LECTURE 1.3.

POINCARÉ INEQUALITY (UNDER RICCI CURVATURE BOUNDS)

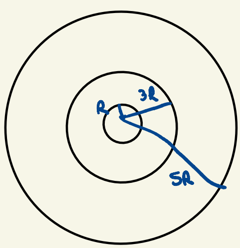
Proposition [Cf. Lemma 6.1. SY94]

Let (M^n,g) be complete Riemannian manifold with $Ric \geq -(n-1)\delta$, $\delta>0$. Let $p\geq 1$, $O\in M$, $R>0$.
Assume $B_{5R}(O) \subsetneq M$.

Hence $\exists$ C_1, C_2 depending only on p and n such that

$$\int_{B_R(O)} |\varphi|^p \leq C_1 R^p e^{C_2\sqrt{\delta}\cdot R}\int_{B_R(O)} |D\varphi|^p \quad \forall \varphi\in C_c^\infty(B_O(R))$$

Pf. (Idea)



take $x_0 \in \partial B_{3R}(O)$ and let $\rho(x) := d(x,x_0)$.
Hence by Laplacian comparison and $2R\leq\rho\leq 4R$ on $B_O(R)$,

$\Delta\rho \leq \alpha(n,\delta,R)$ on $B_R(O)$

From Laplacian comparison again

$\Delta e^{-2\alpha\rho} \geq 2\alpha^2 e^{-2\alpha\rho}$ on $B_R(O)$

Now test the previous with $\varphi\geq 0$, $\varphi\in C_c^\infty(B_R(O))$ and use $2R\leq\rho\leq 4R$ on $B_R(O)$ to conclude the proof for $p=1$ and $\varphi\geq 0$. For obtaining with $|\cdot|$ notice $|D|\varphi||=|D\varphi|$ a.e. and with $p\neq 1$ use it on $|\varphi|^p$ + Hölder. ∎

CHENG-YAU GRADIENT ESTIMATES

$$\left(\Delta u = -\lambda u \right.$$
under Ricci lower bounds)

Prop. [cf. 3.1. SY94; Theorem 1.1. Li04]

Let (M^n, g) be complete Riemannian manifold with $Ric \geq -(n-1)k$, $k > 0$. Let $p \in M$, $R > 0$, $\lambda \geq 0$. Assume

- $f \in C^\infty(B_{2R}(p))$,
- $f > 0$ on $B_{2R}(p)$,
- $\Delta f = -\lambda f$ on $B_{2R}(p)$

Then, $\forall\ 0 < \varepsilon < 2$

$$\sup_{B_R(p)} \frac{|\nabla f|^2}{|f|^2} \leq \frac{(4(n-1)^2 + 2\varepsilon)k}{4 - 2\varepsilon} + C(n)\left((1+\varepsilon^{-1})R^{-2} + \lambda\right).$$

E.g., for $\varepsilon = 1$

$$\sup_{B_R(p)} \frac{|\nabla f|^2}{|f|^2} \leq C(n)\left(k + R^{-2} + \lambda\right)$$

Proof. [Starting idea] $h := \log f$. then $\Delta h = -|\nabla h|^2 - \lambda$.
Study $G := \phi |\nabla h|^2$, ϕ cut-off function, at maximum point.

SOLVABILITY OF $\Delta u = c$

Proposition Let (M^n, g) be a complete noncompact Riemannian manifold. Let $p \in M$. Let $c \in \mathbb{R}$. Then the following hold for a.e. $R \in (0, +\infty)$

- $\partial B_R(p)$ is a Lipschitz submanifold
- Given $f \in \mathrm{Lip}(\bar{B}_R(p))$,

$\exists\, u_R \in C^\infty(B_R(p)) \cap C^0(\bar{B}_R(p))$ s.t

$$(D) \quad \begin{cases} \Delta u_R = c & B_R(p) \\ u_R = f & \partial B_R(p) \end{cases}$$

"**Proof**": The first item is a consequence of Rifford, see [Rif 04]

The existence for the Dirichlet problem (D) is classical [for $C^{2,\alpha}$ domains, with bdry datum in C^α, with $\Delta u \in C^{2,\alpha}$, the solution is $C^{2,\alpha}$ in the interior and C^α to the bdry] see [GT, Theorem 6.13-6.14-6.15]. In the case we are dealing with, i.e. Lipschitz bdry, the solution is still smooth in the interior but just $C^\gamma(\bar{B}_R(p))$ for some $\gamma > 0$, see [Chapter 2.6, Fernandez Real - Ros Oton 20]

MOSER ITERATION

$$\left(\begin{array}{c} \Delta u \geq -\lambda u \text{ on } B_2 \Rightarrow \sup_{B_1} |u| \lesssim \fint_{B_1} u^2 \\ \text{under Ricci lower bounds} \end{array} \right)$$

Preliminaries

Thm I [Heb 00, Theorem 3.2. page 54. ; Varopoulos; Coulhon - Saloff-Coste] (see also)

(M, g) smooth, complete, n-dim Riem. manifold.

Assume $Ric \geq k$ (I), $|B_1(x)| \geq v > 0$ (II) $\forall x \in M$.

Then $\forall u \in C_c^\infty(M)$

$$\left(\int_M |u|^{n/n-1} \, d\mathrm{Vol} \right)^{n-1/n} \leq C(n, k, v) \left(\int_M |Du| + |u| \, d\mathrm{Vol} \right)$$

Pf. [Sketch, see Heb00 Lemma 3.2. p. 57 ; Lemma 3.3. p. 59]

(I) $\Rightarrow$ Poincaré

$\forall x \in M \, \forall r \leq R \, \forall u \in C_c^\infty(B_r(x))$ $\displaystyle\int_{B_r(x)} |u - \bar{u}_r(x)| \leq C(n, k, R) \int_{B_r(x)} |Du|$

Then with (II)

$\forall \Omega$ with $\mathrm{Vol}(\Omega) \leq \eta(n, k, v)$. $\mathrm{Per}(\Omega) \geq C(n, k, v) \, \mathrm{Vol}(\Omega)^{\frac{n-1}{n}}$

Then by Correa

$$\forall x\in M\ \forall u\in C_c^\infty(B_{\underset{\delta(n,k,v)}{\delta}}(x)) \qquad \left(\int_M |u|^{n/n-1}\right)^{n-1/n} \le A(n,k,v)\int |Du|$$

And then conclude by a patching argument. $\lrcorner$

Cor. I

(M,g) smooth, complete, n-dim Riem. manifold.

Assume $\mathrm{Ric}\geq k^{(\mathrm{I})}$, $|B_1(x)|\geq v>0^{(\mathrm{II})}$ $\forall x\in B_1(p)$

Then $\forall u\in C_c^\infty(B_1(p))$

$$\left(⨍_{B_1(p)}|u|^{n/n-1}\,d\mathrm{vol}\right)^{n-1/n}\le C(n,k,v)⨍_{B_1(p)}|Du|\,d\mathrm{vol}$$

Pf. Localize the previous argument, use Bishop-Gromov, and the (1,1)-Poincaré.

Cor. II

(M,g) smooth, complete, n-dim Riem. manifold.

Assume $\mathrm{Ric}\geq k^{(\mathrm{I})}$, $|B_1(x)|\geq v>0^{(\mathrm{II})}$ $\forall x\in B_R(p)$

Then $\forall r\leq R\ \forall u\in C_c^\infty(B_r(p))$

$$\left(⨍_{B_r(p)}|u|^{n/n-1}\,d\mathrm{vol}\right)^{n-1/n}\le C(n,k,v,R)\,r⨍_{B_r(p)}|Du|\,d\mathrm{vol}$$

Pf. Scaling

Rk. From the Thm I we obtain with a classical argument [page 27, Heb00] that $\forall q \in [1, n)$ $\forall u \in C_c^\infty(M)$ we have [under the same hypotheses there]

$$\left(\int_M |u|^{q^*}\right)^{1/q^*} \leq C(n,k,v)\left[\left(\int_M |\nabla u|^q\right)^{1/q} + \left(\int |u|^q\right)^{1/q}\right]$$

where $q^* = \dfrac{nq}{n-q}$.

Analogously in Cor 2I, with the same hypotheses in there we have

$\forall q \in [1, n)$ $\forall r \leq R$ $\forall u \in C_c^\infty(B_r(p))$

$$\left(\fint_{B_r(p)} |u|^{q^*}\right)^{1/q^*} \leq C(n,k,v,R,q)\, r \left(\fint_{B_r(p)} |\nabla u|^q\right)^{1/q}$$

Thm II [cf. Li 96, Lemma 11.1 ; MOSER ITERATION]

(M,g) smooth, complete, n-dim. Riemannian manifold

Assume $Ric \geq k$, $Vol(B_1(x)) \geq v > 0 \quad \forall x \in B_R(p)$

Assume $u \geq 0$ smooth on $B_R(p)$,

$\exists \lambda > 0$ s.t. $\Delta u \geq -\lambda u$ on $B_R(p)$

Then $\forall \vartheta > 0 \quad \exists C := C(\vartheta, n, k, R, v, \lambda)$

$$\sup_{x \in B_{\frac{R}{2}}(p)} |u(x)| \leq C \left(\fint_{B_R(p)} |u|^\theta \right)^{1/\theta}$$

Proof.

By scaling I can assume $Vol(B_R(p)) = 1$

Fix $a \geq 1$. From $\Delta u \geq -\lambda u$, $\forall \phi \in C_c^\infty(B_R(p))$

$$-\int \Delta u \, \phi^2 u^{2a-1} \leq \int \lambda \phi^2 u^{2a}$$

|| integration by parts

$$2\int \langle \nabla u, \nabla \phi \rangle \phi u^{2a-1} + (2a-1) \int \phi^2 u^{2a-2} |\nabla u|^2$$

$$\geq$$

$$2\int \langle \nabla u, \nabla \phi \rangle \phi u^{2a-1} + a \int \phi^2 u^{2a-2} |\nabla u|^2$$

$$||$$

$$\frac{1}{a}\left[\int |\nabla(\phi u^a)|^2 - \int |\nabla \phi|^2 u^{2a} \ldots \right]$$

Hence

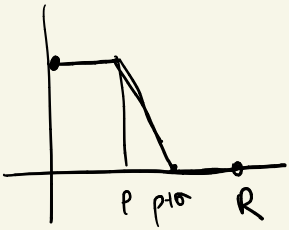
$$\int |D(\phi u^a)|^2 \le \int |D\phi|^2 u^{2a} + a\int \lambda \phi^2 u^{2a}$$

$$\ge$$

SOBOLEV CONSTANT

$$\frac{C_s}{R^2}\left[\int (\phi u^a)^{2^*}\right]^{2/2^*}.$$

Taking the cut-off



ϕ as in the picture (odp) we have $\forall\, 0<\rho<\rho+\sigma<R$,

$$\left(\int_{B_\rho(p)} u^{2^*a}\right)^{2/2^*} \le \frac{R^2}{C_s}\left(a\lambda + \frac{1}{\sigma^2}\right)\int_{B_{\rho+\sigma}(p)} u^{2a}$$

and then, being $\|u\|_{m,\rho} := \left(\int_{B_\rho(p)} |u|^m\right)^{1/m}$,

$$\|u\|_{\overbrace{2^*a}^{\left[2a\,\frac{2^*}{2}\right],\ \frac{2^*}{2}\to\mu},\rho} \le \left(\frac{R^2}{C_s}\left(a\lambda + \frac{1}{\sigma^2}\right)\right)^{1/2a}\|u\|_{2a,\rho+\sigma}$$

For simplicity, let's give the proof only for $k=2$.

Define
$$\begin{cases} a_i = \mu^i & \forall i\ge 0 \\ \sigma_i = \dfrac{1}{2^{i+2}}R & \forall i \ge 0 \\ \rho_i = R - \sum_{j=0}^{i} \sigma_j & \forall i\ge 0 \end{cases}$$

$\mu = \frac{m}{m-2}$

. Notice $\rho_i \xrightarrow[i\to+\infty]{} R/2$

Hence $\forall i \geq 0$,

$$\|u\|_{2a_{i+1}, \rho_i} \leq \left[\prod_{j=0}^{i} \left(\frac{R^2}{C_S} \left(a_i \lambda + \frac{1}{\sigma_i^2} \right) \right) \right]^{1/2a_i} \|u\|_{2,R}$$

$$= \prod_{j=0}^{i} \left(\frac{R^2}{C_S} \left(\lambda \mu^i + \frac{2^{2i+4}}{R^2} \right) \right)^{1/2\mu^i} \|u\|_{2,R}$$

$$\leq \prod_{j=0}^{i} \left(\frac{R^2}{C_S} \lambda + \frac{16}{C_S} \right)^{1/2\mu^i} \max\{\mu, 4\}^{i/2\mu^i} \cdot \|u\|_{2,R}$$

For $i \to +\infty$, LHS $\to \|u\|_{\infty, R/2}$

$$\text{LHS} \leq \left[\left(\frac{R^2}{C_S} \lambda + \frac{16}{C_S} \right)^{1/2} \right]^{\frac{\mu}{\mu-1}} \cdot \tilde{C} \cdot \|u\|_{2,R}$$

Rk. The same Moser-Iteration can be performed on annuli! ∎

SOME FACTS ABOUT GH-DISTANCE & CONVERGENCE

References [BBI01, Chapters 7-8]
[Petersen 16, Chapter 11]
[Villani 09, Chapter 27]
[GMS15]

• The GH-distance

Def. Let (X,d^X), (Y,d^Y) be metric spaces. Then

$$d_{GH}((X,d^X),(Y,d^Y)) = \inf_{(Z,d^Z)} \left\{ d_{H,Z}(X,Y) : \begin{array}{l} i_X : (X,d^X) \hookrightarrow (Z,d^Z) \\ i_Y : (Y,d^Y) \hookrightarrow (Z,d^Z) \end{array} \right\}$$

isometric embeddings

one can give the pointwise analogue by adding $+d(x,y)$ in the RHS

Hausdorff distance on (Z,d^Z)

Rk. It works ok for cpt metric spaces. I.e., if X,Y are cpt, then $d_{GH}(X,Y)=0 \iff X \cong Y$ isometric

Moreover $\left(\{\text{cpt. metric spaces}\}/\sim, d_{GH} \right)$ is a complete, separable metric space

$\sim$ isometry

• Approach with the GH-approximations

Def. We say that $f: (X,d^X) \to (Y,d^Y)$ is ε-GH-approximation if the following hold

(i) $\sup_{(x_1,x_2)\in X\times X} |d^Y(f(x_1),f(x_2)) - d^X(x_1,x_2)| \le \varepsilon$;

(ii) $\forall y \in Y \ \exists x \in X \quad d^Y(f(x),y) \le \varepsilon$.

The GH-distance can be measured, up to constants, by the GH-approximations

Lemma Given (X, d^X), (Y, d^Y) metric spaces,

$$\frac{2}{3} d_{GH}(X,Y) \le \inf\{\varepsilon : \exists\, f: X \to Y \ \varepsilon\text{-GH-approx}\} \le d_{GH}(X,Y)$$

Rk $d_{GH}(X,Y)$ might be $+\infty$ if X, Y not cpt... Some adjustments needed in the noncompact case.

The GH-convergence in the noncompact case

All the spaces from now on will be complete and separable. A general definition of pGH convergence is found in [BBI01, Def. 8.1.1.]

Def. We say that $(X_i, d_i, x_i) \xrightarrow[pGH]{} (Y, d^Y, y)$ if $\exists\ (Z, d_Z)$ and $\Psi_i : (X_i, d_i) \to (Z, d^Z)$, $\Psi : (Y, d^Y) \to (Z, d^Z)$ isometric embeddings s.t. the following holds. $\forall \varepsilon, R > 0\ \exists\, i_0(\varepsilon, R)$ s.t.

ε-closed tubular neighborhood

$$\Psi_i(B_R(x_i)) \subseteq \overline{B}_\varepsilon(\Psi(B_R(y))) \quad \forall i \ge i_0$$

$$\Psi(B_R(y)) \subseteq \overline{B}_\varepsilon(\Psi_i(B_R(x_i))) \quad \forall i \ge i_0$$

Rk. • Such a convergence is metrizable [GMS15, Thm 3.15] with a distance $\mathbb{D}$ [Morally an average of the pGH distances between $(\overline{B}_R(x_i), x_i)$ and $(\overline{B}_R(y), y)$, $R \to +\infty$] and the equivalence class of pointed metric space endowed with $\mathbb{D}$ is complete and separable (GMS15, Thm 3.17)

• The previous notion is equivalent to [BBI01, Def. 8.8.1] in the class of locally uniformly doubling pointed metric spaces [GMS15, Thm 3.30-3.38

Rk. Given a realization space (Z, d^Z), one can define convergence of points $x_n \in X_n \to x \in X$ and of sets $E_n \subseteq X_n \to E \subseteq X$ just by reading them in the realization (Z, d^Z).

Def. Assume $(X_j, d_j, x_j) \xrightarrow[pGH]{} (Y, d^Y, y)$ and let (Z, d^Z) be a realization. We say that $f_j : X_j \to \mathbb{R}$ converge pointwise to $f : Y \to \mathbb{R}$ if the following holds

"$\forall x_j \overset{\in X_j}{\to} y^{\in Y}$, $f_j(x_j) \to f(y)$"

We say that the convergence is uniform if $\forall \varepsilon \, \exists N$ s.t. the following holds

"$\forall j \geq N \, \forall x_j \in X_j \, \forall y \in Y \quad d^Z(x_j, y) \leq N^{-1} \Rightarrow |f_j(x_j) - f(y)| \leq \varepsilon$"

Thm. (Ascoli-Arzelà)

Assume $(X_j, d_j, x_j) \xrightarrow[pGH]{} (Y, d^Y, y)$ and let (Z, d^Z) be a realization. Assume $f_j : X_j \to \mathbb{R}$ are L-lipschitz and $\sup_{j \in \mathbb{N}} |f_j(x_j)| < +\infty$. Hence $\exists f_\infty : X_\infty \to \mathbb{R}$ L-lipschitz s.t. $f_j|_{B_R(x_j)} \xrightarrow{\text{uniformly}} f|_{B_R(y)}$ $\forall R > 0$ up to subsequences.

Prop. Gromov Compactness

Let $\mathcal{C}$ be a class of compact metric spaces with $\operatorname{diam} X \leq D$ $\forall\, X \in \mathcal{C}$. TFAE:

- $\mathcal{C}$ is pre-compact in the GH-topology
- $\exists\, N_1 : (0,D) \to (0,+\infty)$ s.t. $\operatorname{Cap}_X(\varepsilon) \leq N_1(\varepsilon)$ $\forall \varepsilon \in (0,D)$ $\forall X \in \mathcal{C}$

  Maximum number of disjointed $\frac{\varepsilon}{2}$-balls in X
- $\exists\, N_2 : (0,D) \to (0,+\infty)$ s.t. $\operatorname{Cov}_X(\varepsilon) \leq N_2(\varepsilon)$ $\forall \varepsilon \in (0,D)$ $\forall X \in \mathcal{C}$

  Minimum number of ε-balls needed to cover X

Localizing the previous criterion and using Bishop-Gromov comparison one has the following

Prop. The class of pointed complete Riemannian manifolds with $\mathrm{Ric} \geq k$ for some $k \in \mathbb{R}$ is precompact in the pointed Gromov-Hausdorff topology

SPIN-OFF

(SOME)

FUNDAMENTAL REFERENCES ON

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