The almost splitting theorem

LECTURE 2.1

References [See the references for the 1st lecture, and add the following ones]

Articles

[CheegerGromoll71] "The splitting theorem for manifolds of nonnegative Ricci curvature", (1971), by J. Cheeger and D. Gromoll

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[CheegerNaber13] "Lower bounds on Ricci curvature and quantitative behaviour of singular sets", (2013), by J. Cheeger, A. Naber

[CheegerJiangNaber21] "Rectifiability of singular sets in noncollapsed spaces with Ricci curvature bounded below", (2021), by J. Cheeger, W. Jiang, A. Naber

[AntonelliBruèSemola19] "Volume bounds for the quantitative singular strata of noncollapsed RCD metric measure spaces", (2019), by G. Antonelli, E. Bruè, D. Semola

[KapovitchMondino19] "On the topology and the boundary of N-dimensional RCD(k,N) spaces", (2019), by V. Kapovitch, A. Mondino

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[BruePasqualettoSemola19] "Rectifiability of the reduced boundary for sets of finite perimeter over RCD(k,N) spaces" (2019) by E. Bruè, E. Pasqualetto, D. Semola"

[GigliDePhilippis18] "Non-collapsed spaces with Ricci curvature bounded from below" (2018) by N. Gigli, G. De Philippis

[ColdingNaber12] "Sharp Hölder continuity of tangent cones for spaces with a lower Ricci curvature bound" (2012), by T. Colding, A. Naber

[BruèSemola20] "Constancy of the dimension for RCD(k,N) spaces via regularity of Lagrangian flows", (2020), by E. Bruè, D. Semola

[Deng20] "Hölder continuity of tangent cones in RCD(k,N) spaces and applications to non-branching", (2020), by Q. Deng

[PanWei21] "Examples of Ricci limit spaces with non-integer Hausdorff dimension" (2021), by J. Pan and G. Wei

[GigliPasqualetto18] "Behaviour of the reference measure on RCD spaces under charts", (2018), by N. Gigli and E. Pasqualetto

[KellMondino18] "On the volume measure of non-smooth spaces with Ricci curvature bounded below", (2018), by M. Kell, A. Mondino

[DePhilippisMarcheseRindler17] "On a conjecture of Cheeger" (2017), by G. De Philippis, A. Marchese, F. Rindler

[Naber20] "Conjectures and open questions on the structure and regularity of spaces with lower Ricci curvature bounds" (2020), by A. Naber

[Honda20] "Collapsed Ricci limit spaces as non-collapsed RCD spaces" (2020), by S. Honda

[Fukaya87] "Collapsing of Riemannian manifolds and eigenvalues of the Laplace operator" by K. Fukaya, (1987).

[Anderson90] "Convergence and rigidity of manifolds under Ricci curvature bounds" by M.T. Anderson (1990)

For the RCD setting, which is NOT the main topic of these lectures, the fundamental references are

[LottVillani09] "Ricci curvature for metric-measure spaces via optimal transport" (2009), by J. Lott, C. Villani

[SturmI06] "On the geometry of metric measure spaces I" (2006) by K.T. Sturm.

[SturmII06] "On the geometry of metric measure spaces II" (2006) by K.T. Sturm.

[Ambrosio18] "Calculus, heat flow, and curvature-dimension bounds in metric measure spaces" (2018), by L. Ambrosio

THE SPLITTING THEOREM

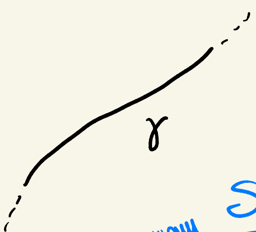
Thm (Cheeger-Gromoll '71)

Let (M^n, g) be a complete Riemannian manifold with $Ric \geq 0$. Assume $\gamma : (-\infty, +\infty) \to M$ is a line, i.e.,

$$d(\gamma(s), \gamma(t)) = |t - s| \quad \forall t, s \in \mathbb{R}$$

Hence M is isometric to $\mathbb{R} \times N$.

Proof. (Sketch)



Step 1 Let $b_t(x) := d(x, \gamma(t)) - |t|$ and $b_\pm := \lim_{t \to \pm\infty} b_t$ be the Busemann functions.

Ric≥0 + Lapl. comp. — by construction

Step 2 $\Delta(b_+ + b_-) \leq 0$ & $b_+ + b_- \equiv 0$ on γ

maximum principle — Implies $\Delta b_+ = \Delta b_- = 0$.

Bochner & step 2 — "pseudo-distance" — by construction

Step 3 $|\nabla b_+|^2$ subharmonic + $|\nabla b_+| \leq 1$ + $|\nabla b_+| \equiv 1$ on $\gamma_{[0,+\infty)}$

Implies $|\nabla b_+|^2 \equiv 1$.

Step 4 Hence Step 2 + Step 3 gives $\nabla^2 b_+ = 0$

Conclusion Let $N := \{b_+ = 0\}$ and $\Phi : N \times \mathbb{R} \to M$, $(y, s) \to$ flow at time s starting at y of the v.f. ∇b_+, gives the isometry

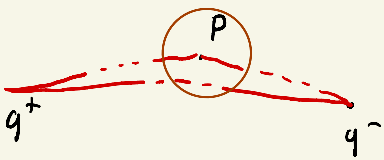
THE ALMOST SPLITTING THEOREM

Thm [Cheeger-Colding '96]

$\forall \varepsilon \exists \delta$ s.t. the following holds

If (M, g) satisfies $\mathrm{Ric} \geq -(n-1)\delta$ and

p, q^+, q^- are three points such that



(a) $\min\{d(p,q^+), d(p,q^-)\} \geq \delta^{-1}$

(b) $E(p) := d(p,q^+) + d(p,q^-) - d(q,q^-) \leq \delta$

Then

$$d_{pGH}\left((B_1(p), p), (B_1(x, 0'), (x, 0')) \right) \leq \varepsilon$$

where $(x, 0) \in X \times \mathbb{R}$, X being a metric boundedly compact metric space.

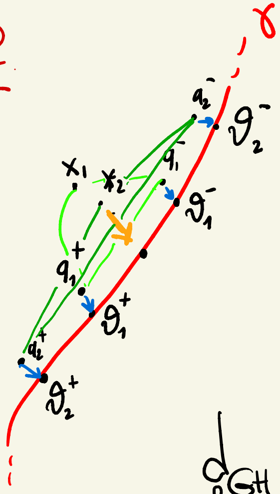
SPLITTING THEOREM FOR RICCI-LIMITS

Theorem (Cheeger - Colding '97)
Let (M_i^n, g_i) be complete Riemannian manifolds such that $\mathrm{Ric}_{M_i^n} \geq -(n-1)\delta_i$ with $\delta_i \to 0$. Let (X, d) such that

$$(M_i^n, d_{g_i}, x_i) \longrightarrow (X, d, x)$$

in the pGH sense, with some $x_i \in M_i$ and $x \in X$. Assume $\gamma : (-\infty, +\infty) \to X$ is a line. Hence there exists a complete, boundedly compact, length space Y s.t. X is isometric to $\mathbb{R} \times Y$.

Pf



Without loss of generality $x \in \gamma(\mathbb{R})$.
Work in a realization of the pGH-convergence
Up to subsequences, you can find $R_i \to +\infty$, and $Z_i \subseteq M_i^n$ such that

$$d_{pGH}\left(B_{R_i}(x_i), B_{R_i}^{\mathbb{R} \times Z_i}((0, z_i)) \right) \xrightarrow[i \to +\infty]{} 0 \qquad (GH)$$

Up to subsequence, by Gromov precompactness,

$$(Z_i, d_{M_i^n}|_{Z_i}, z_i) \longrightarrow (Z, d_Z, z).$$

From $(M_i^n, d_{g_i}, x_i) \to (X, d, x)$, the previous convergence, the stability of pGH convergence with products and (GH) we conclude

$$d_{pGH}\big(B_R(x), B_R((0,z))\big) = 0 \quad \forall R>0.$$

Hence $X \cong \mathbb{R} \times Z$.

What is this used for?

Coupled with a "reduction of the dimension" argument one can prove, by iteration,

THM [ChCo97] Let (X,d) be a noncollapsed Ricci-limit-space, i.e. $(M_i^n, d_{g_i}, x_i) \xrightarrow[pGH]{} (X,d,x)$ for some manifolds with $Ric_{M_i^n} \geq -1$ and $v = \inf_i Vol_i(B_1(x_i)) > 0$.

Let $S_k := \{\bar{y} \in X : \bar{y}$ is not regular and every tangent cone at $\bar{y}$ does not split isometrically $\mathbb{R}^{k+1}\}$

Hence $\dim S_k \leq k$.

For the RCD version see [GigliDePhilippis18]

What is a tangent cone?

Given (X,d) a metric space, $x \in X$, we say that (Y,d,y) is a tangent space at $x \in X$ if there exists $\{r_i\}_{i \in \mathbb{N}}$ infinitesimal such that $(X, r_i^{-1}d, x) \longrightarrow (Y,d,y)$ in the pGH-topology.

Def. $\mathcal{R}_k := \{x \in X :$ every tangent cone is isometric to $\mathbb{R}^k\}$

[Cheeger Colding '97 Fukaya 87]

Theorem (Colding-Naber 13) Let X be a RLS. Hence there exists a Radon measure ν on X such that $\left(M_i^n, d_{g_i}, \frac{\mathcal{H}^N}{\mathcal{H}^N(B_1(x_i))}, x_i\right) \longrightarrow (X, d, \nu, x)$ in the pmGH topology. Moreover $\exists!$ $k \leq n$ s.t. $\nu(X \setminus \mathcal{R}_k) = 0$

$(X_i, d_i, m_i, x_i) \xrightarrow[\text{pmGH}]{} (Y, d, m, y)$ means the pGH convergence + $(\Psi_i)_\# m_i \to (\Psi)_\# m$ in the realization (duality with $C_c(Z)$))

Generalized in the RCD setting in [Bruè Semola 20]

The Hölder continuity property of [Colding Naber 13] generalized in the RCD setting by [Deng 21]

ACHTUNG! In the collapsed case $\dim_H X$ migth be greater than the rectifiability dimension [Pan Wei 21]

Structure of the measure in [Gigli Pasqualetto 18, Kell Mondino 18, De Philippis Marchese Rindler 17, Mondino Naber 19]

More refined result are in [Cheeger-Naber] [Cheeger-Jiang-Naber]. about the singular set.

Def. $\mathcal{S}^k_{\eta,r} := \{ y \in X \mid d_{pGH}(B_s(y), B_s((0,z))) \geq \eta s$ for all $\mathbb{R}^{k+1} \times Z$, for all $r \leq s \leq 1 \}$

Thm I [Cheeger-Naber '13]

$$\mathrm{Vol}(\mathcal{S}^k_{\eta,r} \cap B_{\frac{1}{2}}(x)) \leq c(n,v,\eta) r^{n-k-\eta}$$

Remark: Stronger than $\dim_H \mathcal{S}^k \leq k$, see [Lemma 2.5, A.-Bruè-Semola]

[Cheeger-Naber '13] was improved in

Thm II [Cheeger-Jiang-Naber '21]

- $\mathrm{Vol}(\mathcal{S}^k_{\eta,r} \cap B_{\frac{1}{2}}(x)) \leq c(n,v,\eta) r^{n-k}$
- $\mathrm{Vol}(B_r(\bigcap_r \mathcal{S}^k_{\eta,r}) \cap B_{\frac{1}{2}}(x)) \leq c(n,v,\eta) r^{n-k}$
- $\mathcal{H}^k((\bigcap_r \mathcal{S}^k_{\eta,r}) \cap B_{\frac{1}{2}}(x)) \leq c(n,v,\eta)$

Moreover $\mathcal{S}^k = \bigcup_\eta \bigcap_r \mathcal{S}^k_{\eta,r}$ is k-rectifiable and for $\mathcal{H}^k$-a.e. $x \in \mathcal{S}^k$, every tangent cone at x splits isometrically $\mathbb{R}^k$

Already known from [Cheeger-Colding '97 Theorem A.1.8.]

Moreover X is biHölder homeo to a smooth manifold outside a rectifiable set of codimension 2.

after [Anderson 90]

For the version of [Cheeger-Naber '13] on $RCD(k,N)$ spaces X with measure $\mathcal{H}^N$ and $\mathcal{H}^N(B_1(x)) \geq v > 0 \ \forall x \in X$, see [A.-Bruè-Semola]

For generalizations of [Cheeger-Jiang-Naber '21] in the setting above, and related results in the codimension two case, see

[Kapovitch-Mondino] (A)

[Bruè-Naber-Semola] (B)

In (A) generalization of [CC97, Theorem A.1.8.] on RCD spaces

In (B) generalizations of [Cheeger-Jiang-Naber '21] for $k = n-2$ on RCD spaces

$$\partial X = \overline{\mathcal{S}^{n-1} \setminus \mathcal{S}^{n-2}}$$

"A" BOUNDARY

For conjectures see [Naber20], [Honda20]

SPLITTING THEOREM FOR RCD SPACES

Theorem (Gigli '13)

Let (X, d, m) be an $RCD(0,N)$ metric measure space. Assume that X contains a line $\gamma : (-\infty, +\infty) \to X$.
$\gamma \subseteq \operatorname{spt} m$

Hence (X, d, m) is isomorphic, as a metric measure space, to $(Y \times \mathbb{R}, d^Y \otimes d_{eu}, m^Y \otimes m_{eu})$ where (Y, d^Y, m^Y) is an $RCD(0, N-1)$ metric measure space.

Remark. We have a dimensional bound on Y that was not available in the Ricci-limit scenario!
Moreover, we have the splitting of the measure!

δ- SPLITTING MAPS

Def. Given $k \in \mathbb{N}$, $\varepsilon, r > 0$, $p \in M$ a (k, ε)-splitting map on $B_r(p)$ is a smooth map $(u_1, \dots, u_k) : B_r(p) \to \mathbb{R}^k$ such that:

(i) $\Delta u_i = 0 \qquad \forall i = 1, \dots, k$

(ii) $\sup_{B_r(p)} |\nabla u_i| \le 1 + \varepsilon \qquad \forall i = 1, \dots, k$

(iii) $\fint_{B_r(p)} |\langle \nabla u_i, \nabla u_j \rangle - \delta_{ij}| \le \varepsilon \qquad \forall i, j = 1, \dots, k$

(iv) $r^2 \fint_{B_r(p)} |\nabla^2 u|^2 \le \varepsilon \qquad \forall i = 1, \dots, k$

Remark If a function u satisfies (i), (iii) and (iv) at a scale r and $Ric \ge -\varepsilon$ on M, then $\sup_{B_r(p)} |\nabla u| \le C \Rightarrow \sup_{B_{\frac{r}{2}}(p)} |\nabla u| \le 1 + \Psi(\varepsilon)$

[Cheeger-Naber '15, (3.42)-(3.46)]

Idea: [See also Bruè-Naber-Semola p.17]

Take φ a cut off function $\begin{cases} \varphi \equiv 0 & B_r^c(p) \\ \varphi \equiv 1 & B_{\frac{3}{4}r}(p) \\ r^2 |\Delta \varphi| + r |\nabla \varphi| \le C_n \end{cases}$

and consider $f_t(y) := \int (|\nabla u|^2 - 1) \varphi(z) p_t(y, z) dm(z)$

where p_t is the heat kernel.

Heat kernel estimates $\rightarrow \frac{d}{dt} f_t(y) \gtrsim -\frac{\varepsilon^{1/2}}{r^2} \qquad \begin{matrix} \forall y \in B_{\frac{1}{2}r}(x) \\ \forall t \in [0, r^2] \end{matrix}$

FROM SPLITTING MAP TO SPLITTING SPACES

Thm $\forall \varepsilon\ \exists \delta$ s.t the following holds.
If (M,g) satisfies $\mathrm{Ric} \geq -(n-1)\delta$
and $u : B_{16}(p) \to \mathbb{R}^k$ is a (k,δ)-splitting map, hence $\exists$ Z complete length metric space s.t.

$$d_{pGH}\Big((B_1(p), p) \ ,\ (B_1(z, 0^k), (z, 0^k)) \Big) < \varepsilon$$

Modern version [Bruè - Pasqualetto - Semola '19
Proposition 3.7 "Rectifiability of the reduced boundary for sets of finite perimeter over RCD(k,N) spaces"]

$\forall \varepsilon\ \exists \delta$ s.t the following holds.
If (X,d,m) is an $RCD(-\delta, N)$ s.t.
$u : B_{\delta^{-1}}(x) \to \mathbb{R}^k$ is δ-splitting on $B_s(x)$ $\forall 0 < s < \delta^{-1}$
hence

$$d_{pmGH}\Big((X,d,m,x), (\mathbb{R}^k \times Z, (0^k, z)) \Big) < \varepsilon$$

for some $RCD(0, N-k)$ space (Z, d_Z, m_Z).

FROM SPLITTING SPACES TO SPLITTING MAPS

Thm $\forall \varepsilon\ \exists \delta$ s.t the following holds.
If (M,g) satisfies $\mathrm{Ric} \geq -(n-1)\delta$
and $d_{pGH}\left((B_{\delta^{-1}}(p), p), (B_{\delta^{-1}}(z,0^k), (z,0^k))\right) < \delta$
for some Z complete length metric space, then there exists
$u : B_1(p) \to \mathbb{R}^k$ a (k,ε)-splitting map

Modern version [Bruè - Pasqualetto - Semola '19
Proposition 3.9 "Rectifiability of the reduced boundary for sets of finite perimeter over RCD(k,N) spaces"]

SOME WORDS ABOUT
GH-CLOSE TO $B_1^{\mathbb{R}^n}(0)$ $\Longleftrightarrow$ VOLUME-CLOSE TO $B_1^{\mathbb{R}^n}(0)$

Thm [Colding 97] Given $\varepsilon>0$, $n\in\mathbb{N}$, $\exists\,\Lambda=\Lambda(\varepsilon,n)$, $\delta=\delta(\varepsilon,n)$ such that if $\mathrm{Ric}_{M^n}\geq-\Lambda$, $d_{pGH}(B_1(p), B_1^{\mathbb{R}^n}(0))<\delta$ (with $p\in M$)
then $|\mathrm{Vol}(B_{1/2}(p))-V(n,0,\frac{1}{2})|<\varepsilon$.

Given $\varepsilon>0$, $n\in\mathbb{N}$ $\exists$ $\Lambda=\Lambda(\varepsilon,n)$, $\delta=\delta(\varepsilon,n)$ s.t.
$\mathrm{Ric}_{M^n}\geq-\Lambda$, $|\mathrm{Vol}(B_1(p))-V(n,0,1)|\leq\delta \Rightarrow d_{pGH}(B_{\frac{1}{2}}(p), B_{\frac{1}{2}}^{\mathbb{R}^n}(0))<\varepsilon$

From this you have non trivial consequences.

Cor I [Reifenberg] Given $\varepsilon>0$, $n\in\mathbb{N}$ $\exists\,\delta=\delta(\varepsilon,n)$ such that if $\mathrm{Ric}_{M^n}\geq-\delta$, $d_{pGH}(B_1(p), B_1^{\mathbb{R}^n}(0))<\delta$ (with $p\in M$), hence
$d_{pGH}(B_r(x), B_r^{\mathbb{R}^n}(0))<\varepsilon r \quad \forall\, B_r(x)\subset\subset B_{1/2}(p)$

Cor. II If at a point of a RLS one tangent cone is $\mathbb{R}^n$, hence all the tangent cones are $\mathbb{R}^n$.

Cor III [ChCo] If $(M_i^n, d_i, x_i)\xrightarrow{pGH}(X,d,x)$ and $\mathrm{Ric}_{M_i^n}\geq-1$ and $\inf_i \mathrm{Vol}(B_1(x_i))=V>0$ $(*)$, we have that
$(M_i^n, d_i, x_i, \mathcal{H}^n)\xrightarrow{pmGH}(X,d,x,\mathcal{H}^n)$. If $(*)$ does not hold $\dim_H X\leq n-1$.

The analogous versions of these results in $RCD(k,N)$ spaces with reference measure $\mathcal{H}^N$ are in [Gigli-DePhilippis 18]

NOTATION ALERT!

$$\underline{\psi}(\delta_1, \dots, \delta_k \mid \eta_1, \dots, \eta_\ell)$$

stands for a quantity depending on the positive numbers $\delta_1, \dots, \delta_k, \eta_1, \dots, \eta_\ell$ such that

$$\psi(\delta_1, \dots, \delta_k \mid \eta_1, \dots, \eta_\ell) \longrightarrow 0$$

as $\quad \delta_1 + \dots + \delta_k \longrightarrow 0$

This quantity, in the estimate, may change from line to line!