LECTURE 1.2.

VOLUME/AREA COMPARISON & BISHOP GROMOV MONOTONICITY

Thm (Bishop-Gromov Comparison)

Let (M^n, g) be a complete Riemannian manifold with $\mathrm{Ric} \geq (n-1)k\, g$, $k \leq 0$

[Assume it for simplicity. In the case $k>0$ all the results hold in the interval $[0, \pi/\sqrt{k}]$ instead of $[0,+\infty)$, and M is cpt.]

Fix $p \in M$. Then

(i) $r \longmapsto \dfrac{\mathrm{Per}(B_r(p))}{s(n,k,r)}$ is essentially non-increasing

$r \in [0,+\infty)$

i.e., $\exists\, \mathcal{J}$ full measure in $[0,+\infty)$ s.t. we have

$$\frac{\mathrm{Per}(B_{r_1}(p))}{s(n,k,r_1)} \geq \frac{\mathrm{Per}(B_{r_2}(p))}{s(n,k,r_2)} \qquad \forall\, 0 < r_1 < r_2, \quad r_1, r_2 \in \mathcal{J}$$

Moreover

$$\frac{\mathrm{Per}(B_r(p))}{s(n,k,r)} \nearrow 1 \quad r \to 0$$

(ii) $\forall\, 0 \le r_1 < r_2 \le r_3 < r_4$ we have

$$\frac{\mathrm{Vol}(B_{r_4}(p)) - \mathrm{Vol}(B_{r_3}(p))}{v(n,k,r_4) - v(n,k,r_3)} \le \frac{\mathrm{Vol}(B_{r_2}(p)) - \mathrm{Vol}(B_{r_1}(p))}{v(n,k,r_2) - v(n,k,r_1)}$$

(iia) In particular, we have that

$$\underset{[0,+\infty)}{\overset{r}{\cap}} \longmapsto \frac{\mathrm{Vol}(B_r(p))}{V(n,k,r)} \quad \text{non-increasing}$$

and moreover $\quad \dfrac{\mathrm{Vol}(B_r(p))}{v(n,k,r)} \underset{r\to 0}{\nearrow} 1$

(iib) In particular,

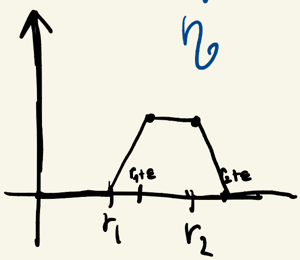
$$\frac{\mathrm{Per}(B_{r_2}(p))}{s(n,k,r_2)} \le \frac{\mathrm{Vol}(B_{r_2}(p)) - \mathrm{Vol}(B_{r_1}(p))}{v(n,k,r_2) - v(n,k,r_1)} \le \frac{\mathrm{Per}(B_{r_1}(p))}{s(n,k,r_1)} \quad \forall\, 0 < r_1 < r_2$$

(iii) We have

$$\frac{\mathrm{Per}(B_r(p))}{s(n,k,r)} \le \frac{\mathrm{Vol}(B_r(p))}{v(n,k,r)} \qquad \forall\, r \in [0,+\infty)$$

(Rigidity) If $\exists\, \bar r \in [0,+\infty)$ s.t. $\mathrm{Vol}(B_{\bar r}(p)) = v(n,k,\bar r)$, hence $(B_{\bar r}(p), g) \cong (B_{\bar r}(0) \subseteq \mathbb{H}^n_k, g_k)$.

Proof. (i) Let $\psi(x) := \eta(d(p,x))$.



Consider $\varphi(x) := \dfrac{\psi(x)}{s(n,k,d(p,x))} \in \mathrm{Lip}_c(M)$

and plug it into $\Delta r \le (n-1)\dfrac{sn_k'(r)}{sn_k(r)}$

$$\overset{\shortparallel}{\dfrac{s'(n,k,r)}{s(n,k,r)}}$$

Using $|\nabla r| = 1$ and simplifying we get

$$\frac{1}{\varepsilon}\int_{B_{r_2+\varepsilon}\setminus B_{r_2}} \frac{1}{s(n,k,d(p,x))}\, d\mathrm{Vol}(x) - \frac{1}{\varepsilon}\int_{B_{r_1+\varepsilon}\setminus B_{r_1}} \frac{1}{s(n,k,d(p,x))}\, d\mathrm{Vol}(x) \le 0$$

Then use the coarea formula $\left[\int_A f\,|DF|\,d\mathrm{Vol} = \int_{\mathbb{R}}\int_{F^{-1}(t)\cap A} f\, d\mathcal{H}^{n-1}\right]$ and take $\varepsilon \to 0$, where r_1, r_2 are Lebesgue point of $r \mapsto \dfrac{\mathrm{Per}(B_r(p))}{s(n,k,r)}$.

(ii) Consequence of (i) and a variant of Cheeger–Gromov. ∎

Taylor Lemma : $\longrightarrow$

$\rightarrow$ " $f, g \in L^1_{loc}([0,+\infty))$, $\dfrac{f}{g}$ non increasing $\Rightarrow \dfrac{\int_{r_3}^{r_4} f}{\int_{r_3}^{r_4} g} \le \dfrac{\int_{r_1}^{r_2} f}{\int_{r_1}^{r_2} g}$ "

Pf.
$$\int_{r_3}^{r_4} f \int_{r_1}^{r_2} g = \int_{r_3}^{r_4} \frac{f}{g}\, g \int_{r_1}^{r_2} g \le \frac{f(r_3)}{g(r_3)} \int_{r_3}^{r_4} g \int_{r_1}^{r_2} g$$

$$= \int_{r_3}^{r_4} g \int_{r_1}^{r_2} \frac{f(r_3)}{g(r_3)} \frac{g}{f}\, f \le \int_{r_3}^{r_4} g \int_{r_1}^{r_2} f$$

(iia) Plug $[r_1 = 0 \ \& \ r_2 = r_3 = r \ \& \ r_4 = R]$ in (ii)

The convergence to 1 is a consequence of the fact that $\frac{\mathrm{vol}(B_r(p))}{\omega_n r^n} \xrightarrow[r \to 0]{} 1$ and $\frac{v(n,k,r)}{\omega_n r^n} \xrightarrow[r\to 0]{} 1$ [Same for the perimeter]

(iib) Plug $r_2 = r_3$ + Coarea and $r_4 \downarrow r_2$ in (ii)

(iii) Coarea

$$\mathrm{Vol}(B_r(p)) = \int_0^r \mathrm{Per}(B_s(p))\,ds = \int_0^r \frac{\mathrm{Per}(B_s(p))}{s(n,k,s)}\, s(n,k,s)\,ds$$

$$\geq \frac{\mathrm{Per}(B_r(p))}{s(n,k,r)} \int_0^r s(n,k,s)\,ds = \frac{\mathrm{Per}(B_r(p))}{s(n,k,r)}\, v(n,k,r).$$

(Rigidity) [Sketch]

= of volumes $\Rightarrow \Delta r = (n-1)\frac{sn_k'(r)}{sn_k(r)}$ on $B_r(p) \subseteq U_p$

$\Rightarrow \nabla^2 r = \frac{sn_k'(r)}{sn_k(r)}(g - dr \otimes dr)$ on $B_r(p)$ $\Rightarrow$ Conclusion ∎

SEGMENT INEQUALITY

[Cheeger-Colding 96, Thm 2.11]
See also Thm 7.1.10. Petersen

Perliminary

(M, g) complete Riemannian.

Hence $\exists\ \mathcal{F} \subset M \times M$ open such that

i) $M \times M \setminus \mathcal{F}$ has measure zero

ii) $\forall (x_1, x_2) \in \mathcal{F}$ $\exists!$ minimizing geodesic

$\gamma_{x_1, x_2} : [0, d(x_1, x_2)] \to M$ between x_1 and x_2

iii) γ_{x_1, x_2} varies continuously on x_1, x_2

Thm In the setting above, ame

a) $Ric \geq (n-1)k$ $k \leq 0$

b) $A, B \subset W \subset M$ s.t. $(x_1, x_2) \in \mathcal{F} \cap (A \times B)$ then

$\gamma_{x_1, x_2}(t) \in W$ $\forall t \in [0, d(x_1, x_2))]$

c) $\operatorname{diam} W \leq D$

Then $\forall f : M \to [0, +\infty)$ bonded Borel $\exists C := C(n, k, D)$

$$\int_{\mathcal{F} \cap (A \times B)} \int_0^{d(x_1, x_2)} f(\gamma_{x_1, x_2}(t))\, dt\, dx_1 dx_2 \leq C\left(\mathrm{Vol}(A) + \mathrm{Vol}(B)\right) \int_W f$$

Iterated segment inequality

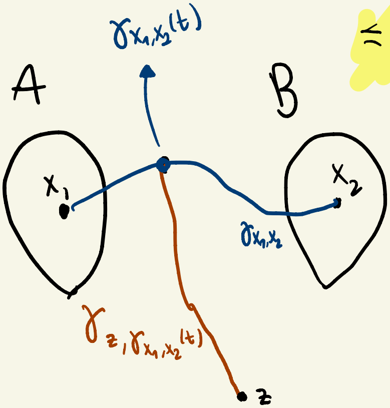
Fix $z \in M$. Hence $\exists\ \mathcal{J}_z$ open full measure such that $\forall\, z' \in \mathcal{J}_z$ $\exists!$ geodesic segment from z to z', call it $\gamma_{z,z'}$. Given $f : M \to [0,+\infty)$ bounded Borel, define on $\mathcal{J}_z \ni z'$

$$F_{f,z}(z') = \int_0^{d(z,z')} f(c_{z,z'}(t))\,dt$$

Hence applying the segment inequality to $F_{f,z}$ we get

$$\int_{\mathcal{J}\cap(A\times B)} \int_0^{d(x_1,x_2)} \int_0^{d(z,\gamma_{x_1,x_2}(t))} f(c_{z,\gamma_{x_1,x_2}(t)}(s))\,ds\,dt\,dx_1\,dx_2$$

$$\le C\left(vol(A)+vol(B)\right)\int_W \int_0^{d(z,x)} f(\gamma_{z,x}(t))\,dt\,dx$$



Remark.

The segment inequality can be used to show in a short way the following Poincaré inequality (see Cor. 7.1.11 Petersen).

Lemma Let (M^n, g) be smooth complete Riemannian manifold with $Ric \geq (n-1)kg$. Let $u: M \to [0, +\infty)$ smooth. Let $D>0$. Hence $\exists\, C := C(N, k, D)$ such that

$$\fint_{B_r(p)} |u - u_{B_r(p)}| \leq C R \fint_{B_{2r}(p)} |\nabla u|$$

where $r \leq D$.

The previous lemma, with a technique by Hajłasz-Koskela can be used to obtain the sharp Sobolev-Poincaré. See Theorem 7.1.13 Petersen, i.e., in the same hypotheses,

$$\left(\fint_{B_r(p)} |u - u_{B_r(p)}|^{\frac{n}{n-1}} \right)^{n-1/n} \leq CR \fint_{B_r(p)} |\nabla u|$$

where $r \leq D$.

MAXIMUM PRINCIPLE

Preliminary $f : M \to \mathbb{R}$ function.

$\Delta f \geq 0$ can have different meanings

I) pointwise, when f is at least C^2

II) distributional; i.e., $\left[\begin{array}{l} f \in W^{1,1}_{loc}(M) \text{ or} \\ f \in Lip_{loc}(M) \ldots \end{array} \right]$

$$-\int \langle \nabla f, \nabla \varphi \rangle \geq 0 \quad \forall \text{ nonnegative } \varphi \in C^\infty_c(M)$$

III) in the barrier sense at $p \in M$, when f is continuous, i.e.,

$\forall \varepsilon \; \exists \; f_\varepsilon : M \to \mathbb{R}$ smooth such that

$$\begin{cases} f(p) = f_\varepsilon(p) \\ f_\varepsilon(x) \leq f(x) \quad \text{in some neighbourhood of } p \\ \Delta f_\varepsilon(p) \geq -\varepsilon \end{cases}$$

IV) in the viscosity sense at $p \in M$, i.e. whenever the following implication holds

$$\begin{cases} \tilde{f} \in C^\infty(U_p) \\ \tilde{f}(p) = f(p) \quad \text{neighborhood of } p \\ \tilde{f} \geq f \text{ on } U_p \end{cases} \Rightarrow \quad \Delta \tilde{f} \geq 0$$

Thm. [Maximum Principle - Barrier. See Thm 7.1.8 - Petersen] (M, g) complete Riemannian manifold, $\Omega \subset M$ open bounded set

i) $f \in C^0(\Omega)$, $\Delta f \geq 0$ in the barrier sense on Ω.

Then f is constant in the neighborhood of any local maximum

ii) If f is also $C^0(\bar{\Omega})$ and Ω is connected, hence f reaches its (maximum in $\bar{\Omega}$) on $\partial\Omega$

For the relations between various notions, see also [MaVU].

Rk. An analogous result can be given, mutatis mutandis, for $\Delta f \leq 0$. It will be a minimum principle

QUANTITATIVE MAX. PRINCIPLE & "good" CUT-OFF FUNCTIONS

Comparison functions [Cf. AG90]

Given $l \in (0,+\infty)$, $k \in (-\infty, 0]$, define

$\frac{l^2}{2(2-n)} + \frac{\rho^2(2-n)-2\rho^{2-n}l^n}{2n(2-n)}$

$$\varphi_k(\rho, l) := \int_\rho^l \frac{v(n,k,l)-v(n,k,t)}{s(n,k,t)}\,dt \qquad \rho \in (0,+\infty)$$

decreasing and convex in ρ when l fixed

$\frac{\rho^2}{2n}$

$$\theta_k(\rho) := \int_0^\rho \frac{v(n,k,t)}{s(n,k,t)}\,dt \qquad \rho \in (0,+\infty)$$

increasing

Fix M^n_k, $p \in M^n_k$. $h(q) := \varphi_k(d(p,q), l)$ then

$$\begin{cases} \Delta h = 1 & \text{on } B_l(p)\setminus\{p\} \\ h(q) = 0,\ \partial h(q) = 0 & \text{on } \partial B_l(p) \end{cases}$$

If instead $g(q) := \theta_k(d(p,q))$ we have

$$\Delta g = 1 \quad \text{on } M^n_k$$

Prop. Let (M^n, g) complete Riemannian manifold with $Ric \geq (n-1)k$, $k \leq 0$, $p \in M$.

Then $h(q) := \varphi_k(d(p,q), l)$; $g(q) = \theta_k(d(p,q))$ we have

$$\Delta h \geq 1 \quad \text{on } B_l(p)\setminus\{p\}$$

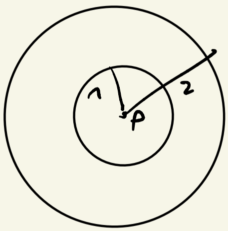
$$\Delta g \leq 1 \quad \text{on } M$$

Proof. Use comparison for Δr.

Such functions allow for versions of "quantitative" maximum principles, cf. [Chapter 8, Cheeger 01]

Let us give an example, through a toy lemma.

Lemma Let (M^n, g) be complete with $Ric \geq -(n-1)\delta$, $\delta > 0$. Let $p \in M^n$.



Let $f \in C^\infty(B_2(p) \setminus \bar{B}_1(p)) \cap C^0(\overline{B_2(p) \setminus B_1(p)})$ such that $\Delta f \leq 2$ $B_2(p) \setminus B_1(p)$ and $f \equiv 0$ on $\partial(B_2(p) \setminus B_1(p))$.

Then

$$\tilde{C}(n) \underset{\text{if } \delta < 1}{\leq} -2\varphi_{-\delta}(1,2) \leq \inf_{B_2(p) \setminus B_1(p)} f$$

Pf. Let $h(q) := \varphi_{-\delta}(d(p,q), 2)$;

Hence $\Delta h \geq 1$ on $B_2(p) \setminus \{p\}$

Thus $\Delta(f - 2h) \leq 0$ on $B_2(p) \setminus B_1\{p\}$ and thus by M.P.

$f(x) - 2h(x) \geq -2\varphi_{-\delta}(1,2)$. $\forall x \in B_2 \setminus B_1$

Such fuctions can be used to build good cutoff fuctions

Lemma (v1) Let (M^n, g) complete, Riemannian manifold with $Ric \geq -(n-1)\delta$, $\delta > 0$. Fix $p \in M$, $0 < R_1 < R_2$.

$\exists\ \phi \in C^\infty(M)$ s.t.

$$\begin{cases} \phi \equiv 1 & B_{R_1}(p) \\ \phi \equiv 0 & \text{on } M \setminus B_{R_2}(p) \\ |\nabla \phi| + |\Delta \phi| & \leq c(n, \delta, R_1, R_2) \end{cases}$$

(v2) Same assumption as above. $\forall R > 0$, $\rho > 1$ $\exists$ $C(n, \delta, R, \rho)$ s.t. $\forall r < R$ $\exists\ \phi_r \in C^\infty(M)$ s.t.

$$\begin{cases} \phi \equiv 1 & \text{on } B_r(p) \\ \phi \equiv 0 & \text{on } M \setminus B_{\rho r}(p) \\ r|D\phi_r| + r^2 |\Delta \phi_r| \leq C(n, \delta, R, \rho) \end{cases}$$

Proof Two approaches

(i) Use the comparison fuctions and Cheeg-Yau estimate [Thm 8.16, Cheeger 01]

(ii) Heat-mollify a fuction of the distance and use Bakry-Ledoux Γ-gradient-Laplacian estimate [Lemma 3.1 - MN16] $\left[|DP_t f|^2 + \frac{4k}{N(e^{2kt}-1)} |\Delta P_t f|^2 \leq e^{-2kt} P_t |Df|^2 \right]$

Rk. These cut-off functions can be used to produce similar cut-off functions on annuli. I.e., given $0 < R_1 < R_2 < R_3 < R_4$ $\exists \phi \in C^\infty(M)$ s.t.

$$\begin{cases} \phi \equiv 0 & B_{R_1}(p) \cup M \setminus B_{R_4}(p) \\ \phi \equiv 1 & B_{R_3}(p) \setminus B_{R_2}(p) \\ |\nabla \phi| + |\Delta \phi| \leq C(n, \delta, R_i) & i = 1, \dots, 4. \end{cases}$$