LECTURE 1.1.

Lectures on Ricci-limit spaces

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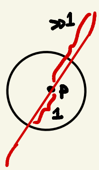
CONTENTS

I. Preliminaries on Riemannian Geometry & Bounds on curvature

II. Almost splitting theorem

On $Ric \geq -\delta$ with $\delta \ll 1$

$$d_{GH}\Big($$



$$,\ B_1^{\mathbb{R}\times X}\Big) \ll 1$$

III. Almost volume cone $\Rightarrow$ Almost metric cone

...

References [1st lecture]

Books

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[DoCarmo92] "Riemannian geometry" (1992) by M. Do Carmo

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Articles

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Based on my master thesis "Limits of Riemannian manifolds with Ricci curvature bounded from below" (Advisors: Luigi Ambrosio & Luciano Mari).

DISTANCE FUNCTIONS ON RIEMANNIAN MANIFOLDS

Reference: [Petersen16, Chapters 5-13
DoCarmo92, Chapters 3-7-9]

Setting (M,g) Riemannian manifold with Levi-Civita connection ∇

Def. $\gamma : I \to M$ is a geodesic if

$$\nabla_{\dot\gamma}\dot\gamma \equiv 0 \quad \text{on } I$$

Def. Given $p, q \in M$

$$d(p,q) := \inf\left\{ \int_I g(\dot\gamma(t),\dot\gamma(t))^{1/2}\,dt : \begin{array}{c}\gamma \text{ A.C. curve} \\ \gamma(0)=p,\ \gamma(1)=q\end{array} \right\}$$

Def. The exponential map $\exp : U \subseteq TM \to M$ is defined as follows

$$\exp(\underset{\substack{\uparrow \\ T_pM}}{v}) := \sigma_v(1)$$

where σ_v is the geodesic with $\begin{cases}\sigma_v(0)=p \\ \dot\sigma_v(0)=v\end{cases}$

Remark

1) Geodesic are "locally" length minimizing

11) d induces the manifold topology.

111) [Hopf-Rinow] TFAE:

   a) (M,d) is complete metric space

   b) $\forall p$ $\exp_p$ is defined on all T_pM

   c) $\exists p$ $\exp_p$ is defined on all T_pM

   d) closed and bounded sets of M are compact

If any a)-d) holds then $\forall p,q \in M$ $\exists$ a minimizing (segment) geodesic $\gamma: I \to M$ connecting p to q. It might not be unique...

[From now on (M,d) complete]

Def. $p \in M$, $v \in T^1_pM$ $\boxed{g_p(v,v)=1}$

$C_p(v) := \sup\{t>0 \mid \exp_p(tv) \text{ is minimizing in } (0,t)\}$

$seg(p) := \{tv \in T_pM \mid v \in T^1_pM,\ t \in [0, c_p(v)]\}$

$seg^0(p) := \{tv \in T_pM \mid v \in T^1_pM,\ t \in [0, c_p(v))\}$

$U_p := \exp_p(seg^0(p))$ $\qquad$ $cut(p) := M \setminus U_p$.

Thm.

i) $c: T^1 M \to \mathbb{R}^+$ is continuous

ii) $\forall p \in M$ $\quad cut(p)$ is closed and negligeble

iii) $\forall p \in M$ $\quad seg^\circ(p)$ is open and $\exp_p|_{seg^\circ(p)}$ is a diffeo with its image

iv) $\forall p \in M$ $\quad r(x) := d(x,p)$ is smooth on $U_p \setminus \{p\}$ and $r(x) = \| \exp_p^{-1}(x) \|_g$

Def. Fix $p \in M$. Take $\{e_1, \dots, e_m\}$ orthonormal basis for (T_pM, g_p). For $J = 1, \dots, m$ define

$$\tilde{x}_J \left(\sum_{i=1}^m v_i e_i \right) := v_J$$

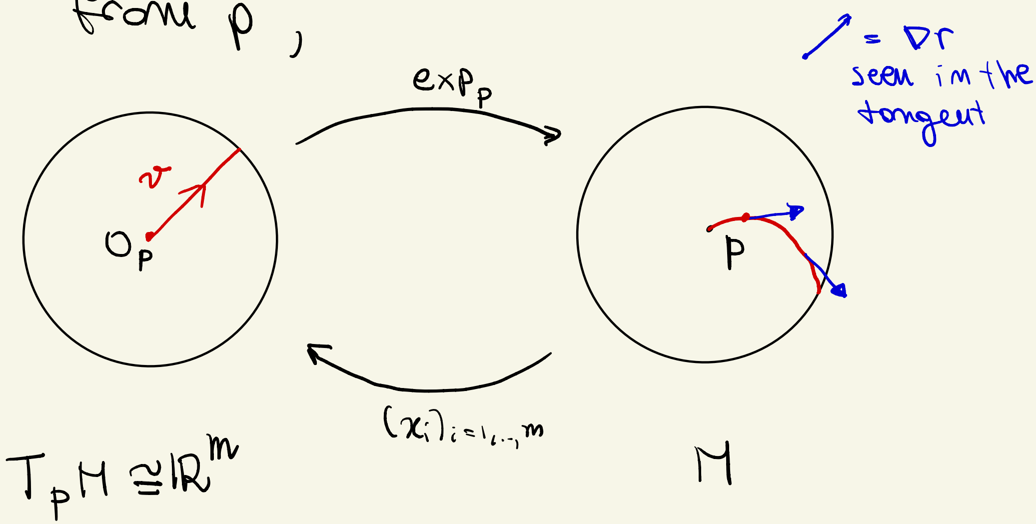
and

$$(x_i)_{i=1,\dots,m} : U_p \longmapsto \mathbb{R}^m$$
$$q \longmapsto (\tilde{x}_i \circ (\exp_p)^{-1}(q))_{i=1,\dots,m}$$

Remark. i) $p \in M$, $r(x) := d(x,p)$ in exponential normal chart is $\quad r(x) = \sqrt{\left(\sum_{i=1}^m x_i^2\right)}$

ii) Geodesics $\gamma(t) = \exp_p(tv)$ are rays through the origin

III) $\nabla r = \frac{x_i}{r}\partial_i$ is the velocity of any unit speed emanating from p,



Remark

I) It holds $\mathcal{L}_{\nabla r}\, g = 2\nabla^2 r$

II) In exponential normal chart $g_{ij} = \delta_{ij} + O(r^2)$

III) Thus $\nabla^2 r = \frac{1}{r}(g - dr \otimes dr) + O(r)$

and $\Delta r = \frac{m-1}{r} + O(r)$

RADIAL MODELS OF CONSTANT SECTIONAL CURVATURE

Notation $k \in \mathbb{R}$, $n \in \mathbb{N}$

$$M := (0, T_k) \times \mathbb{S}^{n-1} \qquad T_k := \begin{cases} \frac{\pi}{\sqrt{k}} & k > 0 \\ +\infty & k \leq 0 \end{cases}$$

$r \in (0, T_k)$, $\theta \in \mathbb{S}^{n-1}$

$d^2\theta :=$ standard metric on $\mathbb{S}^{n-1}$

$$\underset{\substack{\cup \\ [0,+\infty)}}{\operatorname{sn}_k(r)} := \begin{cases} \frac{\sinh(\sqrt{-k}\, r)}{\sqrt{-k}} & k < 0 \\ r & k = 0 \\ \frac{\sin(\sqrt{k}\, r)}{\sqrt{k}} & k > 0 \end{cases}$$

Thm The completion of $(M, dr^2 + \operatorname{sn}_k^2(r)\, d^2\theta)$ is denoted by (M^n_k, g_k) and it is a simply connected Riemannian manifold of constant sectional curvature $\equiv k$.

Useful formulae

Let $r(p) := d(p, o)$

$$\begin{cases} \nabla r = \partial_r \\ \nabla^2 r = \dfrac{sn_k'(r)}{sn_k(r)}\left(g_k - dr^2\right) \\ \Delta r = (n-1)\dfrac{sn_k'(r)}{sn_k(r)} \end{cases}$$

Notation

$s(n,k,r)$ = surface of the geodesic ball of radius r in M_k^n.

$= n\,\omega_n (sn_k(r))^{n-1}$

volume of unit ball in $\mathbb{R}^n$

$v(n,k,r)$ = volume of the geodesic ball of radius r in M_k^n

$= \int_0^r s(n,k,s)\,ds$

BOCHNER FORMULA

Thm $f \in C^3(M)$, it holds

$$\frac{1}{2} \Delta |\nabla f|^2 = |\nabla^2 f|^2 + \langle \nabla f, \nabla \Delta f \rangle + Ric(\nabla f, \nabla f)$$

Pf. It's a particular case of

$$\frac{1}{2} \Delta |X|^2 = |\nabla X|^2 + \langle X, \nabla \operatorname{div} X \rangle + Ric(X, X)$$

for a vector field X such that ∇X symmetric

Let $\{\partial_i\}$ be coordinate fields of a normal chart at x. Hence, at x

$$\Delta \frac{1}{2} |X|^2 = \frac{1}{2} \partial_i \partial_i |X|^2 = \partial_i \langle \nabla_{\partial_i} X, X \rangle =$$

$$= \langle \nabla_{\partial_i} \nabla_{\partial_i} X, X \rangle + |\nabla X|^2 .$$

Now let's compute

$$\langle \nabla_{\partial_i} \nabla_{\partial_i} X, X \rangle = X^k \langle \nabla_{\partial_i} \nabla_{\partial_i} X, \partial_k \rangle$$

$$= X^k \left(\partial_i \langle \nabla_{\partial_i} X, \partial_k \rangle \right) =$$

$$= X^k \left(\partial_i \langle \nabla_{\partial_k} X, \partial_i \rangle \right) =$$

$$= X^k \langle \nabla_{\partial_i} \nabla_{\partial_k} X, \partial_i \rangle =$$

$$= X^k \langle \nabla_{\partial_k} \nabla_{\partial_i} X + R(\partial_i, \partial_k) X, \partial_i \rangle$$

$$= X^k \langle \nabla_{\partial_k} \nabla_{\partial_i} X, \partial_i \rangle + \langle R(\partial_i, X) X, \partial_i \rangle$$

$$= X^k \partial_k \langle \nabla_{\partial_i} X, \partial_i \rangle + Ric(X, X)$$

$$= X^k \partial_k \operatorname{div} X + Ric(X, X)$$

$$= \langle \nabla \operatorname{div} X, X \rangle + Ric(X, X).$$

Hence ( ) follows and Bochner's formula is ( ) with $X = \nabla f$

LAPLACIAN OF THE DISTANCE

[Reference: Bowler]

Thm (M,g) complete Riemannian manifold. Fix $p \in M$. Let $\Omega \subset M$ be open. bounded.

Let $r(x) := d(x,p)$.

Let $h : (0,+\infty) \to \mathbb{R}$ be a smooth monotone (either non-increasing or non-decreasing) function, and let

$$f(x) := h(r(x))$$

I) ∇f exists almost everywhere on Ω and $\forall Z \in C^1_c(\Omega; T\Omega)$ we have

$$\int_\Omega \langle Z, \nabla f \rangle = -\int_\Omega \operatorname{div}(Z) f$$

II) $\exists!$ signed Radon measure Δf s.t. $\forall \varphi \in C^2_c(\Omega)$ we have [Def. 1.40(b)]

$$\int_\Omega f \Delta\varphi = -\int_\Omega \langle \nabla f, \nabla \varphi \rangle = \int_\Omega \varphi \, d\Delta f$$

III) $(\Delta f)_- \ll \mathrm{vol}$ (or $(\Delta f)_+ \ll \mathrm{vol}$)

IV) If f is smooth $\Delta f = (\Delta f)\mathrm{vol}$.

For other results related to Δr

see (MMU14)

LAPLACIAN COMPARISON

[Reference: Elaborated versions of results in [Petersen (b)]]

Thm Let (M^n, g) be a smooth complete Riemannian manifold with $\mathrm{Ric} \geq (n-1)k\, g$. Let $p \in M$ and let U_p be the maximal domain of normal coordinates at p.

Let $r_p := d(\cdot, p)$. Hence

$$\Delta r_p \leq (n-1) \frac{sn_k'(r_p)}{sn_k(r_p)}$$

holds

(i) in the classical sense on U_p;

(ii) in the barrier sense on M

(iii) distributionally on M

[see after in these notes]

Pf. (i) On U_p, r_p is smooth.

Recall $|\nabla r_p| \equiv 1$ on U_p. By Bochner Formula

$$0 = \frac{1}{2}\Delta|\nabla r_p|^2 = |\nabla^2 r_p|^2 + \langle \nabla r_p, \nabla \Delta r_p \rangle + Ric(\nabla r_p, \nabla r_p) \geq$$

$$\geq \frac{(\Delta r)^2}{n-1} + \partial_r \Delta r_p + (n-1)k.$$

Hence on a ray $\gamma : [0, l) \to U_p$ emanating from p we have, calling $f = \Delta r_p(\gamma(t))$

$$\begin{cases} \dfrac{f^2}{n-1} + f' + (n-1)k \leq 0 \\ f(t) = \dfrac{n-1}{t} + o(t) \quad \text{as } t \to 0 \end{cases}$$

By Riccati comparison we have the conclusion. Cf. Lemma 7.1.2. Petersen

(ii) Cf. Lemma 7.1.9. Petersen

(iii) By approximation, using $cut(p)$ negligible, and U_p starshaped, it follows. ▫

Remark Slightly working on the proofs above, one has

(i) If $F : (0, +\infty) \to \mathbb{R}$ is smooth non-decreasing, hence

$$\Delta F(r_p) \leq (n-1) \frac{sn_k'(r_p)}{sn_k(r_p)} F'(r_p) + F''(r_p)$$

(ii) If $F : (0, +\infty) \to \mathbb{R}$ is smooth non-increasing, hence

$$\Delta F(r_p) \geq (n-1) \frac{sn_k'(r_p)}{sn_k(r_p)} F'(r_p) + F''(r_p)$$

Both the inequalities can be interpreted either

(i) in the classical sense on U_p;

(ii) in the barrier sense on M.

(iii) distributionally on M.