Forward

Complex analysis is a nexus for many mathematical fields, including:

1. Algebra (theory of fields and equations);
2. Algebraic geometry and complex manifolds;
3. Theory of Riemann surfaces;
4. Topology (notion of a manifold; surfaces, covering spaces, mapping–class groups; Schoenflies theorem);
5. Geometry (Platonic solids; flat tori; hyperbolic manifolds of dimensions two and three);
6. Lie groups, discrete subgroups and homogeneous spaces (e.g. $\mathbb{H}/\operatorname{SL}_2(\mathbb{Z})$;
7. Dynamics (iterated rational maps, Kleinian groups);
8. Number theory and automorphic forms (elliptic functions, zeta functions);
9. Several complex variables;
10. Real analysis and PDE (harmonic functions, elliptic equations and distributions).

This course covers some basic material on both the geometric and analytic aspects of complex analysis in one variable.

Prerequisites: Background in real analysis and basic differential topology (such as covering spaces and differential forms), and a first course in complex analysis.

Exercises

(These exercises are review.)

1. Let $T \subset \mathbb{R}^3$ be the spherical triangle defined by $x^2 + y^2 + z^2 = 1$ and $x, y, z \geq 0$. Let $\alpha = z\,dx\,dz$.

   (a) Find a smooth 1-form β on $\mathbb{R}^3$ such that $\alpha = d\beta$.

(b) Define consistent orientations for T and ∂T.

(c) Using your choices in (ii), compute $\int_T \alpha$ and $\int_{\partial T} \beta$ directly, and check that they agree. (Why should they agree?)

*2. Let $f(z) = (az + b)/(cz + d)$ be a Möbius transformation. Show the number of rational maps $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that

$$g(g(g(g(g(z))))) = f(z)$$

is 1, 5 or ∞. Explain how to determine which alternative holds for a given f.

3. Let $\sum a_n z^n$ be the Taylor series for $\tanh(z)$ at $z = 0$.

   (a) What is the radius of convergence of this power series?

   (b) Show that $a_5 = 2/15$.

   (c) Give an explicit value of N such that $\tanh(1)$ and $\sum_0^N a_n$ agree to within $\epsilon = 10^{-1000}$. Justify your answer.

4. Let $f : U \to V$ be a proper local homeomorphism between a pair of connected open sets $U, V \subset \mathbb{C}$. Prove that f is a covering map. (Here *proper* means that $f^{-1}(K)$ is compact whenever $K \subset V$ is compact.)

*5. Let $f : \mathbb{C} \to \mathbb{C}$ be given by a polynomial of degree 2 or more. Let

$$V_1 = \{f(z) \ : \ f'(z) = 0\} \subset \mathbb{C}$$

be the set of critical values of f, let $V_0 = f^{-1}(V_1)$, and let $U_i = \mathbb{C} - V_i$ for $i = 0, 1$. Prove that $f : U_0 \to U_1$ is a covering map.

6. Give an example where U_0/U_1 is a normal (or Galois) covering, i.e. where $f_*(\pi_1(U_0))$ is a normal subgroup of $\pi_1(U_1)$.

7. Let $H = \mathbb{R}[i, j, k] \cong \mathbb{R}^4$ denote the real quaternions, with $ij = k$ and $i^2 = j^2 = k^2 = -1$. True or false: For any nonzero $x \in H$, the transformation of $\mathbb{R}^4$ defined by $f(y) = xy$ is conformal.

   (A conformal map is the composition of an isometry of $\mathbb{R}^4$ with multiplication by a real number.)