ON THE ISOMETRIC CONJECTURE OF BANACH

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ABSTRACT. We show that a real Banach space of dimension $N = 4k+2 \geq 6$, $N \neq 134$, all of whose codimension 1 subspaces are isometrically isomorphic to each other, is a Hilbert space. This gives a partial answer to a conjecture of Stefan Banach from 1932.

CONTENTS

1. INTRODUCTION

1.1. The main theorem. S. Banach asked in 1932 the following question:

Let X be a Banach space, real or complex, finite or infinite dimensional, all of whose n-dimensional subspaces, for some fixed integer n, $2 \leq n < \dim(X)$, are isometrically isomorphic to each other. Is it true that X is a Hilbert space? (See [Ba], remarks on Chap. XII, property (5), p. 244.)

The question[1] has been answered affirmatively in the following cases. In 1935, Auerbach, Mazur and Ulam [AMU] gave a positive answer in case V is a real 3-dimensional Banach space and $n = 2$. In 1959, A. Dvoretzky [Dv] proved a theorem, from which follows an affirmative answer for all real infinite dimensional V and $n \geq 2$. Dvoretzky's theorem was extended in 1971 to the complex case by V. Milman [Mi]. In 1967, M. Gromov [Gr] gave an affirmative answer in case V is finite dimensional, real or complex, except when n is odd

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[1]Following a long established tradition starting with [Gr], we rename Banach's question a 'conjecture' in this article, although Banach himself, as far as we know, did not conjecture a positive answer to his question.

and $\dim(V) = n+1$ in the real case, or n is odd and $n < \dim(V) < 2n$ in the complex case. A recent, and very thorough, account of the history behind this conjecture can be read in section 6, p. 388, of [So]. This article also discusses many related problems in convex geometry. It is also worthwhile to see [Pe] and the notes of Section 9 of [MMO], p. 206.

For a finite dimensional real Banach space V, by considering the closed unit ball $B = \{\|x\| \leq 1\} \subset V$, since a finite dimensional Banach space is a Hilbert space if and only if its unit ball is an ellipsoid, Banach's question can be reformulated as follows:

Let $B \subset \mathbb{R}^N$ be a symmetric convex body, all of whose sections by n-dimensional subspaces, for some fixed integer n, $2 \leq n < N$, are linearly equivalent. Is it true that B is an ellipsoid?

We give an affirmative answer to 'one half' of the remaining cases of this question, as follows.

Theorem 1.1 (Main theorem)**.** *Let $B \subset \mathbb{R}^{n+1}$, $n = 4k+1 \geq 5$, $n \neq 133$, be a convex symmetric body, all of whose sections by n-dimensional subspaces are linearly equivalent. Then B is an ellipsoid.*

Remark 1.2. The reason for the strange exception $n \neq 133$ will become clearer during the proof (133 is the dimension of the exceptional Lie group E_7).

In fact, using Theorem 1 of [Mo1], one can drop the symmetry assumption on B in Theorem 1 above, obtaining:

Our main convex geometry theorem. *Let $B \subset \mathbb{R}^{n+1}$, $n = 4k+1 \geq 5$, $n \neq 133$, be a convex body, all of whose sections by n-dimensional affine subspaces through a fixed interior point are affinely equivalent. Then B is an ellipsoid.*

1.2. **Sketch of the proof of the main theorem.** Our proof of Theorem 1.1 combines two main ingredients: convex geometry and algebraic topology. To describe these, we need to recall first some standard definitions.

A *symmetric convex body* is a compact convex subset of a finite dimensional real vector space with a nonempty interior, invariant under $x \mapsto -x$. A *hyperplane* is a codimension 1 linear subspace. An *affine hyperplane* is the translation of a hyperplane by some vector. A *hyperplane section* of a subset in a vector space is its intersection with a hyperplane. Two sets, each a subset of a vector space, are *linearly* (respectively, *affinely*) *equivalent* if they can be mapped to each other by a linear (respectively, affine) isomorphism between their ambient vector spaces. An *ellipsoid* is a subset of a vector space which is affinely equivalent to the unit ball in euclidean space.

A symmetric convex body $K \subset \mathbb{R}^n$ is a *symmetric body of revolution* if it admits an *axis of revolution*, i.e., a 1-dimensional linear subspace L such that each section of K by an affine hyperplane A orthogonal to L is an $n-1$ dimensional closed euclidean ball in A, centered at $A \cap L$ (possibly empty or just a point). If L is an axis of revolution of K then $L^\perp$ is the associated *hyperplane of revolution*. An *affine symmetric body of revolution* is a convex body linearly equivalent to a symmetric body of revolution. The images, under the linear equivalence, of an axis of revolution and its associated hyperplane of revolution of the body of revolution are an axis of revolution and associated hyperplane of revolution of the affine body of revolution (not necessarily perpendicular anymore). Clearly, an ellipsoid centered at the origin is an affine symmetric body of revolution and any hyperplane serves as a hyperplane of revolution.

With these definitions understood, the convex geometry result that we use in the proof of Theorem 1.1 is the following characterization of ellipsoids.

Theorem 1.3. *A symmetric convex body $B \subset \mathbb{R}^{n+1}$, $n \geq 4$, all of whose hyperplane sections are linearly equivalent affine symmetric bodies of revolution, is an ellipsoid.*

The main ingredient in the proof of this theorem is the following result, possibly of independent interest.

Theorem 1.4. *Let $B \subset \mathbb{R}^{n+1}$, $n \geq 4$, be a symmetric convex body, all of whose hyperplane sections are affine symmetric bodies of revolution. Then, at least one of the sections is an ellipsoid.*

Note that in Theorem 1.4, unlike Theorem 1.3, we do not assume that all hyperplane sections of B are necessarily linearly equivalent to each other. If we add this assumption then it follows from Theorem 1.4 that *all* hyperplane sections of B are ellipsoids. The following well known characterization of ellipsoids then implies that B itself is an ellipsoid, thus proving Theorem 1.3.

Proposition 1.5. *Let $B \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a symmetric convex body, all of whose hyperplane sections are ellipsoids. Then B is an ellipsoid.*

In fact, this result is known to hold even without the symmetry assumption on B (see, e.g., Theorem 2.12.4 of [MMO], p. 43). At the end of Section 2 we give a proof of the above symmetric case which is somewhat simpler than the general case.

It is an open question whether a symmetric convex body all of whose sections are affine symmetric bodies of revolution is itself a body of revolution (the converse of Lemma 2.2). In Remark 2.5 we briefly discuss this question and explain why Theorem 1.4 may be considered as a first step towards an affirmative answer.

Theorem 1.4 and Proposition 1.5 are proved in Section 2. The rest of the article consists of using topological methods to show that, under the hypotheses of Theorem 1.1, all hyperplane sections of B are necessarily affine symmetric bodies of revolution. The link to topology is via a beautiful idea that traces back to the work of Gromov [Gr]. It consists of the following key observation.

Lemma 1.6. *Let $B \subset \mathbb{R}^{n+1}$ be a symmetric convex body, all of whose hyperplane sections are linearly equivalent to some fixed symmetric convex body $K \subset \mathbb{R}^n$. Let $G_K := \{g \in GL_n(\mathbb{R}) | g(K) = K\}$ be the* group of linear symmetries *of K. Then the structure group of S^n can be reduced to G_K.*

See Section 3.1 below for a proof of this lemma, as well as a brief reminder about structure groups of differentiable manifolds and their reductions. Lemma 1.6 can be interpreted through the notion of *a field of convex bodies* tangent to S^n. See, for example, Mani [Ma] and [Mo1], as well as Remark 3.2 below.

Following Lemma 1.6, our task is to understand the possible reductions of the structure group of S^n (a classical problem in topology). The results we need are contained in the next purely topological theorem which, when applied to Lemma 1.6 with the dimension hypothesis of Theorem 1.1, implies that K is an affine symmetric body of revolution.

But first, another definition. We say that a subgroup $G \subset GL_n(\mathbb{R})$ is *reducible* if the induced action on $\mathbb{R}^n$ leaves invariant a k-dimensional linear subspace, $1 < k < n$; otherwise, it is an *irreducible* subgroup of $GL_n(\mathbb{R})$. (Beware of the potentially confusing use of the notions 'reducible' and 'can be reduced' in the statement of the following theorem.)

Theorem 1.7. *Let $n \equiv 1 \mod 4$, $n \geq 5$, and suppose that the structure group of S^n can be reduced to a closed connected subgroup $G \subset SO_n$. Then:*

(a) If G is reducible then it is conjugate to a subgroup of the standard inclusion $SO_{n-1} \subset SO_n$, acting transitively on S^{n-2}.
(b) If G is irreducible then $G = SO_n$, or $n = 133$ and $G \subset H \subset SO_{133}$, where H is the adjoint representation of the simple exceptional Lie group E_7.

We prove Theorem 1.7 in Section 3.3 by applying to our situation some known results from the literature about structure groups on spheres, mainly from [St], [Le] and [CC]. In case (b) (the irreducible case), we need to supplement these results with several basic facts about the representation theory and topology of compact Lie groups.

* * *

In summary, Theorem 1.1 is a consequence of the above results, as follows. Since all hyperplane sections of B are linearly equivalent to each other, they are linearly equivalent to some fixed symmetric convex body $K \subset \mathbb{R}^n$. By Lemma 1.6, the structure group of S^n can be reduced to G_K. It is easy to see that it can be further reduced to the identity component $G_K^0 \subset G_K$ (see Lemma 3.1 below). For a convex body K, G_K and G_K^0 are compact (Lemma 2.6) and are therefore conjugate to subgroups of O_n (Lemma 2.7); hence, by passing to a convex body linearly equivalent to K, we can assume that $G_K^0 \subset SO_n$. Next, Theorem 1.7 applied to $G = G_K^0$, implies that K is a symmetric body of revolution: in case (a), $\mathbb{R}e_n$ is an axis of revolution of K; in case (b), K is a euclidean ball. Thus all hyperplane sections of B are linearly equivalent to the symmetric body of revolution K. It follows, by Theorem 1.3, that B is an ellipsoid. □

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2. Affine bodies of Revolution

2.1. **Proof of Theorem 1.4.** The aim of this section is to prove the following theorem, announced in the introduction.

Theorem 1.4. *Let $B \subset \mathbb{R}^{n+1}$, $n \geq 4$, be a symmetric convex body, all of whose hyperplane sections are affine symmetric bodies of revolution. Then, at least one of the sections is an ellipsoid.*

To prove Theorem 1.4 we need the following four lemmas about affine symmetric bodies of revolution. Their proofs are given at the end of this section.

Lemma 2.1. *A symmetric affine body of revolution $K \subset \mathbb{R}^n$, $n \geq 3$, admitting two different hyperplanes of revolution, is an ellipsoid.*

Lemma 2.2. *Let $K \subset \mathbb{R}^n$, $n \geq 3$, be an affine symmetric body of revolution. Then any section $K' = \Gamma \cap K$ with a k-dimensional linear subspace $\Gamma \subset \mathbb{R}^n$, $1 < k < n$, is an affine symmetric body of revolution in Γ. Furthermore, if L is an axis of revolution of K and H the associated hyperplane of revolution then*

(a) If $\Gamma \subset H$ then K' is an ellipsoid.
(b) If $\Gamma \not\subset H$ then $H' := \Gamma \cap H$ is a hyperplane of revolution of K'.

(c) If $L \subset \Gamma$ then L is also the axis of revolution of K' associated to the hyperplane of revolution $\Gamma \cap H$.

Lemma 2.3. *Let $B \subset \mathbb{R}^{n+1}$ be a symmetric convex body, $n \geq 4$, $\Gamma_1, \Gamma_2 \subset \mathbb{R}^{n+1}$ two distinct hyperplanes, such that the hyperplane sections $K_i := \Gamma_i \cap B$, $i = 1, 2$, are affine symmetric bodies of revolution, with axes and associated hyperplanes of revolution L_i, H_i (respectively). If $L_1 \subset H_2$ then K_1 is an ellipsoid.*

Lemma 2.4. *Let $B \subset \mathbb{R}^{n+1}$ be a symmetric convex body, all of whose hyperplane sections are non-ellipsoidal affine symmetric bodies of revolution. For each $x \in S^n$ let L_x be the (unique) axis of revolution of $x^\perp \cap B$. Then $x \mapsto L_x$ is a continuous function $S^n \to \mathbb{R}P^n$.*

Proof of Theorem 1.4. Let $B \subset \mathbb{R}^{n+1}$ be a symmetric convex body, all of whose hyperplane sections are affine symmetric bodies of revolution. If none of the sections is an ellipsoid then, by Lemma 2.1, for each $x \in S^n$ the section $x^\perp \cap B$ has a unique axis of revolution $L_x \subset x^\perp$. By Lemma 2.4, $x \mapsto L_x$ defines a continuous function $S^n \to \mathbb{R}P^n$, i.e., a line subbundle of TS^n. (Note that for even n this is already a contradiction, so we proceed for odd n.) Now every line bundle on S^n, $n \geq 2$, is trivial, i.e., admits a non-vanishing section, hence one can find a continuous function $\psi : S^n \to S^n$ such that $\psi(x) \in L_x$ for all $x \in S^n$. Since $\psi(x) \perp x$, the function $F(t, x) := (t\psi(x) + (1-t)x)/\|t\psi(x) + (1-t)x\|$, $0 \leq t \leq 1$, is well defined (the denominator does not vanish), defining a homotopy between $\psi = F(1, \cdot)$ and the identity map $F(0, \cdot)$. It follows that ψ is a degree 1 map and is thus *surjective*.

Now let $\Gamma_2 \cap B$ be a hyperplane section of B, with hyperplane of revolution $H_2 \subset \Gamma_2$. Let $L_1 \subset H_2$ be any 1-dimensional subspace. Then the surjectivity of ψ implies that B admits a hyperplane section $K_1 = \Gamma_1 \cap B$ with axis of revolution L_1. By Lemma 2.3, K_1 is an ellipsoid, in contradiction to our assumption that none of the hyperplane sections of B is an ellipsoid. □

Remark 2.5. Lemma 2.2 says that any hyperplane section of an affine symmetric convex body of revolution B is again an affine symmetric convex body of revolution. The converse of this result, as far as we know, is an open problem. Let us state a somewhat more general question:

Let $B \subset \mathbb{R}^{n+1}$, $n \geq 4$, be a convex body containing the origin in its interior. If every hyperplane section of B is an affine body of revolution, is B necessarily an affine body of revolution?

An obvious necessary condition for B to be an affine body of revolution is that one of its hyperplane sections is an ellipsoid (take the hyperplane of revolution of B). Thus, Theorem 1.4 can be viewed as a first step for a positive answer to the above question (at least, under the further assumption of symmetry). Since Theorem 1.4 assumes $n \geq 4$, we dare only ask the above question under the same dimension restriction.

The case $n = 2$ has a different flavour altogether, where 'axis of revolution' of a plane section is replaced by 'axis of symmetry'. (For example, there are convex plane regions with several different axes of symmetry which are not ellipses; this is the reason we proved Theorem 1.4 only for $n \geq 4$). Yet there is a result in this dimension, somewhat related to Theorem 1.4. It is Theorem 2.1 of [Mo2]: *Let $B \subset \mathbb{R}^3$ be a convex body such that every plane section through some fixed interior point of B has an axis of symmetry. Then at least one of the sections is a disk.*

2.2. **Proofs of Proposition 1.5 and Lemmas 2.1 - 2.4.**

Proposition 1.5. *Let $B \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a symmetric convex body, all of whose hyperplane sections are ellipsoids. Then B is an ellipsoid.*

Proof. (See Figure 1.) Let us fix a unit vector $u \in \mathbb{R}^{n+1}$ and the hyperplane $\Gamma := u^\perp$. We can then map B linearly to a convex symmetric body $B' \subset \mathbb{R}^{n+1}$, intersecting Γ in the unit euclidean ball in Γ, with support planes $\Gamma \pm u$ at $\pm u \in B'$. Let v be a unit vector in Γ and P the 2-plane spanned by u, v. Then $P \cap B'$ is a solid ellipse in P, centered at the origin with support lines $\mathbb{R}v \pm u$, whose boundary is an ellipse passing through $\pm u, \pm v$. Thus $P \cap B'$ is the unit disk in P centered at the origin. As v varies along the unit sphere in Γ, the unit disks $P \cap B'$ fill up the unit ball in $\mathbb{R}^{n+1}$. Thus B' is a ball and B is an ellipsoid. □

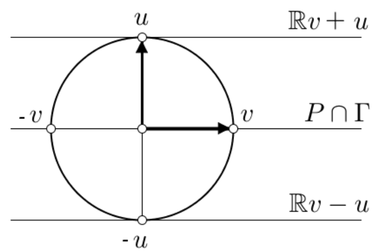


FIGURE 1. The proof of Proposition 1.5.

In preparation to the proof of Lemma 2.1, we need the two following lemmas.

Lemma 2.6. *Let $K \subset \mathbb{R}^n$ be a symmetric convex body. Then its linear symmetry group $G_K = \{g \in GL_n(\mathbb{R}) \,|\, g(K) = K\}$ is compact.*

Proof. Let $A_K := \{a \in End(\mathbb{R}^n) \,|\, a(K) \subset K\}$. Since K is closed in $\mathbb{R}^n$, A_K is closed in $End(\mathbb{R}^n) \simeq \mathbb{R}^{n^2}$ (this follows easily from the continuity of matrix multiplication $End(\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}^n$). Since K is bounded and 0 is an interior point, there exist $R, r > 0$ such that $B_r \subset K \subset B_R$, where $B_\rho \subset \mathbb{R}^n$ is the closed ball of radius ρ. It follows that for every $a \in A_K$, $a(B_r) \subset B_R$, hence $\|a\| \leq R/r$. Thus $A_K \subset End(\mathbb{R}^n)$ is also bounded and hence compact. It remains to show that $G_K \subset A_K$ is closed. Let $g_i \in G_K$ with $g_i \to g \in End(\mathbb{R}^n)$. Since $(g_i)^{-1} \in A_K$, $(g_i)^{-1}(B_r) \subset B_R$, hence $0 < (r/R)\|v\| \leq \|g_i v\|$ for all i and all $v \neq 0$. Taking $i \to 0$ we get $0 < (r/R)\|v\| \leq \|gv\|$, hence g is invertible, i.e., $g \in G_K$. □

Lemma 2.7. *Every compact subgroup $G \subset GL_n(\mathbb{R})$ is conjugate to a subgroup of O_n.*

Proof. By taking an arbitrary positive inner product on $\mathbb{R}^n$ (e.g., the standard inner product $\sum x_i y_i$) and averaging it over G with respect to a bi-invariant measure, one obtains a G-invariant inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^n$. Now any two inner products on $\mathbb{R}^n$ are isomorphic to each other, hence one can find an element $g \in GL_n(\mathbb{R})$ such that $(u, v) \mapsto \langle gu, gv\rangle$ is the standard inner product on $\mathbb{R}^n$. It follows that $g^{-1}Gg \subset O_n$. For more details see, e.g., Prop. 3.1 on p. 36 of [Ad]. □

Lemma 2.1. *A symmetric affine body of revolution $K \subset \mathbb{R}^n$, $n \geq 3$, admitting two different hyperplanes of revolution, is an ellipsoid.*

Proof. Let $G = G_K^0 \subset GL_n(\mathbb{R})$ be the identity component of the linear symmetry group of K. By Lemmas 2.6 and 2.7, we can assume, by passing to a body of revolution linearly equivalent to K, that $G \subset SO_n$. We will show that in this case K is a ball centered at the origin, by showing that $G = SO_n$.

Now, each hyperplane of revolution of K gives rise to a subgroup of G conjugate in SO_n to SO_{n-1} (the stabilizer of the hyperplane). It is thus enough to show that the only connected subgroup $G \subset SO_n$ satisfying $SO_{n-1} \subsetneq G \subset SO_n$ is $G = SO_n$ (i.e., SO_{n-1} is a *maximal* connected subgroup of SO_n). Since the three Lie groups SO_{n-1}, G, SO_n are connected, $SO_{n-1} \subsetneq G \subset SO_n$ is equivalent to their Lie algebras satisfying $\mathfrak{so}_{n-1} \subsetneq \mathfrak{g} \subset \mathfrak{so}_n$ and $G = SO_n$ is equivalent to $\mathfrak{g} = \mathfrak{so}_n$. Consider the conjugation action of SO_{n-1} on $\mathfrak{so}_n$ (the adjoint representation of SO_n restricted to SO_{n-1}). Then $\mathfrak{so}_{n-1}, \mathfrak{g} \subset \mathfrak{so}_n$ are invariant subspaces, hence $\mathfrak{so}_{n-1} \subsetneq \mathfrak{g}$ implies that $\mathfrak{g}/\mathfrak{so}_{n-1}$ is a non-trivial invariant subspace of $\mathfrak{so}_n/\mathfrak{so}_{n-1}$. Now it is easy to show that $\mathfrak{so}_n$ decomposes under SO_{n-1} as $\mathfrak{so}_{n-1} \oplus \mathfrak{m}$, where the action of SO_{n-1} on the second summand is equivalent to the standard (irreducible) action of SO_{n-1} on $\mathbb{R}^{n-1}$. It follows that $\mathfrak{so}_n/\mathfrak{so}_{n-1} \simeq \mathfrak{m}$ is an irreducible SO_{n-1} representation, hence $\mathfrak{g}/\mathfrak{so}_{n-1} = \mathfrak{so}_n/\mathfrak{so}_{n-1}$. Thus $\mathfrak{g} = \mathfrak{so}_n$ and so $G = SO_n$. □

Lemma 2.2. *Let $K \subset \mathbb{R}^n$, $n \geq 3$, be an affine symmetric body of revolution. Then any section $K' = \Gamma \cap K$ with a k-dimensional linear subspace $\Gamma \subset \mathbb{R}^n$, $1 < k < n$, is an affine symmetric body of revolution in Γ. Furthermore, if L is an axis of revolution of K and H the associated hyperplane of revolution then*

(a) If $\Gamma \subset H$ then K' is an ellipsoid.
(b) If $\Gamma \not\subset H$ then $H' := \Gamma \cap H$ is a hyperplane of revolution of K'.
(c) If $L \subset \Gamma$ then L is also the axis of revolution of K' associated to the hyperplane of revolution $\Gamma \cap H$.

Proof. (a) If $\Gamma \subset H$ then $\Gamma \cap K$ is a linear section of the ellipsoid $H \cap K$, hence is an ellipsoid.

(b) We can assume, by applying an appropriate linear transformation, as in the proof of Proposition 1.5, that K is a symmetric body of revolution with an axis of revolution $L = \mathbb{R}e_n$ and plane of revolution $H = L^\perp = \{x_n = 0\}$, such that $H \cap K$ is the unit ball in H and $H \pm e_n$ are support hyperplanes of K at $\pm e_n$. Furthermore, we can also arrange that $H' := \Gamma \cap H$ is spanned by $e_1, \ldots, e_{k-1}$ and so Γ is spanned by $e_1, \ldots, e_{k-1}, v$, where $v = \lambda e_{n-1} + e_n$ for some $\lambda \in \mathbb{R}$. To show that H' is a hyperplane of revolution of K' with an associated axis of revolution $L' = \mathbb{R}v$, we need to show that every non empty section of K' by an affine hyperplane of the form $H' + tv$, $t \in \mathbb{R}$, is an $(n-2)$-dimensional ball in $H' + tv$, centered at tv. The latter section is the section of the $(n-1)$-dimensional ball $(H + te_n) \cap K$, centered at te_n, by $H' + tv$, an affine hyperplane of $H + te_n$, hence is an $(n-2)$-dimensional ball, centered at tv, as needed.

(c) In the previous item, if $L \subset \Gamma$, we can choose $v = e_n$. □

In preparation to proving Lemma 2.3, we prove the following lemma.

Lemma 2.8. *Let $K \subset \mathbb{R}^n$, $n \geq 3$, be an affine symmetric body of revolution with an axis of revolution L. Suppose a section of K by a linear subspace $\Gamma \subset \mathbb{R}^n$ of dimension ≥ 2 passing through L is an ellipsoid. Then K is an ellipsoid.*

Proof. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. By passing to a linearly equivalent body of revolution, we can assume that K is a symmetric body of revolution with an axis of revolution $L = \mathbb{R}e_n$ and associated hyperplane of revolution $H = L^\perp = \{x_n = 0\}$. Furthermore, we can also assume that $H \cap K$ is the unit ball in H and that $H \pm e_n$ are support hyperplanes of K at $\pm e_n$. We will show that, under these assumptions, K is the unit ball in $\mathbb{R}^n$. To this end, it is enough to show that each section of K by a 2 dimensional subspace Δ containing L is the unit disk in Δ centered at the origin. Let us choose a 2-dimensional subspace $\Delta \subset \Gamma$ containing L and a unit vector v in the 1-dimensional space $\Delta \cap H$. Then $\Delta \cap K$ is a (solid) ellipse, centered at the origin, whose boundary passes through $\pm v, \pm e_n$, with support lines $\mathbb{R}v \pm e_n$ at $\pm e_n$. It follows that $\Delta \cap K$ is the unit disk in Δ centered at the origin. Now since $L = \mathbb{R}e_n$ is an axis of revolution of K, all rotations in $\mathbb{R}^n$ about L leave K invariant. Applying all such rotations to Δ, we obtain all 2-dimensional subspaces containing L, and each of them intersects K in a unit disk centered at the origin, as needed. □

Lemma 2.3. *Let $B \subset \mathbb{R}^{n+1}$ be a symmetric convex body, $n \geq 4$, $\Gamma_1, \Gamma_2 \subset \mathbb{R}^{n+1}$ two distinct hyperplanes, such that the hyperplane sections $K_i := \Gamma_i \cap B$, $i = 1, 2$, are affine symmetric bodies of revolution, with axes and associated hyperplanes of revolution L_i, H_i (respectively). If $L_1 \subset H_2$ then K_1 is an ellipsoid.*

Proof. Let $E := K_1 \cap K_2$. We will show that E is an ellipsoid. This implies, by Lemma 2.8, that K_1 is an ellipsoid, since $E = K_1 \cap \Gamma_2$ and Γ_2 contains L_1, an axis of revolution of K_1.

To show that E is an ellipsoid, we note first that Γ_2 does not contain H_1, else $L_1, H_1 \subset \Gamma_2$ would imply $\Gamma_1 = L_1 \oplus H_1 \subset \Gamma_2$. Hence, by Lemma 2.2(b), $\Gamma_2 \cap H_1$ is a hyperplane of revolution of $E = \Gamma_2 \cap K_1$.

Next we look at $\Gamma_1 \cap \Gamma_2$. This has codimension 1 in Γ_2. If it coincides with H_2, then $E = \Gamma_1 \cap K_2 = H_2 \cap K_2$, which is an ellipsoid, by Lemma 2.2(a). If $\Gamma_1 \cap \Gamma_2 \neq H_2$, then by Lemma 2.2(b), $\Gamma_1 \cap H_2$ is a hyperplane of revolution of $E = \Gamma_1 \cap K_2$.

Now $\Gamma_1 \cap H_2$, $\Gamma_2 \cap H_1$ are two distinct hyperplanes of revolution of E, since L_1 is contained in the first but not in the second. It follows from Lemma 2.1 that E is an ellipsoid. □

In order to show Lemma 2.4, we prove the following lemma. Its statement has appeared elsewhere (e.g., statement III of the proof of Theorem 2.2 of [Mo2]), but no written proof of it was available (perhaps because it is intuitively clear and a hassle to prove).

Lemma 2.9. *Let $B \subset \mathbb{R}^{n+1}$ be a symmetric convex body and $x_i \to x$ a convergent sequence in S^n. Assume each hyperplane section $x_i^\perp \cap B$ is an affine symmetric body of revolution with an axis of revolution $L_i \subset x_i^\perp$. If $\{L_i\}$ is a convergent sequence in $\mathbb{R}P^n$, $L_i \to L$, then $x^\perp \cap B$ is an affine symmetric body of revolution with an axis of revolution L.*

Proof. Let $\Gamma_i := x_i^\perp, \Gamma := x^\perp$, $K_i := \Gamma_i \cap B$, $K := \Gamma \cap B$. Assume, without loss of generality, that $x = e_{n+1}$, so that $\Gamma = \mathbb{R}^n$.

Claim 1. *$K_i \to K$ in the Hausdorf metric.*

We postpone for the moment the proof this claim (and the two subsequent ones). Define $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ by $(x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_n)$. Note that $\pi(K) = K$ and $\pi(L) = L$.

Claim 2. *For large enough i, $\pi|_{\Gamma_i} : \Gamma_i \to \mathbb{R}^n$ is a linear isomorphism.*

We henceforth restrict to a subsequence of $\{K_i\}$ such that each $\pi|_{\Gamma_i}$ is an isomorphism. Let $K_i' := \pi(K_i) \subset \mathbb{R}^n$, $L_i' := \pi(L_i) \subset \mathbb{R}^n$. Then each $K_i' \subset \mathbb{R}^n$ is an affine symmetric body of revolution with an axis of revolution L_i', $L_i' \to L$ and $K_i' \to K$ (by Claim 1). By definition of affine symmetric body of revolution, there exist linear isomorphisms $T_i : \mathbb{R}^n \to \mathbb{R}^n$ such that $K_i'' := T_i(K_i')$ is a (honest) symmetric body of revolution. By postcomposing T_i with appropriate elements of $GL_n(\mathbb{R})$, we can also assume that $\mathbb{R}e_n = T_i(L_i')$ is an axis of revolution of K_i'', that $\mathbb{R}^{n-1} \pm e_n$ are support hyperplanes of K_i'' at $\pm e_n$ and that $K_i'' \cap \mathbb{R}^{n-1}$ is the unit $n-1$ dimensional closed ball in $\mathbb{R}^{n-1}$, centered at the origin.

Claim 3. *$\{T_i\}$ is contained in a compact subset of $GL_n(\mathbb{R})$.*

It follows that there is a subsequence of $\{T_i\}$, which we rename $\{T_i\}$, converging to an element $T \in GL_n(\mathbb{R})$. Let $K'' := T(K)$. Then $\lim K''_i = \lim T_i(K'_i) = (\lim T_i)(\lim K'_i) = T(K) = K''$, and $T(L) = (\lim T_i)(\lim L'_i) = \lim T_i(L_i) = \mathbb{R}e_n$. It is thus enough to show that $\mathbb{R}e_n$ is an axis of revolution of K''. Now $\mathbb{R}e_n$ is an axis of revolution of each K''_i hence $gK''_i = K''_i$ for all $g \in O_{n-1}$ (the elements of O_n leaving $\mathbb{R}e_n$ fixed). Taking the limit $i \to \infty$ we obtain $g(K'') = K''$. Hence $\mathbb{R}e_n$ is an axis of revolution of K''.

Proof of the 3 claims:

(1) Let $\Gamma \subset \mathbb{R}^n$ be a hyperplane and $U \subset \mathbb{R}^n$ an open subset such that $\Gamma \cap B \subset U$. Then there is a $\delta > 0$ such that $\Gamma_\delta \cap B \subset U$, where Γ_δ is the δ-neighbourhood around Γ (this follows since the distance between the compact $\Gamma \cap B$ and the closed $\mathbb{R}^{n+1} \setminus U$ is positive).

For $x, x' \in S^n$, let $\Gamma = x^\perp$ and $\Gamma' = x'^\perp$. For any fixed $R > 0$, the ball of radius R in Γ' will be contained in Γ_δ provided Γ and Γ' are close enough (i.e., provided $\langle x, x'\rangle$ is close enough to 1). Thus $\Gamma' \cap B \subset \Gamma_\delta \cap B$ for Γ and Γ' sufficiently close.

Fix an $\epsilon > 0$ and take $U = K_\epsilon$; then there is $\delta > 0$ such that $\Gamma_\delta \cap B \subset K_\epsilon$, but then $K_i = \Gamma_i \cap B \subset \Gamma_\delta \cap B \subset K_\epsilon$, for all i sufficiently large.

The argument is symmetric, thus $K \subset (K_i)_\epsilon$ for all sufficiently large i. □

(2) $\mathrm{Ker}(\pi) = \mathbb{R}e_{n+1}$, hence $\mathrm{Ker}(\pi|_{\Gamma_i}) \neq 0$ if and only if $e_{n+1} \perp x_i$. But $x_i \to e_{n+1}$ implies $\langle x_i, e_{n+1}\rangle \to 1$, hence $\langle x_i, e_{n+1}\rangle \neq 0$ for all i sufficiently large.

(3) For each pair of constants $c, C > 0$ the set of elements $A \in GL_n(\mathbb{R})$ satisfying $c\|v\| \leq \|Av\| \leq C\|v\|$ for all $v \in \mathbb{R}^n$ is clearly closed. It is also bounded because its elements satisfy $\|A\| \leq C$ (using the operator norm on $\mathrm{End}(\mathbb{R}^n)$). It is thus enough to find constants $c, C > 0$ such that $c\|v\| \leq \|T_i v\| \leq C\|v\|$ for all $v \in \mathbb{R}^n$ and all i.

Denote by B_ρ the closed ball in $\mathbb{R}^n$ of radius ρ centered at the origin. Then there are constants $r', R', r'', R'' > 0$ such that $B_{r'} \subset \pi(B) \subset B_{R'}$ and $B_{r''} \subset K''_i \subset B_{R''}$ for all i. It follows that $T_i(B_{r'}) \subset T_i(K'_i) = K''_i \subset B_{R''}$, thus $\|T_i v\| \leq C\|v\|$ for all $v \in \mathbb{R}^n$ and all i, where $C = R''/r'$.

Next, $(T_i)^{-1}B_{r''} \subset (T_i)^{-1}(K''_i) = K'_i \subset B_{R'}$, hence $\|(T_i)^{-1}w\| \leq c'\|w\|$ for all $w \in \mathbb{R}^n$ and all i, where $c' = R'/r''$. Substituting $w = T_i v$ in the last inequality we obtain $c\|v\| \leq \|T_i v\|$ for all $v \in \mathbb{R}^n$ and all i, where $c = 1/c' = r''/R'$. □

Lemma 2.4. *Let $B \subset \mathbb{R}^{n+1}$ be a symmetric convex body, all of whose hyperplane sections are non-ellipsoidal affine symmetric bodies of revolution. For each $x \in S^n$ let L_x be the (unique) axis of revolution of $x^\perp \cap B$. Then $x \mapsto L_x$ is a continuous function $S^n \to \mathbb{R}P^n$.*

Proof. Let $x_i \to x$ be a converging sequence in S^n. To show that $L_{x_i} \to L_x$ it is enough to show that L_{x_i} is convergent and its limit is an axis of revolution of $x^\perp \cap B$. Since $\mathbb{R}P^n$ is a compact metric space, to show that L_{x_i} is convergent it is enough to show that all its convergent subsequences have the same limit. To show this, it is enough to show that the limit of a convergent subsequence of L_{x_i} is an axis of revolution of $x^\perp \cap B$. This is the statement of Lemma 2.9. □

3. Structure groups of spheres

3.1. **A reminder on structure groups of manifolds and their reduction.** First, let us recall the following basic definitions (see, for example, §5 of Chap. I of [KN], or Part I of [St]).

Let G be a topological group, M a topological space and $P \to M$ a principal G-bundle. A *reduction of the structure group* of $P \to M$ to a closed subgroup $H \subset G$ is a principal H-subbundle of P. Equivalently, it is a continuous section of the bundle $P/H \to M$ associated with the left G-action on G/H. The *frame bundle* of an n-dimensional differentiable manifold M is the $GL_n(\mathbb{R})$-principal bundle $F(M) \to M$, whose fiber at a point $x \in M$ is the set of all linear isomorphisms $\mathbb{R}^n \to T_xM$, with the $GL_n(\mathbb{R})$ right action given by precomposition of linear maps. A *G-reduction* of the structure group of a smooth n-manifold M (or a *G-structure*) is the reduction of the structure group $GL_n(\mathbb{R})$ of its frame bundle to a closed subgroup $G \subset GL_n(\mathbb{R})$. Equivalently, it is given by an open cover of M, together with a trivilization of the restriction of TM to each of the covering open subsets, such that the transition functions between the trivilizations on overlapping members of the cover take values in G (Prop. 5.3 of [KN], p. 53). For $M = S^n$, there is a standard cover by two 'hemispheres', intersecting along a neighborhood of the 'equator' S^{n-1}, hence its structure group is given by a single transition function $\chi_n : S^{n-1} \to GL_n(\mathbb{R})$, called the *characteristic map* (§18 of [St], pp. 96-100).

Here is a standard result on reductions of structure groups on spheres. Let $S^n := \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$.

Lemma 3.1. *If the structure group of S^n, $n \geq 2$, can be reduced to some closed subgroup $G \subset GL_n(\mathbb{R})$ then it can be further reduced to its identity component $G^0 \subset G$.*

Proof. The structure group of S^n can be reduced to G if and only if the characteristic map $\chi_n : S^{n-1} \to GL_n(\mathbb{R})$ is homotopic to a map whose image is contained in G. The maps and homotopies in question are all 'pointed', i.e., they send some fixed point of the equator $* \in S^{n-1} \mapsto e \in GL_n(\mathbb{R})$. Since S^{n-1} is connected, its image under χ_n is connected as well, hence is contained in G^0. □

3.2. **Proof of Lemma 1.6.** Let us recall Lemma 1.6, announced in the introduction.

Lemma 1.6. *Let $B \subset \mathbb{R}^{n+1}$ be a symmetric convex body, all of whose hyperplane sections are linearly equivalent to some fixed symmetric convex body $K \subset \mathbb{R}^n$. Let $G_K := \{g \in GL_n(\mathbb{R}) | g(K) = K\}$ be the* group of linear symmetries *of K. Then the structure group of S^n can be reduced to G_K.*

Proof. Identify for each $x \in S^n$, by parallel translation in $\mathbb{R}^{n+1}$, the tangent space to S^n at x with $x^\perp$. Define the set $P_x \subset F_x(S^n)$ of frames at x as the set of linear isomorphisms $\mathbb{R}^n \to x^\perp$ mapping K to $x^\perp \cap B$. It is easy to check that $P \subset F(S^n)$ is a G_K-structure. □

Remark 3.2. The proof of Lemma 1.6 is deceptively simple and somewhat hard to appreciate. It is a special case of a correspondence between a large class of geometric structures on manifolds, those given by a distribution of linearly equivalent tangent objects, and reductions of their structure group (the proof of Lemma 1.6 clearly does not use any property of K).

For example, choosing an orientation on an (orientable) n-manifold M corresponds to choosing a set of 'positively oriented frames', which amounts to a reduction of the structure group of the manifold to the subgroup $GL_n^+(\mathbb{R}) \subset GL_n(\mathbb{R})$ of matrices with positive determinant. Similarly, choosing a riemannian metric on a manifold is equivalent to deciding which frames are orthonormal, amounting to a reduction of the structure group of the manifold to the orthogonal group $O_n \subset GL_n(\mathbb{R})$. Choosing both a riemannian metric and orientation reduces the structure group to $SO_n(\mathbb{R}) = O_n \cap GL_n^+(\mathbb{R})$.

How far can one reduce the structure group of a manifold is an indication – a sort of a quantitative group theoretic measure – of the triviality of its tangent bundle. In our case, Theorem 1.7 states, roughly speaking, that the structure group of S^{4k+1} cannot be reduced much, and so Lemma 1.6 implies that K must be highly symmetric (its linear symmetry group G_K is 'large').

Here is another example. An odd dimensional sphere $S^{2k+1} \subset \mathbb{C}^{k+1}$, $k \geq 0$, admits a non-vanishing vector field v, coming from scalar multiplication in $\mathbb{C}^{k+1}$ by unitary complex numbers. In the language of the proof of Lemma 1.6, for each $x \in S^{2k+1}$ there is associated the subset $\{v(x)\} \subset T_x S^{2k+1}$, linearly equivalent to $K := \{e_1\} \subset \mathbb{R}^{2k+1}$. Thus, according to Lemma 1.6, the non vanishing vector field v defines a reduction of the structure group of S^{2k+1} to the subgroup of $GL_{2k+1}(\mathbb{R})$ consisting of matrices of the form

$$\begin{pmatrix} 1 & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots \\ 0 & * & \cdots & * \end{pmatrix},$$

the stabilizer of e_1 in $GL_{2k+1}(\mathbb{R})$. If we also consider along with v the standard riemannian metric and orientation on S^{2k+1}, then the structure group will further reduce to $\{1\} \times SO_{2k} \subset SO_{2k+1}$ (note that v is a unit vector with respect to the standard metric).

In fact, one can further reduce to the subgroup $\{1\} \times SU_k \subset \{1\} \times SO_{2k} \subset SO_{2k+1}$, by introducing a corank 1 subbundle $D \subset TS^{2k+1}$, together with an almost complex structure on D (a so called 'CR-structure') and a compatible complex volume form on it. This can be constructed using the Hopf fibration $\pi : S^{2k+1} \to \mathbb{C}P^k$, mapping a point $x \in S^{2k+1}$ to the complex line $\mathbb{C}x \in \mathbb{C}P^k$. Then it is easy to see that $v(x)$ generates the tangent to the fiber through x and that $D(x) := v(x)^\perp \subset T_x S^{2k+1}$ defines a corank 1 subbundle, transverse to the fibers. Restricting $d\pi : TS^{2k+1} \to T\mathbb{C}P^k$ to D, we can pull back the standard almost complex structure and the canonical line bundle from $\mathbb{C}P^k$ to D, thus defining an almost complex structure and a compatible complex volume form on D. Associated with this structure is the claimed reduction to $\{1\} \times SU_k \subset \{1\} \times SO_{2k} \subset SO_{2k+1}$.

For odd $k = 2m+1$, i.e. S^{4m+3}, there is also a further reduction to $\{1_{\mathbb{R}^3}\} \times Sp_m \subset \{1_{\mathbb{R}^3}\} \times SU_{2m} \subset \{1_{\mathbb{R}}\} \times SU_{2m+1} \subset SO_{4m+3}$, arising in a similar fashion from the quaternionic Hopf fibration $S^{4m+3} \to \mathbb{H}P^m$. Yet for even k, which is the case considered in this article, the above reduction to SU_k is the 'smallest possible', with few exceptions (see Corollary 3.10 below).

3.3. **Proof of Theorem 1.7a (the reducible case).** Suppose the structure group of S^n can be reduced to a closed connected subgroup $G \subset SO_{n-1}$, acting reducibly on $\mathbb{R}^n$. Then G is conjugate to a closed connected subgroup $G' \subset SO_k \times SO'_{n-k} \subset SO_n$ for some k, $n/2 \leq k < n$, where SO'_{n-k} denotes the subgroup of SO_n fixing $\mathbb{R}^k = \{x_{k+1} = \ldots = x_n = 0\} \subset \mathbb{R}^n$. If $n \equiv 1 \bmod 4$, then such a reduction is possible only if $k = n-1$, i.e., $G' \subset SO_{n-1}$, acting irreducibly on $\mathbb{R}^{n-1}$ (see [St], §27.14, §27.18, pp. 143-144). In particular, the structure group of S^n reduces to SO_{n-1} but not to SO_{n-2}. Corollary 3.2 of [Le] now implies that G' acts transitively on S^{n-2}. We include the argument.

Consider the standard fibration $SO_{n-2} \to SO_{n-1} \xrightarrow{\pi} S^{n-2}$. If G' does not act transitively on S^{n-2} it means that the composition $G' \overset{i}{\hookrightarrow} SO_{n-1} \xrightarrow{\pi} S^{n-2}$ is not surjective, and is therefore null homotopic. Let $F : G' \times I \to S^{n-2}$ be the homotopy. Then, by the homotopy lifting property, there exists a map $\widetilde{F}$ completing the diagram

$$\begin{array}{ccc} G' & \overset{i}{\hookrightarrow} & SO_{n-1} \\ {\scriptstyle I\times 0}\downarrow & \overset{\widetilde{F}}{\nearrow} & \downarrow{\scriptstyle \pi} \\ G'\times I & \underset{F}{\longrightarrow} & S^{n-2} \end{array}$$

Commutativity of the diagram implies that $\widetilde{F}(x,1) \in SO_{n-2} \subset SO_{n-1}$ for every $x \in G'$. Let $f : G' \to SO_{n-2}$ be defined by $f(x) = \widetilde{F}(x,1)$; then, up to homotopy, the following diagram commutes

$$\begin{array}{ccc} G' & \overset{i}{\hookrightarrow} & SO_{n-1} \\ & {\scriptstyle f}\searrow \quad \nearrow{\scriptstyle j} & \\ & SO_{n-2} & \end{array}$$

But now, precomposing $j \circ f$ with the characteristic map $\chi_n : S^{n-1} \to G'$, yields a reduction of the structure group of S^n to SO_{n-2}, which is a contradiction.

$$\begin{array}{cccc} S^{n-1} \xrightarrow{\chi_n} & G' & \overset{i}{\hookrightarrow} & SO_{n-1} \\ & & {\scriptstyle f}\searrow \quad \nearrow{\scriptstyle j} & \\ & & SO_{n-2} & \end{array}$$

□

3.4. **Proof of Theorem 1.7b (the irreducible case).** We need the following three lemmas.

Lemma 3.3. *For all $n \equiv 1$ mod 4, $n \geq 5$, if the structure group of S^n can be reduced to $G \subset SO_n$, then* $\dim G \geq n-2$.

Proof. This follows readily from Proposition 3.1 of [CC], since – as mentioned above – the structure group of S^n, $n \equiv 1$ mod 4, may be reduced to SO_{n-1} but not to SO_{n-2}. Given that the argument is a simple one, we include it here.

Assume that $\dim G = k < n$. We are going to show that the structure group of S^n reduces to the standard $SO_{k+1} \subset SO_n$. This implies the result.

Consider the characteristic map $\chi_n : S^{n-1} \to SO_n$ of S^n. Assuming that the structure group of S^n reduces to G amounts to the existence of $f : S^{n-1} \to G$ such that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\chi_n} & SO_n \\ & {\scriptstyle f}\searrow \quad \nearrow{\scriptstyle i} & \\ & G & \end{array}$$

The standard inclusion $SO_{k+1} \hookrightarrow SO_n$ induces isomorphisms $\pi_j(SO_{k+1}) \simeq \pi_j(SO_n)$ for every $j < k$ (this follows immediately from the long exact sequences of the fibrations $SO_{k+1+r} \to SO_{k+2+r} \to S^{k+1+r}$ for the range of j's in question).

Now, this implies that $G \hookrightarrow SO_n$ factors (up to homotopy) through SO_{k+1}. One way of seeing this is via obstruction theory. Think of G as a CW-complex. Then the obstruction to extend the inclusion $G \hookrightarrow SO_{k+1}$ from the j-skeleton to the $j+1$-skeleton is a cocycle with coefficients in $\pi_j(SO_{k+1})$. But the inclusion $SO_{k+1} \hookrightarrow SO_n$ induces isomorphisms onto $\pi_j(SO_n)$ $(j < k)$ where we know that the obstruction vanishes. Therefore, there is no obstruction to construct $G \to SO_{k+1}$ such that $G \to SO_{k+1} \hookrightarrow SO_n$ is homotopic to the inclusion $G \hookrightarrow SO_n$. Hence, the structure group of S^n reduces to SO_{k+1}. □

Lemma 3.4. *If $n \geq 8$, then the structure group of S^n cannot be reduced to an irirreducibleeducible subgroup $G \subsetneq SO_n$ isomorphic to SO_k, SU_m or Sp_m, with $k \geq 4, m \geq 2$.*

Proof. This is Corollary 2.2 of [CC]. □

Lemma 3.5. *For all $n \geq 2$, if the structure group of S^n reduces to a closed connected irreducible* maximal *subgroup $H \subsetneq SO_n$, then H is* simple.

Proof. See Theorem 3 of [Le]. □

We now proceed to the proof of Theorem 1.7b, using the above three lemmas. We first treat $n \geq 9$, then $n = 5$.

The case $n \geq 9$. Assume that $G \subset SO_n$ acts irreducibly on $\mathbb{R}^n$ but is not all of SO_n. Then it is contained in some *maximal* connected closed subgroup H, $G \subset H \subsetneq SO_n$. The structure group of S^n then reduces to H, acting also irreducibly on $\mathbb{R}^n$. By Lemma 3.5, H is simple. By Lemma 3.4, H is a non-classical group, i.e., it is isomorphic to either $Spin_m$, $m \geq 7$, or one of the 5 exceptional simple Lie groups: G_2, F_4, E_6, E_7 or E_8. By Lemma 3.3, $n \leq \dim H + 2$. Let V be the complexification of the (irreducible) representation of H on $\mathbb{R}^n$. Since $\dim V$ is odd, V is a complex irreducible representation.

Let us list all the properties of the pair (H, V) that we have so far:

(i) H is a non-classical compact connected group, i.e., $Spin_m$, $m \geq 7$, or one of the five exceptional compact simple Lie groups.
(ii) V is a complex irreducible representation of H of *real type* (i.e., the complexification of a real irreducible representation).
(iii) $\dim V \equiv 1 \mod 4$.
(iv) $\dim V \leq \dim(H) + 2$.
(v) If $H = Spin_m$, then its action on V does not factor through SO_m.

We claim that these 5 conditions on the pair (H, V) are *incompatible*, for $\dim V \geq 9$, except if V is the complexified adjoint representation of $H = E_7$, in which case $\dim V = \dim H = 133 \equiv 1 \mod 4$. We are unable to exclude this case.

For the exceptional groups, one can simply check (e.g., in Wikipedia) that none of them, other than E_7, has a non-trivial irreducible representation satisfying conditions (iii) and (iv). In the following table we list the smallest irreducible representations for them; we have marked in boldface the first dimensions that are $\equiv 1 \pmod 4$.

Group	G_2	F_4	E_6	E_7	E_8
$\dim G$	14	52	78	133	248
Irreps	7	26	27	56	248
	14	52	78	**133**	3875
	27	**273**	351	912	⋮
	64	⋮	**2925**	⋮	**1763125**
	77	⋮	⋮	⋮	⋮

For the spin groups, the next lemma shows that conditions (iii) and (v) are incompatible. (We thank Ilia Smilga for kindly informing us about this lemma and its proof).

Lemma 3.6. *Every irreducible complex representation of $Spin_m$, $m \geq 3$, which does not factor through SO_m is even dimensional.*

Proof. We first review some well-known general facts concerning representations of simple compact Lie groups (see, for example, [Ad]). With each d-dimensional complex representation of a compact semi-simple Lie group G of rank r with a maximal torus T, one can associate its weight system $\Omega \subset \mathfrak{t}^*$, a subset with d points (counting multiplicity). The Weyl group $W = N_G(T)/T$ acts on $\mathfrak{t}^*$, preserving Ω. Thus, to show that d is even, it is enough to show the following:

(a) An irreducible non classical representation V of $Spin_m$ does not have a 0 weight.
(b) The Weyl group of $Spin_m$ contains a subgroup whose order is a positive power of 2, and whose only fixed point in $\mathfrak{t}^*$ is 0.

Note that (a) and (b) imply that d is even, since under the action of said subgroup of W, say W', Ω breaks into the disjoint union of W'-orbits, each with an even number of elements, since, by (a), all stabilizers are strict subgroups of W', hence have even index.

To show (a), note that the T action on the 0 weight space is trivial. Now $-1 \in Spin_m$ is in T (since it is central), but -1 must act on V by $-Id$, else the $Spin_m$ action on V would factor through $SO_m = Spin_m/\{\pm 1\}$.

To show (b), let us first take $m = 2k$. Then $\mathbb{R}^m$ decomposes under T as the direct sum of k 2-planes. Consider the subgroup $N' \subset SO_m$ which leaves invariant each of these 2-planes. Then $N' \simeq S(O_2 \times \ldots \times O_2)$, $T \subset N' \subset N(T)$, and its image $W' = N'/T \subset W = N(T)/T$ acts on $\mathfrak{t}^*$ by diagonal matrices with entries ± 1 on the diagonal, with an even number of -1's. Using this description, it is easy to show that W' has order 2^{k-1} and that its only fixed point in $\mathfrak{t}^*$ is 0.

For $m = 2k+1$ the argument is simpler. Under T, $\mathbb{R}^m$ decomposes as a direct sum of k 2-planes, plus a line. We take an element in SO_m which is a reflection about a line through the origin in each of these planes, and $(-1)^k$ in the line. This is in $N(T)$ and acts on $\mathfrak{t}^*$ by $-Id$, hence its image in W has order 2 and its only fixed point in $\mathfrak{t}^*$ is the origin. □

The case $n = 5$. The only reduction of the structure group of S^5 that cannot be ruled out by Lemmas 3.3, 3.4 or 3.5 is the 5-dimensional irreducible representation of SO_3. This case is eliminated by the next lemma.

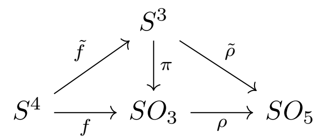
Lemma 3.7. *Let $\rho : SO_3 \to SO_5$ be the irreducible 5 dimensional representation of SO_3. Then, for any $f : S^4 \to SO_3$, the composition $S^4 \xrightarrow{f} SO_3 \xrightarrow{\rho} SO_5$ is null homotopic. It follows that the structure group of S^5 cannot be reduced to ρ.*

Proof. Since the tangent bundle of S^5 is not trivial, the characteristic map $\chi_5 : S^4 \to SO_5$ is not null-homotopic. Consequently, to show that the structure group of S^5 cannot be reduced to ρ it is enough to show that any composition $S^4 \xrightarrow{f} SO_3 \xrightarrow{\rho} SO_5$ is null homotopic. To show this, we use the following three claims.

(a) $\pi_3(S^3) \simeq \pi_3(SO_3) \simeq \pi_3(SO_5) \simeq \mathbb{Z}$, $\pi_4(S^3) \simeq \pi_4(SO_3) \simeq \pi_4(SO_5) \simeq \mathbb{Z}_2$.
(b) The map $\rho_* : \pi_3(SO_3) \to \pi_3(SO_5)$ has a cyclic cokernel of *even* order (the 'Dynkin index' of ρ).
(c) For any topological group G and integers $k, n \geq 2$, the composition of maps $S^n \to S^k \to G$ defines a *bi-additive* map $\pi_k(G) \times \pi_n(S^k) \to \pi_n(G)$, $([f],[g]) \mapsto [f] \circ [g] := [f \circ g]$ (the 'composition product').

Claim (a) is standard (see, e.g., [It], Vol. 2, App. A, Table 6.VII, p. 1745). Claim (b) is a straightforward Lie algebraic calculation, see next subsection. For claim (c), see [Wh], Theorem (8.3), p. 479.

Now let $f : S^4 \to SO_3$ be any (pointed) continuous map and $\tilde{f} : S^4 \to S^3$ its lift to the universal double cover $\pi : S^3 \to SO_3$. By (b), the composition $\tilde{\rho} := \rho \circ \pi : S^3 \to SO_5$ has an even Dynkin index (in fact, it is the same as the index of ρ, since π, being a cover, has index 1). In particular, $[\tilde{\rho}] = 2[u] \in \pi_3(SO_5)$, for some $u : S^3 \to SO_5$. By (c), with $n = 4, k = 3, G = SO_5$, $[\rho \circ f] = [\tilde{\rho} \circ \tilde{f}] = [\tilde{\rho}] \circ [\tilde{f}] = (2[u]) \circ [\tilde{f}] = 2([u] \circ [\tilde{f}]) = 0 \in \pi_4(SO_5) \simeq \mathbb{Z}_2$. ☐



3.5. **The Dynkin index.** Here we prove claim (b) from the proof of Lemma 3.7 of the previous subsection. We begin with some background.

Let $\rho : H \to G$ be a homomorphism of compact simple Lie groups. The third homotopy group of any simple Lie group is infinite cyclic (isomorphic to $\mathbb{Z}$), hence the induced map $\rho_* : \pi_3(H) \to \pi_3(G)$ has a cyclic cokernel of order $j \in \mathbb{N}$, called the *Dynkin index* of ρ (if $\rho_* = 0$ then $j = 0$, by definition). Clearly, j is *multiplicative*, i.e., if $\tilde{H}$ is a simple compact Lie group and $\pi : \tilde{H} \to H$ is a homomorphism, then $j(\rho \circ \pi) = j(\rho)j(\pi)$.

There is a simple Lie algebraic expression for $j(\rho)$. To state it, the Killing form on any simple compact Lie algebra needs to be normalized first by $\langle \delta, \delta \rangle = 2$, where δ is the longest root. Next, the pullback by $\rho : H \to G$ of the Killing form of G is an Ad_H-invariant quadratic form on the Lie algebra of H, hence, by simplicity of H, is a non-negative multiple of the Killing form of H. This multiple turns out to be precisely the Dynkin index of ρ.

Theorem 3.8. *Let $\rho : H \to G$ be a homomorphism of compact simple Lie groups and $\rho_* : \mathfrak{h} \to \mathfrak{g}$ the induced Lie algebra homomorphism. Then*

$$\langle \rho_* X, \rho_* Y \rangle_{\mathfrak{g}} = j(\rho) \langle X, Y \rangle_{\mathfrak{h}} \tag{1}$$

for all $X, Y \in \mathfrak{h}$.

In fact, Dynkin *defined* $j(\rho)$ via Formula 1 (see [Dy, formula (2.2), p. 130]), and showed in the same article that $j(\rho)$ is an integer, without reference to its topological interpretation. Later, it was shown to have an equivalent definition via homotopy groups, as given above (we are not sure who proved it first, we learned it from [On], §2 of Chapter 5, p. 257).

Lemma 3.9. $j(\rho) = 10$ *for the irreducible representation $\rho : SO_3 \to SO_5$.*

Proof. Theorem 3.8 gives an easy to follow recipe for j. To apply it, one needs to compute first the normalization of the Killing forms of SO_3 and SO_5.

Let $\mathfrak{so}_5$ be the set of 5×5 antisymmetric real matrices, the Lie algebra of SO_5, with $\mathfrak{t} \subset \mathfrak{so}_5$ the set of block diagonal matrices of the form $(x_1 J \oplus x_2 J \oplus 0)$, where $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. The roots are $\pm x_1 \pm x_2, \pm x_1, \pm x_2$, with $\delta := x_1 + x_2$. Since $\mathrm{tr}(XY)$ is clearly an Ad-invariant non-trivial bilinear form on $\mathfrak{so}_5$, the normalized Killing form of $\mathfrak{so}_5$ is of the form $\langle X, Y \rangle = \lambda\, \mathrm{tr}(XY)$, for some $\lambda \in \mathbb{R}$. The normalization condition is $\langle \delta^\flat, \delta^\flat \rangle = 2$, where $\delta^\flat \in \mathfrak{t}$ is defined via $\delta(X) = \langle \delta^\flat, X \rangle$ for all $X \in \mathfrak{t}$. Let $\delta^\flat = \lambda'(J \oplus J \oplus 0)$, for some $\lambda' \in \mathbb{R}$. Then for all $X \in \mathfrak{t}$, $\langle \delta^\flat, X \rangle = \lambda \mathrm{tr}(\delta^\flat X) = -2\lambda\lambda' \delta(X)$, thus $-2\lambda\lambda' = 1$, so $\delta^\flat = -\frac{1}{2\lambda}(J \oplus J \oplus 0)$ and $2 = \langle \delta^\flat, \delta^\flat \rangle = \lambda \mathrm{tr}[(\delta^\flat)^2] = -1/\lambda$, hence $\lambda = -1/2$. It follows that $\langle X, Y \rangle_{\mathfrak{so}_5} = -\mathrm{tr}(XY)/2$. For $\mathfrak{so}_3$ we get by a similar argument $\langle X, Y \rangle_{\mathfrak{so}_3} = -\mathrm{tr}(XY)/4$.

Now let $\rho : SO_3 \to SO_5$ be the 5-dimensional irreducible representation on $\mathbb{R}^5$ (conjugation of traceless symmetric 3×3 matrices). Let $X = (J \oplus 0) \in \mathfrak{so}_3$. To calculate $\operatorname{tr}[(\rho_* X)^2]$, we let X act on $S^2((\mathbb{C}^3)^*)$ (complexifying, passing to the dual and adding an extra trivial summand does not affect trace). Now $x_1 \pm i x_2, x_3$ are X eigenvectors in $(\mathbb{C}^3)^*$, with eigenvalue $\pm i, 0$, hence the eigenvalues of the $\rho_* X$ action on $S^2((\mathbb{C}^3)^*)$ are $\pm 2i, \pm i, 0, 0$, and those of $(\rho_* X)^2$ are $-4,-4,-1,-1,0,0$, giving $\operatorname{tr}[(\rho_* X)^2] = -10$. Thus $j(\rho) = \langle \rho_* X, \rho_* X\rangle_{\mathfrak{so}_5}/\langle X, X\rangle_{\mathfrak{so}_3} = 2\operatorname{tr}[(\rho_* X)^2]/\operatorname{tr}(X^2) = 10$, as claimed. □

A byproduct of the proof of Theorem 1.7 is the following corollary that could be of some interest to topologists.

Corollary 3.10. *Suppose that the structure group of S^n can be reduced to a closed connected subgroup $G \subsetneq SO_n$. If $n = 4k+1 \geq 5$, but $n \neq 9, 17$ or 133, then G is conjugate to the standard inclusion of SO_{4k}, U_{2k} or SU_{2k} in SO_{4k+1}. For $n = 9$, G is conjugate to the standard inclusion of SO_8, U_4, SU_4 or $Spin_7 \subset SO_8$ in SO_9.*

Proof. By Theorem 1.7(b), such a G is conjugate to a subgroup of the standard inclusion $SO_{4k} \subset SO_{4k+1}$, acting transitively on S^{4k-1}. The only closed connected subgroups $G \subset SO_{4k}$ acting transitively on S^{4k-1}, in the said dimensions, are the standard linear actions of SO_{4k}, $U_{2k}, SU_{2k}, Sp_k Sp_1, Sp_k U_1, Sp_k$ on $\mathbb{R}^{4k} = \mathbb{C}^{2k} = \mathbb{H}^k$, or the spin representation of $Spin_7$ on $\mathbb{C}^4$ (see, e.g., [Be, 7.13, p. 179]). But the groups $Sp_k Sp_1, Sp_k U_1, Sp_k$, $k \geq 1$, cannot occur as structure groups of S^{4k+1}, since they contain the last one, Sp_k, which is excluded by Theorem 2.1 of [CC]. □

Remark 3.11. For $n = 17$, the group $Spin_9 \subset SO_{16}$ acts transitively on S^{15}, but we do not know if the structure group of S^{17} could be reduced to it. For $n = 133$, as explained before, we do not know if the group $E_7 \subset SO_{133}$ (or some subgroup of it acting irreducibly on $\mathbb{R}^{133}$) may appear as a reduction of the structure group of S^{133}.

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