\perp}$

$\gamma(0) = m$, $\gamma(t) \in M$, $\forall t \Rightarrow f(\gamma(t)) = a \Rightarrow df_m \dot{\gamma} = 0$

$$f = x^2 + y^2 - z^2$$

① $df = (2x, 2y, -2z) = 0$

ssi $(x, y, z) = 0$; pero $f(0, 0, 0) = 0$

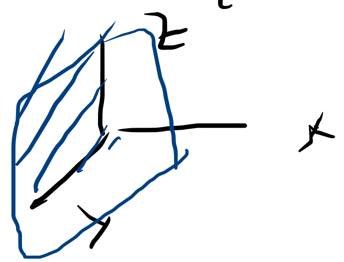
$\Rightarrow$ $H_a \neq 0$ es regular.

② Sea $m = (x_0, y_0, z_0) \in M = f^{-1}(a)$, $a \neq 0$.

$$\Rightarrow T_m M = (x_0, y_0, -z_0)^{\perp} = \{(x, y, z) \in \mathbb{R}^3 \mid x_0 x + y_0 y - z_0 z = 0\}$$

E.g. $m = (\sqrt{a}, 0, 0) \Rightarrow T_m M = \{\sqrt{a}\, x = 0\} = \{x = 0\}$

= "el plano de coord yz"



***9.** (a) Let $x_1, \dots, x_N$ be the standard coordinate functions on $\mathbf{R}^N$, and let X be a k-dimensional submanifold of $\mathbf{R}^N$. Prove that every point $x \in X$ has a neighborhood on which the restrictions of some k-coordinate functions $x_{i_1}, \dots, x_{i_k}$ form a local coordinate system. [HINT: Let $e_1, \dots, e_N$ be the usual basis for $\mathbf{R}^N$. As a linear algebra lemma, prove that the projection of $T_x(X)$ onto the subspace spanned by $e_{i_1}, \dots, e_{i_k}$ is bijective for some choice of $i_1, \dots, i_k$. Show that this implies that $(x_{i_1}, \dots, x_{i_k})$ defines a local diffeomorphism of X into $\mathbf{R}^k$ at the point x.]

(b) For simplicity, assume that $x_1, \dots, x_k$ form a local coordinate system on a neighborhood V of x in X. Prove that there are smooth functions $g_{k+1}, \dots, g_N$ on an open set U in $\mathbf{R}^k$ such that V may be taken to be the set

$$\{(a_1, \dots, a_k, g_{k+1}(a), \dots, g_N(a)) \in R^N : \ a = (a_1, \dots, a_k) \in U\}.$$

That is, if we define $g : U \longrightarrow \mathbf{R}^{N-k}$ by $g = (g_{k+1}, \dots, g_N)$, then V equals the graph of g. Thus every manifold is locally expressible as a graph.

① $V \subset \mathbb{R}^N \Rightarrow i_1, \dots, i_k$ t.q

dim k

$$\mathbb{R}^n \ni (x_1, \dots, x_n) \xmapsto{\pi} (x_{i_1}, \dots, x_{i_k}) \in \mathbb{R}^k$$

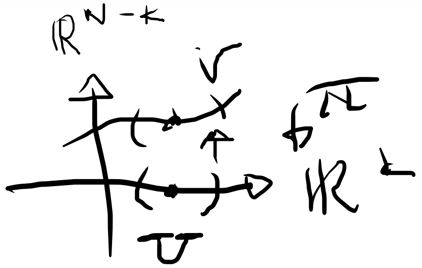
rest. a V es isomorfismo.

② $\Rightarrow$ $\pi|_X : X \to \mathbb{R}^k$ es difeo en una vec. de x.

X

$(\pi|_V) : V \to U \subset \mathbb{R}^k$

$g = (\ )^{-1}$



Dem de ②: basta ver que $d(\pi|_X)_x$ es iso. lin.

(si es iso lin para x, tambien lo es en a vec. de x)

$\underset{\text{TFI}}{\Longrightarrow}$ es difeo en una vec. de x

Luego, π es lineal $\pi = d\pi_x : \mathbb{R}^N \to \mathbb{R}^k \Rightarrow d\pi_x|_{T_xX} = d(\pi|_X)_x$

(RC)

$$M = \begin{pmatrix} & k \times n & \end{pmatrix} \begin{matrix} \leftarrow v_1 \\ \leftarrow v_2 \\ \vdots \\ v_k \end{matrix} \qquad V = \mathrm{sp}\{v_1, \dots, v_k\}$$

$\Longrightarrow$ rango de filas $= k$

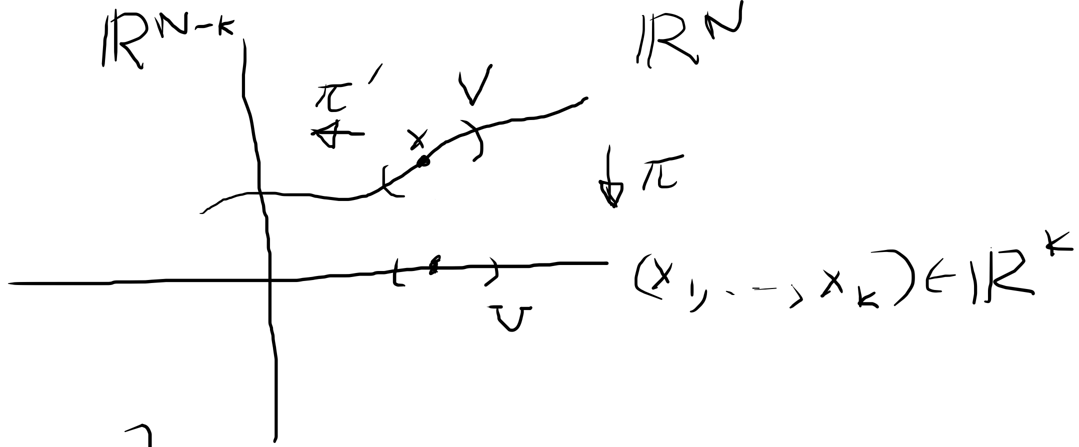
$\shortparallel$

rango de columnas

$\Longrightarrow$ $\exists$ k columnas l.i., con índices $i_1, \dots, i_k$.

$\Longrightarrow$ $(x_1, \dots, x_n) \mapsto (x_{i_1}, \dots, x_{i_k})$, rest. a V, es iso.

9b



$$g = (g_{k+1}, \dots, g_N) = ?$$

***9.** (a) Let $x_1, \dots, x_N$ be the standard coordinate functions on $\mathbf{R}^N$, and let X be a k-dimensional submanifold of $\mathbf{R}^N$. Prove that every point $x \in X$ has a neighborhood on which the restrictions of some k-coordinate functions $x_{i_1}, \dots, x_{i_k}$ form a local coordinate system. [HINT: Let $e_1, \dots, e_N$ be the usual basis for $\mathbf{R}^N$. As a linear algebra lemma, prove that the projection of $T_x(X)$ onto the subspace spanned by $e_{i_1}, \dots, e_{i_k}$ is bijective for some choice of $i_1, \dots, i_k$. Show that this implies that $(x_{i_1}, \dots, x_{i_k})$ defines a local diffeomorphism of X into $\mathbf{R}^k$ at the point x.]

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That is, if we define $g : U \longrightarrow \mathbf{R}^{N-k}$ by $g = (g_{k+1}, \dots, g_N)$, then V equals the graph of g. Thus every manifold is locally expressible as a graph.

(a) $\Rightarrow \pi|_V : V \to U$ es difeo.

$\Rightarrow (\pi|_V)^{-1}$ existe y es suave.

$U \to V$.

$\underbrace{i_V \circ (\pi|_V)^{-1}} : U \to \mathbb{R}^N$, $i_V : V \to \mathbb{R}^N$ la inclusión.

$(a_1, \dots, a_k) \mapsto (a_1, \dots, a_k, \overbrace{g_{k+1}(a), \dots, g_N(a)}^{g(a)}) \overset{\pi'}{\longmapsto} g(a)$

$$\boxed{g := \pi' \circ i_V \circ (\pi|_V)^{-1}}$$

$$\begin{tikzcd}
\mathbb{R}^{N-k} & \mathbb{R}^N \arrow[l, "\pi'"'] & V \arrow[l] \arrow[r, shift left, "\pi|_V"] & U \arrow[l, shift left] \arrow[lll, bend left, "g"]
\end{tikzcd}$$

$\mathrm{graph}(g) \subset U \times \mathbb{R}^{N-k} \subset \mathbb{R}^k \times \mathbb{R}^{N-k} = \mathbb{R}^N$