The canonical bundle of a hermitian manifold

Gil Bor & Luis Hernández-Lamoneda *
Centro de Investigación en Matemáticas, A.C. (CIMAT)
Guanajuato, México

7 october, 1998

Abstract

This note contains a simple formula (Proposition 1 in Section 3) for the curvature of the canonical line bundle on a hermitian manifold, using the Levi-Civita connection (instead of the more usual hermitian connection, compatible with the holomorphic structure). As an immediate application of this formula we derive the following: the six-sphere does not admit a complex structure, orthogonal with respect to any metric in a neighborhood of the round one. Moreover, we obtain such a neighborhood in terms of explicit bounds on the eigen-values of the curvature operator. This extends a theorem of LeBrun.

Keywords: Hermitian manifold, almost-complex structure, canonical bundle, curvature.
AMS Subject classification number: 53C55.

1 Introduction

First, some standard definitions. An almost-complex structure on an even-dimensional manifold $M^{2n}$ is a smooth endomorphism $J : TM \to TM$, such that $J^2 = -Id$. The standard example is $M = \mathbb{C}^n$ with $J$ given by the usual scalar multiplication by $i$. A holomorphic map between two almost-complex manifolds $(M_1, J_1)$ and $(M_2, J_2)$ is a smooth map $f : M_1 \to M_2$ satisfying $df \circ J_1 = J_2 \circ df$. An almost-complex structure is said to be integrable, or is called simply a complex structure, if it is locally holomorphically diffeomorphic to the standard example; in other words, for each $x \in M$ there exist neighborhoods $U \subset M$, $x \in U$, and $V \subset \mathbb{C}^n$, and a holomorphic diffeomorphism $f : U \to V$.

Given an even-dimensional manifold, how is one to decide if it admits a complex structure? There are some, more or less obvious, necessary conditions

---

*Both authors received support from grants 0329-P-E of CONACyT and E130.728 of CONACyT-CSIC
(e.g. the existence of an almost-complex structure, which can be tested by characteristic classes), but in general there is no known answer to this question. A well-known example, so far undecided, is the 6-sphere (this is the only interesting dimension, because in all other dimensions $n \neq 2, 6$, the $n$-sphere does not admit even an almost-complex structure). This space admits a non-integrable almost-complex structure, but it is unknown as yet if it admits a complex structure.

A related question is that of the existence of an orthogonal complex structure. Here the set-up is the following: given an even-dimensional riemannian manifold $(M, g)$, one is looking for an integrable almost-complex structure $J$ which is orthogonal with respect to $g$; that is, $g(X, Y) = g(JX, JY)$, for all $X, Y \in T_m M$ and $m \in M$. One calls such a pair $(g, J)$ a hermitian structure. The problem here is then that of extending a given riemannian structure to a hermitian one.

One way of analyzing the problem of the existence of orthogonal complex structures is to consider the space of all orthogonal almost-complex structures. These are sections of a bundle over $M$, whose fiber at a point of the manifold consists of all (linear) orthogonal complex structures on the tangent space at that point. The total space $Z$ of this bundle is called the twistor space associated to $(M, g)$ and it admits a tautological almost-complex structure. Then the idea is to translate differential geometric problems on $M$ to complex-geometric problems on $Z$. For example, an orthogonal almost-complex structure on $M$ is given, by definition, by a section of $Z \to M$; it will be integrable if the section is holomorphic, thus embedding $M$ as a complex sub-manifold of $Z$. In other words, the problem of orthogonal complex structures on $M$ is translated into that of certain complex submanifolds of $Z$. This approach leads to the proof of C. LeBrun of non-existence of an orthogonal complex structure on $S^6$ relative to the round metric [2]. The twistor space $Z$ in this case turns out to be Kähler, so that an orthogonal complex structure on $S^6$ would give an embedding of $S^6$ as a complex submanifold of a Kähler one, thus inheriting a Kähler structure, which is clearly impossible for $S^6$ (since $H^2(S^6) = 0$). For more information on this approach to orthogonal complex structures we recommend the survey article of S. Salamon [3].

Here we suggest a different construction, considerably more elementary. This is based on the observation that the curvature of a connection on a complex line bundle is a closed two-form (representing the first Chern class of the line bundle, up to a constant), so one can try to use the given data $(g, J)$ on $M$ to construct a line bundle with connection whose curvature two-form is non-degenerate, i.e. a symplectic form. On certain manifolds this might be impossible (e.g. on a compact manifold with $H^2 = 0$), so if one uses a connection coming from the Levi-Civita connection on $(M, g)$ then one obtains in this way a curvature obstruction for the existence of an orthogonal complex structure.
A natural complex line bundle to consider, for a given complex structure, is the so-called canonical line bundle $K := \Lambda^{n,0}(M)$ – the bundle of $(n,0)$-forms, or the top exterior power of the holomorphic cotangent bundle. Now there are two natural ways to use the hermitian structure on $M$ to equip $K$ with a connection. First, the complex structure on $M$ induces a holomorphic structure on $K$ and the riemannian metric on $M$ induces a hermitian metric on $K$; these two in turn determine uniquely a canonical hermitian connection (a metric-preserving connection whose $(0,1)$-part coincides with the $\bar{\partial}$-operator of the complex structure on $M$; see for example Griffiths and Harris [1], p. 73). The other choice of a connection on $K$ comes from the Levi-Civita connection on $TM$, extended (by the Leibniz rule) to the bundle of exterior $n$-forms $\Lambda^n(M)$, complexified, then projected orthogonally to the sub-bundle $K \subset \Lambda^n_c(M)$.

Unless the orthogonal complex structure happens to be Kähler (i.e. the Kähler 2-form $\omega = g(J\cdot, \cdot)$ is closed), these two choices of a connection are different. We make here the second choice, the one coming from the Levi-Civita connection, as it seems to us more natural from a Riemannian geometric point of view, e.g. for relating the resulting curvature 2-form of the canonical bundle with the Riemann curvature tensor of $(M,g)$.

The outcome then is a rather simple formula for the curvature of the canonical line bundle on a hermitian manifold (Proposition 1 of Section 3). From this formula it becomes obvious that a complex structure compatible with the round metric on the sphere will render the curvature 2-form of the corresponding canonical line bundle a symplectic form (in fact Kähler), and that this property will be maintained for nearby metrics (Corollaries 2 and 3 of Section 4).  

We shall now outline the details of the calculation indicated above. We need to recall first some standard terminology.

Let $E \to M$ be a complex hermitian vector bundle over a differentiable manifold, with a hermitian connection $D : \Gamma(E) \to \Gamma(T^*(M) \otimes E)$, i.e.
\[
d(s_1, s_2) = \langle Ds_1, s_2 \rangle + \langle s_1, Ds_2 \rangle
\]
for any two sections $s_1, s_2 \in \Gamma(E)$.

The curvature $R$ of $(E, D)$ is defined by first extending $D$ to $\Gamma(\Lambda^k(M) \otimes E) \to \Gamma(\Lambda^{k+1}(M) \otimes E)$,
\[
D(\alpha \otimes s) := d\alpha \otimes s + (-1)^k \alpha \otimes Ds,
\]
then
\[
R := D^2 \in \Gamma(\Lambda^2(M) \otimes \text{End}(E)).
\]

---

1Claude LeBrun has informed us recently that his proof also extends to metrics near the round one, but this requires embedding the usual twistor space inside a larger one. Also, after completing the work described here we found two articles ([4] and [5]) containing ideas close to ours.
If $E_0 \subset E$ is a sub-bundle then there is an induced hermitian connection on $E_0$ as follows: let $s_0$ be a section of $E_0$, and let $(E_0) \perp$ be the orthogonal complement of $E_0$ in $E$, then decompose orthogonally

$$Ds_0 = D_0s_0 + \Phi s_0,$$

with

$$D_0s_0 \in \Gamma(T^*(M) \otimes E_0), \quad \Phi s_0 \in \Gamma(T^*(M) \otimes (E_0)\perp).$$

One then verifies easily that $D_0$ defines a hermitian connection on $E_0$ and that $\Phi$ is “tensorial”, i.e. a section of $T^*(M) \otimes \text{Hom}(E_0, (E_0)\perp)$, called the second fundamental form of $E_0$ in $E$.

Now there is a well-known formula for the curvature of $(E_0, D_0)$ in terms of the curvature $R$ of $(E, D)$ and the second fundamental form $\Phi$ of $E_0$ in $E$. It is given by

$$\Omega = \pi_0 \circ R \circ \pi_0^* + \Phi^* \wedge \Phi,$$

where $\pi_0 : E \to E_0$ is orthogonal projection. The (easy) calculation can be found for example in [1], p.78.

In our case, starting with the Levi-Civita connection on $\Lambda^n(M)$ and projecting onto the canonical line-bundle $K = \Lambda^n,0(M) \subset \Lambda^n(M)$, we find out the following:

1. $\pi_0 \circ R \circ \pi_0^* = iR(\omega)$, where $\omega$ is the Kähler form and $R$ is the interpretation of the Riemann curvature tensor of $M$ as an operator in $\text{End}(\Lambda^2(M))$ (see the corollary to Lemma 1 in Section 3).

2. The second fundamental form $\Phi \in \Lambda^1(M) \otimes \text{Hom}(\Lambda^n,0, (\Lambda^n,0)\perp)$ is of type $(1,0)$, hence $\Phi^* \wedge \Phi$ is non-positive (see Lemmas 2 and 3 in Section 3; see next section, Definition 2, for the sign convention).

The first fact does not require even the integrability of the orthogonal almost-complex structure, i.e. it holds also for almost-hermitian manifolds. The second one does depend on the integrability (in fact, it can be shown to be equivalent to the integrability of the almost-complex structure).

We use these two basic results to deduce rather easily the non-degeneracy of the 2-form $\Omega$ in the proof of the above mentioned theorem of LeBrun, as well as its extension to metrics which are nearby the round one (see Section 4).

2 Some definitions and notation

First, to make sense of Formula (1) in the Introduction, we need to review some terminology.

Let $V$ be a real $2n$-dimensional vector space with a euclidean inner product $(\cdot, \cdot)$ and a linear orthogonal almost-complex structure $J$. We extend the inner
product \((\cdot, \cdot)\) on \(V\) in the usual way to the real exterior algebra \(\Lambda^*(V^*)\), by declaring the \(k\)-forms \(\{\eta_1 \wedge \ldots \wedge \eta_k\} | 1 \leq i_1 < \ldots < i_k \leq 2n\} \) an orthonormal basis of \(\Lambda^k(V^*)\), where \(\{\eta_1, \ldots, \eta_{2n}\}\) is the dual basis of an orthonormal basis of \(V\).

We denote also by \((\cdot, \cdot)\) the complex-linear extension of the euclidean inner product \((\cdot, \cdot)\) to the complexified vector spaces \(\Lambda_k^*(V^*) = \Lambda^k(V^*) \otimes \mathbb{C}\). The hermitian inner-product on these spaces is thus given by \(\langle \phi, \psi \rangle = (\phi, \overline{\psi})\).

Next, let \(W\) be a complex vector space with an hermitian inner product \((\cdot, \cdot)\) and denote by \(\text{End}_C(W)\) the complex-linear endomorphisms of \(W\). Denote by \(\text{End}(V)\) the real endomorphisms of \(V\).

All tensor products, unless denoted otherwise, are over the reals.

**Definition 1** Let \(V\) and \(W\) be as above, and \(\alpha, \beta \in V^* \otimes \text{End}_C(W)\) two endomorphism-valued 1-forms.

1. The wedge product \(\alpha \wedge \beta \in \Lambda^2(V^*) \otimes \text{End}_C(W)\) is defined by
   \[
   \alpha \wedge \beta(X, Y) := \alpha(X) \circ \beta(Y) - \alpha(Y) \circ \beta(X).
   \]
   Equivalently, if \(\alpha = a \otimes A, \beta = b \otimes B\), where \(a, b \in V^*\) and \(A, B \in \text{End}_C(W)\), then \(\alpha \wedge \beta = (a \otimes b) \circ (A \circ B)\).

2. The adjoint \(\alpha^* \in V^* \otimes \text{End}_C(W)\) is defined by
   \[
   \alpha^*(X) = [\alpha(X)]^*.
   \]
   Equivalently, for \(\alpha = a \otimes A\), \(a^* = a \otimes A^*\).

Note that when extending the notation to complex forms in \(V_C^* \otimes \text{End}_C(W)\), one has that \(\alpha^*(Z) = [\alpha(Z)]^*, Z \in V_C\), so that if \(\alpha = \phi \otimes A\), where \(\phi \in V_C^*\), then \(\alpha^* = \phi \otimes A^*\). (Proof: if \(Z = X + iY\), then \(\alpha^*(Z) = \alpha^*(X) + i\alpha^*(Y) = [\alpha(X)]^* + i[\alpha(Y)]^* = [\alpha(X) - i\alpha(Y)]^* = [\alpha(Z)]^*\). Hence if \(\alpha\) is of type \((1, 0)\) then \(\alpha^*\) is of type \((0, 1)\) etc.

Next, we need to make some convention concerning positivity (watch for a confusing error in [1], pp. 29 & 79, around this definition).

**Definition 2**

1. A 2-form \(\omega \in \Lambda^2(V^*)\) is called positive, \(\omega > 0\), if \(B(X, Y) = \omega(X, JY)\) is a symmetric positive bilinear form. Equivalently: \(\omega\) is positive if it is a real 2-form of type \((1, 1)\) (that’s the “symmetric” requirement) and \(\omega(X', X')/i > 0\) for all non-zero \(X' \in V^{1, 0}\), where \(V_C = V^{1, 0} \oplus V^{0, 1}\) is the decomposition of the complexification of \(V\) into \(\pm i\) eigen-spaces of \(J\). Obviously, a positive (or negative) 2-form is non-degenerate.
2. Now let \( \Omega \in \Lambda^2(V^*) \otimes \text{End}_C(W) \) be a 2-form on \( V \) with values in anti-hermitian endomorphisms on \( W \), so \( i\Omega \) is an hermitian-valued 2-form (we have in mind the curvature of a hermitian connection). Then \( \Omega \) is called positive, \( \Omega > 0 \), if \( i\Omega w, w \) is a positive 2-form for all non-zero \( w \in W \). Equivalently, \( \Omega > 0 \) if it is an \( \text{End}(W) \)-valued \((1,1)\)-form such that \( \Omega(X', \overline{X'}) \) is a positive hermitian operator for all non-zero \( X' \in V^{1,0} \).

We define similarly \( \Omega \geq 0, \Omega < 0 \), etc.

A word of caution: According to the last definition, the Kähler form \( \omega = (J, \cdot, \cdot) \) on \( V \) is a real positive 2-form, whereas \( i\omega \) is an imaginary negative form.

Definition 3

Let \( A \in \text{End}(V) \) be an antisymmetric endomorphism on \( V \), i.e. \( (Av, w) = -(v, Aw) \) for all \( v, w \in V \). Define

1. \( \hat{A} \in \Lambda^2(V^*) \) by \( \hat{A}(v, w) = (Av, w) \).

2. \( A^* \in \text{End}(V^*) \) by \( (A^* \eta)(v) = \eta(Av) \), as well as its extension to \( \Lambda^*(V^*) \) as a derivation:

\[
A^*(\alpha \wedge \beta) = (A^* \alpha) \wedge \beta + \alpha \wedge (A^* \beta).
\]

We use throughout the article the shorthand notation \( \Lambda^k(M) \) for the bundle of alternating \( k \)-forms \( \Lambda^k(T^*M) \).

Definition 4

1. Let \( R \in \Lambda^2(V^*) \otimes \text{End}(V) \) (we have in mind the curvature tensor of the Levi-Civita connection on a riemannian manifold). Define \( \mathcal{R} \in \text{End}(\Lambda^2(V^*)) \) as follows: if \( R = \sum_j \alpha_j \otimes A_j \), where \( \alpha_j \in \Lambda^2(V^*) \) and \( A_j \in \text{End}(V) \), then

\[
\mathcal{R}(\beta) = -\sum_j \alpha_j(\hat{A}_j, \beta), \quad \beta \in \Lambda^2(V^*).
\]

2. Applying this definition to the curvature tensor of a riemannian manifold \( R \in \text{End}(\Lambda^2(M) \otimes \text{End}(TM)) \), we obtain the so-called curvature operator \( \mathcal{R} \in \text{End}(\Lambda^2(T^*M)) \).

Another word of caution concerning sign conventions: we have made the choice of signs in the above definitions so as to make \( \mathcal{R} \) coincide with the curvature operator as defined in riemannian geometry. Thus, for example, the round sphere has a positive curvature operator (in fact, it is the identity operator). This is also tied up with our definition \( R = D^2 \), where there seems to be a conflict in the literature. In complex geometry it is usual to define the curvature of a connection by \( D^2 \), as we did in the Introduction. Thus, the curvature of the canonical bundle of \( \mathbb{CP}^1 \) is \( i \) times the area form. In riemannian geometry on the other hand, probably for historical reasons, the curvature tensor of the Levi-Civita connection is defined by the formula \( \nabla_{[X,Y]} - [\nabla_X, \nabla_Y] \), which amounts to defining \( R = -D^2 \). Our sign choice in Definition 4 is made so as to reconcile this conflict.
3 Three lemmas on hermitian structures

The first lemma is quite simple, and has probably appeared elsewhere. The second is essentially in Griffiths and Harris ([1], p.79, after overcoming the positivity confusion). The third is a curious fact about the Levi-Civita connection on a hermitian manifold, probably known, though we could not find it in the literature.

Let $V$ be, as in the last section, a euclidean $2n$-dimensional real vector space with an orthogonal almost-complex structure $J$, and let $\omega = (J \cdot, \cdot)$ denote the associated Kähler 2-form.

Lemma 1 Let $A \in \text{End}(V)$ be an antisymmetric endomorphism of $V$ and let $\pi_0 : \Lambda^n_0(V^*) \to \Lambda^n,0(V^*)$ denote orthogonal projection. Then

$$\pi_0 \circ A^* \circ \pi_0^* = i(\hat{A},\omega),$$

where $A^* \in \text{End}(\Lambda^n_0(V^*))$ and $\hat{A} \in \Lambda^2(V^*)$ are given above in Definition 3.

Proof. Choose a unitary basis $\theta_1, \ldots, \theta_n$ for $(V^*)^{1,0}$, so that

$$\omega = i(\theta_1 \wedge \bar{\theta}_1 + \cdots + \theta_n \wedge \bar{\theta}_n).$$

Now $\psi = \theta_1 \wedge \cdots \wedge \theta_n$ is a unitary element of $\Lambda^n_0(V^*)$, hence

$$\pi_0 \circ A^* \circ \pi_0^* = (A^* \psi, \bar{\psi}) = (A^* \theta_1 \wedge \theta_2 \wedge \cdots \wedge \theta_n, \bar{\theta}_1 \wedge \bar{\theta}_2 \wedge \cdots \wedge \bar{\theta}_n) + \cdots$$

$$= (A^* \theta_1, \bar{\theta}_1) + \cdots + (A^* \theta_n, \bar{\theta}_n).$$

Now, given any $\alpha, \beta \in V^*$, one can check easily from our definition of $\hat{A}$ that

$$(A^* \alpha, \beta) = -(\hat{A}, \alpha \wedge \beta),$$

hence

$$\pi_0 \circ A^* \circ \pi_0^* = -(\hat{A}, \theta_1 \wedge \bar{\theta}_1 + \cdots + \theta_n \wedge \bar{\theta}_n) = i(\hat{A},\omega),$$

as claimed. □

Corollary 1 Let $(M, g, J)$ be an almost-hermitian manifold and let $\pi_0 : \Lambda^2(M) \otimes \Lambda^n_0(M) \to \Lambda^2(M) \otimes \Lambda^{n,0}(M)$ denote orthogonal projection in the second factor. Then

$$\pi_0 \circ R \circ \pi_0^* = iR(\omega),$$

where $\omega = g(J \cdot, \cdot)$ is the Kähler form, $R \in \Gamma(\Lambda^2(M) \otimes \text{End}(\Lambda^n_0(M))$ is the curvature of the connection induced on $\Lambda^n_0(M)$ by the Levi-Civita connection on $TM$, and $R$ is the curvature operator associated to the riemannian metric (as in Definition 4 above).
follows immediately from the previous lemma and the definition of $A$.

**Proof.** The main point to notice is that if the curvature tensor of a connection on $TM$ is given (locally) by $\sum \alpha_j \otimes A_j$, where $\alpha_j \in \Gamma(\Lambda^2(M))$ and $A_j \in \Gamma(\text{End}(TM))$, then the curvature tensor of the induced connection on $\Lambda^2(M)$ is given by $-\sum \alpha_j \otimes A_j^\ast$, with $A_j^\ast$ given by Definition 3. The result now follows immediately from the previous lemma and the definition of $\mathcal{R}$. □

**Lemma 2** If $\Phi \in \Lambda^{1,0}(V^\ast) \otimes \text{End}_C(W)$, where $W$ is a hermitian vector space, then $\Phi^\ast \wedge \Phi \leq 0$.

**Proof.** As noted above (Section 2, after Definition 1), $\Phi^\ast$ is of type $(0,1)$, hence $\Phi^\ast \wedge \Phi$ is of type $(1,1)$. Next, for any $X' \in V^{1,0}$,

$$(\Phi^\ast \wedge \Phi)(X', \bar{X}') = \Phi^\ast(X')\Phi(\bar{X}') - \Phi^\ast(\bar{X}')\Phi(X') = -\Phi^\ast(\bar{X}')\Phi(X') = -[(\Phi(X'))^\ast \Phi(X')]$$

and the claim follows since $A^\ast A$ is a hermitian non-negative operator for any $A \in \text{End}_C(W)$. □

**Lemma 3** Let $M^{2n}$ be a riemannian manifold with an orthogonal complex structure (i.e. a hermitian manifold). Denote by $\nabla$ the Levi-Civita connection on $TM$, as well as its extension to $\Lambda^\ast(M)$ (using the Leibniz rule). Then the second fundamental form of the canonical bundle $K = \Lambda^{n,0}(M) \subset \Lambda^2(M)$, with respect to the Levi-Civita connection, is of type $(1,0)$ (as in the previous Lemma).

**Proof.** In fact, the statement is true for all the sub-bundles $\Lambda^{k,0}(M) \subset \Lambda^k_C(M)$, $k = 1, 2, \ldots, n$, and follows from the case $k = 1$. To see this, let $\theta_1, \ldots, \theta_n$ be a local framing of $\Lambda^{1,0}(M)$, and

$$\nabla \theta_i = \sum_j \left( \alpha_{ij} \otimes \theta_j + \beta_{ij} \otimes \bar{\theta}_j \right), \quad \alpha_{ij}, \beta_{ij} \in \Lambda^1_C(M).$$

Then for $k = 1$ the claim is that the 1-forms $\beta_{ij}$ are of type $(1,0)$. If this is true, then for any $k \geq 1$,

$$\nabla (\theta_1 \wedge \ldots \wedge \theta_k) = (\nabla \theta_1) \wedge \theta_2 \wedge \ldots \wedge \theta_k + \theta_1 \wedge (\nabla \theta_2) \wedge \theta_3 \wedge \ldots \wedge \theta_k + \ldots = \sum_j \beta_{1i,j} \otimes (\theta_1 \wedge \theta_2 \wedge \ldots \wedge \theta_k) + \beta_{2i,j} \otimes (\theta_1 \wedge \bar{\theta}_j \wedge \ldots \wedge \theta_k) + \ldots$$

$$\ldots + \text{(something in } \Lambda^1_C \otimes \Lambda^{k,0})$$

so that the second fundamental form of $\Lambda^{k,0}(M) \subset \Lambda^k_C(M)$ is of type $(1,0)$.

Now for the case $k = 1$, i.e. to see that the 1-forms $\beta_{ij}$ above are of type $(1,0)$, we argue as follows. First, we pick the frame $\theta_1, \ldots, \theta_n$ to be a unitary frame, i.e. $(\theta_i, \theta_j) = \delta_{ij}$. It then follows that

$$0 = d(\theta_i, \theta_j) = (\nabla \theta_i, \theta_j) + (\theta_i, \nabla \theta_j) = \beta_{ij} + \beta_{ji},$$

8
i.e. $\beta_{ij} = -\beta_{ji}$.

Next, by the torsion-freeness of $\nabla$, we have

$$d\theta_i = \text{anti-symmetrization of } \nabla \theta_i = \sum_j (\alpha_{ij} \wedge \theta_j + \beta_{ij} \wedge \bar{\theta}_j).$$

Now, for an integrable almost-complex structure, we have the vanishing of the $(0,2)$-component of $d : \Lambda^{1,0} \to \Lambda^2$, hence, taking $(0,2)$-components of the last equation we have

$$0 = \sum_j \beta''_{ij} \wedge \bar{\theta}_j,$$

where $\beta''_{ij}$ denotes the $(0,1)$-component of $\beta_{ij}$. If we write $\beta''_{ij} = \sum_k \beta''_{ijk} \bar{\theta}_k$, then the last equation reads

$$0 = \beta''_{ijk} - \beta''_{ikj},$$

i.e. $\beta''_{ijk} = \beta''_{ikj}$, and this, combined with the previous $\beta''_{ijk} = -\beta''_{jik}$ yields $\beta''_{ij} = 0$ (we use here "the $S_3$-lemma": any tensor $T_{ijk}$ which is symmetric in one pair of indices and anti-symmetric in another pair is identically zero). □

In summary, we have obtained the following:

**Proposition 1** Let $M^{2n}$ be a hermitian manifold (a riemannian manifold with an orthogonal complex structure). If we equip its canonical bundle $\Lambda^{n,0}(M)$ with the connection induced by the Levi-Civita connection of $M$, then its curvature $\Omega$ is given by the formula

$$\Omega = iR(\omega) + \Phi^* \wedge \Phi,$$

where $\omega$ is the Kähler 2-form associated with the hermitian structure, $R$ is the curvature operator associated to the riemannian structure (see Definition 4 in Section 2), and

$$\Phi^* \wedge \Phi \leq 0.$$

This last inequality means, recalling our sign conventions of Definition 2 in Section 2, that $i(\Phi^* \wedge \Phi)$ is a non-positive real $(1,1)$-form.

**Remark.** There is an analogous statement for an almost-Kähler manifold, i.e. when the almost-complex structure is not necessarily integrable but the Kähler form is closed (so we have a symplectic manifold). The difference is that in this case the second fundamental form is of type $(0,1)$, hence the correction term $\Phi^* \wedge \Phi$ in the curvature formula is non-negative.
4 Some applications

As an immediate corollary to Proposition 1 we obtain the following result of LeBrun [2]:

**Corollary 2** There is no complex structure on $S^6$ which is orthogonal with respect to the round metric.

**Proof.** For the round metric on a sphere, $R$ is the identity operator. Therefore, for any orthogonal complex structure, the formula of Proposition 1 for the curvature of the canonical line bundle gives

$$\Omega = i\omega + \Phi^\ast \wedge \Phi \leq i\omega < 0.$$ 

It follows that the closed 2-form $\Omega$ is non-degenerate, i.e. symplectic, which is impossible since $H^2(S^6) = 0$. □

The next corollary extends the above conclusion to a $C^2$-neighbourhood of the round metric on $S^6$.

**Corollary 3** Let $g$ be a riemannian metric on $S^6$ satisfying the following conditions:

- The curvature operator $R$ is positive (i.e. all its eigen values are positive).
- At each point $x \in S^6$, the ratio of the largest eigen-value $\lambda_{\text{max}}$ of $R$ to the lowest eigen-value $\lambda_{\text{min}}$ satisfies $\lambda_{\text{max}}/\lambda_{\text{min}} < 7/5 = 1.4$.

Then $(S^6, g)$ does not admit an orthogonal complex structure.

The proof of the last corollary is based on the following

**Lemma 4** Let $V$ be a $2n$-dimensional euclidean vector space with an orthogonal complex structure $J$, and let $\omega = (J \cdot, \cdot)$ be the associated Kähler form. Let $\Omega_0$ be an imaginary $(1,1)$-form satisfying $\Omega_0 \leq i\omega$. (See Definition 2 in Section 2 for the sign convention. Note that in particular, since $i\omega < 0$, $\Omega_0$ is also negative, hence non-degenerate).

Then, if $\Omega$ is any imaginary 2-form satisfying

$$\|\Omega - \Omega_0\| < \frac{1}{2\sqrt{n}},$$

$\Omega$ is non-degenerate.

**Proof.** First, a brief reminder about norms. We use the euclidean norm on $V$ to embed $\Lambda^2(V^\ast) \subset \text{End}(V)$ as antisymmetric endomorphisms: $\alpha \mapsto A$, where $A$ is given by $(Av, w) = \alpha(v, w)$. In fact, this is the inverse of our map of Definition 3 in Section 2, $A \mapsto A = \alpha$. 

10
Next, the euclidean structure on $V$ induces a euclidean norm $\| \cdot \|_E$ on $\text{End}(V)$ by $\|A\|_E^2 = \sum |A_{ij}|^2$, where $A_{ij}$ are the components of an element $A \in \text{End}(V)$ with respect to an orthonormal basis of $V$. This norm is multiplicative, i.e. $\|AB\|_E \leq \|A\|_E \|B\|_E$. Using this multiplicativity property, one can show that if $A \in \text{End}(V)$ satisfies $\|A\|_E < 1$, then $I + A + A^2 + \ldots$ is convergent, thus giving an inverse to $I - A$.

Unfortunately, the norm $\| \cdot \|_E$ induces on $\Lambda^2(V^*)$ a norm which differs by a constant from the standard norm on $\Lambda^2(V^*)$: for any 2-form $\beta$, $\|\beta\|_E = \sqrt{2} \|\beta\|$. For example, the Kähler form $\omega$ has (standard) norm $\sqrt{n}$, whereas the corresponding endomorphism, namely $J$, has norm $\sqrt{2n}$. In what follows, we will work with the $\| \cdot \|_E$ norm on 2-forms.

Now, we can diagonalize $\omega$ and $\Omega_0$ simultaneously (over $\C$), obtaining

$$\omega = i \sum \theta_j \wedge \bar{\theta}_j, \quad \Omega_0 = \sum \lambda_j \theta_j \wedge \bar{\theta}_j,$$

with $\{\theta_j\}$ a unitary frame, and the condition $\Omega_0 \leq i \omega$ implies $\lambda_j \leq -1$. Then

$$\Omega_0^{-1} = \sum \frac{1}{\lambda_j} \theta_j \wedge \bar{\theta}_j,$$

thus

$$\|\Omega_0^{-1}\|_E^2 = 2 \sum \left| \frac{1}{\lambda_j} \right|^2 \leq 2n.$$

Now,

$$\Omega = \Omega_0 + (\Omega - \Omega_0) = \Omega_0 (I + \Omega_0^{-1}(\Omega - \Omega_0)),$$

and our condition of $\|\Omega - \Omega_0\| < 1/(2\sqrt{n})$ translates to

$$\|(\Omega - \Omega_0)\|_E < \frac{1}{\sqrt{2n}},$$

hence

$$\|\Omega_0^{-1}(\Omega - \Omega_0)\|_E \leq \|\Omega_0^{-1}\|_E \|(\Omega - \Omega_0)\|_E < \sqrt{2n} \cdot \frac{1}{\sqrt{2n}} = 1,$$

and so $\Omega$ is non-degenerate.

Proof of Corollary 3. Let us suppose there is a complex structure on $S^6$ which is orthogonal with respect to a metric $g$ whose curvature operator satisfies the said conditions. From Proposition 1, the curvature of the associated canonical line bundle is given by

$$\Omega = i \mathcal{R}(\omega) + \Phi^* \wedge \Phi = i(\mathcal{R}\omega - \omega) + \Omega_0,$$

where $\Omega_0 = i \omega + \Phi^* \wedge \Phi \leq i \omega$. Now we apply the previous lemma. We conclude that $\Omega$ is non-degenerate provided $\|\mathcal{R}\omega - \omega\| < 1/(2\sqrt{3})$ (pointwise).
Now, by rescaling the metric if necessary (this does not affect of course the orthogonality of the complex structure), we can bring the eigen-values of $R$ to the range $(5/6, 7/6)$, so that the eigen-values of $R - I$ are in the range $(-1/6, 1/6)$. This implies that $\|(R - I)\alpha\| < (1/6)\|\alpha\|$ (pointwise) for any $\alpha \in \Lambda^2(M)$, so in particular

$$\|R\omega - \omega\| < \frac{1}{6} \|\omega\| = \frac{1}{6} \sqrt{3} = \frac{1}{2\sqrt{3}}.$$ 

And so, according to the lemma above, the closed 2-form $\Omega$ is non-degenerate, i.e. symplectic, which is impossible since $H^2(S^6) = 0$. □

Remarks.
1. It is tempting to generalize Corollary 3 to the case of a hermitian structure on a $2n$-dimensional manifold with a positive curvature operator satisfying $\lambda_{\text{max}}/\lambda_{\text{min}} < (2n + 1)/(2n - 1)$. Unfortunately, such a generalization is useless, because of the well-known “sphere-theorems”, which implies that the universal cover of a complete riemannian manifold satisfying our curvature bound is a sphere, on which a complex structure is in question only in dimension 6 (because in all dimensions except 2 and 6 the $n$-sphere does not admit even an almost-complex structure), so we are back to our case.
2. However, we believe that one should be able to use Proposition 1 beyond what we have done here, because of the following argument. The condition of orthogonality of a complex structure with respect to a riemannian metric is obviously conformally invariant. On the other hand, the curvature restriction in Corollary 3 is not conformally invariant. Thus, Corollary 3 can be improved by including any metric on $S^6$ which is conformal to a metric satisfying the given curvature condition, but one hopes for a more explicit condition, say in terms of the Weyl tensor. So far, we were not able to derive such a condition.
3. Another direction in which one could possibly use Proposition 1 is by applying it to some specific classes of hermitian structures. In such cases one may be able to give a more delicate estimate of the terms in the formula of Proposition 1, especially the $R\omega$ term.

References


