The almost splitting theorem

LECTURE 2.2.

FROM SPLITTING MAP TO SPLITTING SPACES

Thm $\forall \varepsilon\ \exists \delta$ s.t the following holds.
If (M,g) satisfies $\mathrm{Ric} \geq -(n-1)\delta$
and $u : B_{16}(p) \to \mathbb{R}^k$ is a (k,δ)-splitting map, hence $\exists$ Z complete length metric space s.t.

$$d_{pGH}\Big((B_1(p), p) \,,\, (B_1(z,0^k), (z,0^k)) \Big) < \varepsilon$$

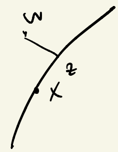
Let's prove it!

Strategy mainly elaborated from

- Bowler's notes
- Cheeger's book [Cheeger 01]
- My master thesis
- Some notes online from UCSD seminar on Cheeger-Colding theory, Ricci flow, Einstein metrics and related topics (Fall 2020)

LEMMA 1 Assume u is δ-splitting on $B_{16}(p)$, and $Ric \geq -(n-1)$ * discuss

Take $x, z, w \in B_2(p)$ s.t. Clearly I can put $B_{10}(p)$ also.



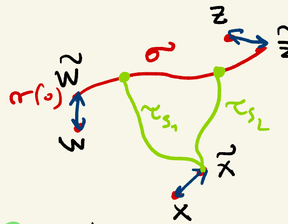
(a) $|u(x) - u(z)| \leq \delta$

(b) $\big| |u(z) - u(w)| - d(z,w) \big| \leq \delta$

Hence $|d^2(x,z) + d^2(z,w) - d^2(x,w)| \leq \Psi(\delta|n)$

Proof. Let's work in the simpler case $k=1$. Take $h := |\nabla^2 u|^2 + \big| |\nabla u|^2 - 1 \big|$.

By the segment inequality the following holds.



$\exists\, \tilde x, \tilde z, \tilde w \in B_4(p)$ s.t.

(!) $d(\tilde x, x) + d(\tilde y, y) + d(\tilde z, z) + \big| |\nabla u|(\tilde x) - 1 \big| \leq \Psi$

(!!) From $\tilde w$ to $\tilde z$ $\exists!$ minimal geodesic σ s.t.

$$\int_0^{d(\tilde w, \tilde z)} h(\sigma(s))\, ds \leq \Psi$$

(!!!) There is $U \subseteq [0, d(\tilde w, \tilde z)]$ full measure s.t. * discuss

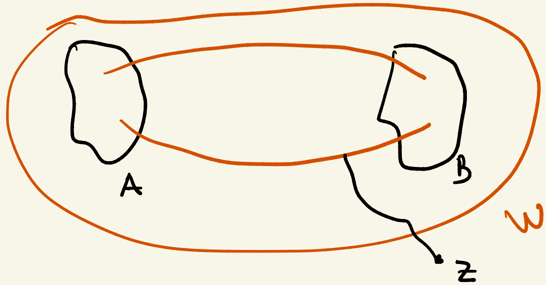
(■) $\forall s \in U$ $\exists!$ τ_s minimal geodesic from $\tilde x$ to $\sigma(s)$

(■■) $$\int^{d(\tilde w, \tilde z)} \int_0^{d(\tilde x, \sigma(s))} h(\tau_s(t))\, dt\, ds \leq \Psi$$

See the argument at page 36-37 of my master thesis

INTERMEZZO How to use segment inequality?

$$\int_{\mathcal{G}\cap(A\times B)} \int_0^{d(x_1,x_2)} f(\gamma_{x_1,x_2}(t))\,dt\,dx_1\,dx_2 \le C\left(\mathrm{vol}(A)+\mathrm{vol}(B)\right)\int_W f$$

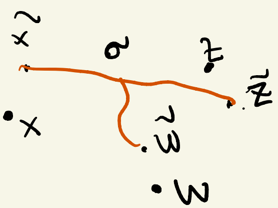


$\mathrm{diam}\, W \le D$

$C := C(n,\delta,D)$

$$\int_{\mathcal{G}\cap(A\times B)} \int_0^{d(x_1,x_2)} \int_0^{d(z,\gamma_{x_1,x_2}(t))} f(c_{z,\gamma_{x_1,x_2}(t)}(s))\,ds\,dt\,dx_1\,dx_2 \le C\left(\mathrm{vol}(A)+\mathrm{vol}(B)\right)\int_W \int_0^{d(z,x)} f(\gamma_{z,x}(t))\,dt\,dx$$

What do we want to do?



Given 3 points x, z, w and given f s.t. $⨍_B f \le \psi$ on a sufficiently big ball containing these three points, we want to find new points $\tilde{x}, \tilde{w}, \tilde{z}$ s.t. $d(x,\tilde{x}) + d(y,\tilde{y}) + d(z,\tilde{z}) \le \tilde{\psi}$ s.t.

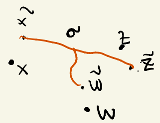
(i) $$\int_0^{d(\tilde{x},\tilde{z})} f(\gamma_{\tilde{x},\tilde{z}}(t))\,dt \le \tilde{\psi}$$

(ii) $$\int_0^{d(\tilde{x},\tilde{z})} \int_0^{d(\tilde{w},\gamma_{\tilde{x},\tilde{z}}(t))} f(\gamma_{\tilde{w},\gamma_{\tilde{x},\tilde{z}}(t)}(s))\,ds\,dt \le \tilde{\psi}$$

Remark: You can also ask $f(\tilde{w}) \le \tilde{\psi}$.

Let us suppose $\fint_{B_{16}(p)} f \le \Psi$ (H), $x, w, z \in B_1(p)$

and all the constructions we are going to do will be localite in $B_{16}(p)$, up to choosing Ψ small enough.



<u>How?</u> Let us call B_x, B_z, B_w the balls of centers x, z, w and radius $\tilde{\Psi}$ to determine. ($\subseteq B_2(p)$)

<u>Step 1</u> Choose one $\bar{w}$. From (SI) we have

$$\int_{B_w \times B_4(p)} \int f(\gamma_{x,y}(t))\,dt\,dx\,dy \le C\left(|B_w| + |B_4(p)|\right) \fint_{B_8(p)} f \le$$

$$\le C(|B_w| + |B_4(p)|)|B_{16}(p)| \fint_{B_{16}(p)} f \le C\Psi\left(|B_w| + |B_4(p)|\right)|B_{16}(p)|$$

Hence $\exists\, \tilde{w} \in B_w$ s.t.

$$(\gamma)\quad \int_{B_4(p)} \int f(\gamma_{\tilde{w},y}(t))\,dt\,dy \le C\Psi \frac{\left(|B_w| + |B_4(p)|\right)}{|B_w|} |B_{16}(p)|$$

<u>Step 2</u> By Markov's inequality $\exists\, (\bar{x}, \bar{z}) \in B_x \times B_z$ s.t.

$$(\alpha)\quad \left[\iint f(\gamma_{\tilde{w},\gamma_{\tilde{x},\tilde{z}}(t)}(s))\,ds\,dt\right] \le 3 \fint_{B_x \times B_z} [\ \ ]\,dx\,dz$$

$$(\beta)\quad \left[\int f(\gamma_{\tilde{x},\tilde{z}}(t))\,dt\right] \le 3 \fint_{B_x \times B_z} [\ \ ]\,dx\,dz$$

Now by (α) + (ISI) + (γ)

$$\iint f(\gamma_{\tilde{w},\gamma_{\tilde{x},\tilde{z}}(t)}(s))\,ds\,dt \le C \frac{|B_x| + |B_z|}{|B_x||B_z|} \frac{|B_w| + |B_4(p)|}{|B_w|} |B_{16}(p)|\, \Psi$$

$$\iint f(\gamma_{\tilde w,\gamma_{x,z}(t)}(s))\,ds\,dt \le C\,\frac{|B_x|+|B_z|}{|B_x||B_z|}\,\frac{|B_w|+|B_u(p)|}{|B_w|}\,|B_{16}(p)|\,\Psi \qquad (\delta)$$

Then by using (β) + (SI) + (H) we have

$$\int f(\gamma_{x,z}(t))\,dt \le C\,\frac{|B_x|+|B_z|}{|B_x||B_z|}\,|B_{16}(p)|\,\Psi \qquad (\varepsilon)$$

Look (ε). $\frac{|B_x|+|B_z|}{|B_x||B_z|} = \frac{1}{|B_x|}+\frac{1}{|B_z|}$

We have $B_1(p)\subseteq B_2(z)$. So $\frac{|B_1(p)|}{|B_z|} \le \frac{|B_2(z)|}{|B_z|} \le \frac{v(n,-\delta,2)}{v(n,-\delta,\tilde\Psi)}$

Thus $\frac{1}{|B_z|} \le \frac{v(n,-\delta,2)}{v(n,-\delta,\tilde\Psi)}\,\frac{1}{|B_1(p)|}$ and

$$(\varepsilon) \le C\,\frac{v(n,-\delta,2)}{v(n,-\delta,\tilde\Psi)}\,\frac{|B_{16}(p)|}{|B_1(p)|}\,\Psi \le \tilde C\,\frac{\Psi}{v(n,-\delta,\tilde\Psi)}$$

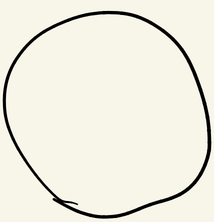
Choose $\tilde\Psi$ w.r.t. to Ψ s.t. $\le \Psi$.

Same reasoning for (δ)

$$\le C\,\frac{\Psi}{v(n,-\delta,\tilde\Psi)}\left(1+\frac{|B_u(p)|}{|B_w|}\right)$$

$$\le C\,\frac{\Psi}{v(n,-\delta,\tilde\Psi)}\left(1+\tilde C / v(n,-\delta,\tilde\Psi)\right)$$

Choose $v(n,-\delta,\tilde\Psi)\sim \Psi^{1/3}$



For the result, notice

$$\fint_{B_w} f = \frac{|B_8(p)|}{|B_w|}\fint_{B_8(p)} f \le \frac{|B_{16}(w)|}{|B_w|}\,\Psi \lesssim \frac{\Psi}{v(n,-\delta,\tilde\Psi)}$$

and use Markov together with ←

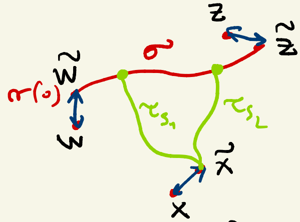
Proof (of Lemma 1)

Assume wlog $u(z) > u(w) = 0$ translating

let $\tilde d := d(\tilde z, \tilde w)$

By the fact that $|\nabla u| \underset{\delta\text{-split}}{\le} 1+\delta$ we have that

$$|u(\tilde w)| \le \Psi(\delta|n). \quad (\simeq)$$



Moreover (b) $\Rightarrow u(z) - u(w) \ge d(z,w) - \delta$, from which by (!) and $|\nabla u| \underset{\delta\text{-split}}{\le} 1+\delta$, we have

$$u(\tilde z) - u(\tilde w) \ge \tilde d - \Psi$$

Similarly $u(\tilde z) - u(\tilde w) \le \tilde d + \Psi$ and thus

$$|u(\tilde z) - u(\tilde w) - \tilde d| \le \Psi, \quad (\sim)$$

that is
$$\left| \int_0^{\tilde d} (\langle \nabla u(\sigma(s)), \dot\sigma(s) \rangle - 1)\, ds \right| \le \Psi. \quad (\bullet)$$

Now

(**)
$$\int_0^{\tilde d} |\nabla u(\sigma(s)) - \dot\sigma(s)|^2 ds = \tilde d + \int_0^{\tilde d} |\nabla u(\sigma(s))|^2 - 2\int_0^{\tilde d} \langle \nabla u(\sigma(s)), \dot\sigma(s) \rangle ds$$

$$= -\tilde d + \int_0^{\tilde d} |\nabla u(\sigma(s))|^2 - 2\left[\int_0^{\tilde d} (\langle \nabla u(\sigma(s)), \dot\sigma(s)\rangle - 1) ds\right]$$

$$\overset{(\bullet)}{\le} \int_0^{\tilde d} \left[|\nabla u|(\sigma(s))^2 - 1 \right] + \Psi \underset{\delta\text{-split}}{\overset{|\nabla u - 1| \le \delta}{\le}} \Psi \quad (\bullet\bullet)$$

(Φ) Moreover $\int_0^{\tilde d} |\langle \nabla u(\sigma(s)), \dot\sigma(s) \rangle - 1| ds \le \Psi$. Indeed

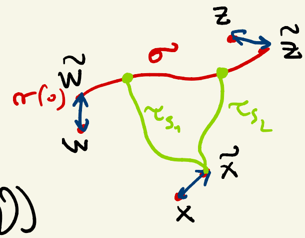
$$\int_0^{\tilde d} |\langle \nabla u \circ \sigma, \dot\sigma \rangle - 1| \le \int_0^{\tilde d} \left| |\nabla u|^2 \circ \sigma - 1 \right| + \int_0^{\tilde d} |\nabla u| \circ \sigma \, |\nabla u \circ \sigma - \dot\sigma|$$

$$\overset{\text{Hölder } (!!) + (**)}{\le} \Psi.$$

Hence

N

$$|u(\sigma(t)) - u(\sigma(0)) - t| = \left|\int_0^t (\langle Du(\sigma(s)), \dot\sigma(s)\rangle - 1)\, ds\right| \le \underline{\psi} \quad \forall t \in [0, \tilde d]$$



And then $|u(\sigma(t)) - t| \le \psi + |u(\sigma(0))| \le$

$$\le \psi + |u(\tilde w)| \le \psi \quad (\partial)$$

Recall $\ell(s) := d(\tilde x, \sigma(s))$, and $\ell'(s) = \langle \dot\sigma(s), \dot\tau_s(\ell(s))\rangle$

Hence

$$\frac{1}{2}(\ell^2(\tilde d) - \ell^2(0)) = \int_0^{\tilde d} \ell(s)\ell'(s)\, ds = \int_0^{\tilde d} \langle \dot\sigma(s), \dot\tau_s(\ell(s))\rangle \ell(s)\, ds$$

$$= \int_0^{\tilde d} \langle Du(\overset{\tau_s(\ell(s))}{\overset{\shortparallel}{\sigma(s)}}), \dot\tau_s(\ell(s))\rangle \ell(s)\, ds + \int_0^{\tilde d} \langle \dot\sigma(s) - Du(\sigma(s)), \dot\tau_s(\ell(s))\rangle \ell(s)\, ds$$

$$= \int_0^{\tilde d}\int_0^{\ell(s)} \langle Du(\tau_s(\ell(s))), \dot\tau_s(\ell(s))\rangle\, dt\, ds + \int_0^{\tilde d} \langle \dot\sigma(s) - Du(\sigma(s)), \dot\tau_s(\ell(s))\rangle \ell(s)\, ds$$

$$= \overbrace{\int_0^{\tilde d}\int_0^{\ell(s)} \langle Du(\tau_s(t)), \dot\tau_s(t)\rangle\, dt\, ds}^{I} + \overbrace{\int_0^{\tilde d} \langle \dot\sigma(s) - Du(\sigma(s)), \dot\tau_s(\ell(s))\rangle \ell(s)\, ds}^{II}$$

$$+ \overbrace{\int_0^{\tilde d}\int_0^{\ell(s)}\int_t^{\ell(s)} D^2u(\dot\tau_s(r), \dot\tau_s(r))\, dr\, dt\, ds}^{III}$$

$$|III| \le \left[\max_{s\in[0,\tilde d]} \ell(s)\right] \int_0^{\tilde d}\int_0^{\ell(s)} |D^2u|(\dot\tau_s(r), \dot\tau_s(r))\, dr\, ds \overset{(!!)}{\le} \Psi$$

$$|II| \le \Psi \quad \text{by } (\bullet\bullet)$$

$$I + \frac{1}{2}\tilde d = \int_0^{\tilde d} [u(\sigma(s)) - u(\tilde x)]\, ds + \frac{1}{2}\tilde d^2 \simeq \int_0^{\tilde d} (s - \tilde d)\, ds + \frac{1}{2}\tilde d^2 \pm \psi$$

$$= \pm\psi$$

- $u(\sigma(s)) = s \pm \psi$ by (∂)
- $u(\tilde x) \underset{(a)}{=} u(\tilde z) \pm \psi \underset{(\simeq)}{=} \tilde d + \psi$

Hence $\frac{1}{2}\left(d^2(\tilde{x},\tilde{z}) - d^2(\tilde{x},\tilde{w})\right) = \frac{1}{2}\left(\ell^2(\bar{d}) - \ell^2(0)\right)$ N

$= \left(\mathrm{I} + \frac{1}{2}\tilde{d}^2\right) + \mathrm{II} + \mathrm{III} - \frac{1}{2}\tilde{d}^2 = \pm\psi - \frac{1}{2}d^2(\tilde{w},\tilde{z})$

Hence $\left|\frac{1}{2}\left(d^2(\tilde{x},\tilde{z}) + \tilde{d}(\tilde{w},\tilde{z}) - d^2(\tilde{x},\tilde{w})\right)\right| \leq \Psi$

from which we have the conclusion since

$d(x,\tilde{x}) + d(z,\tilde{z}) + d(w,\tilde{w}) \leq \Psi$.

If $k \neq 1$ consider

$$u^* := \sum_J \frac{u_J(w) - u_J(z)}{|u(w) - u(z)|} u_J$$

and reduce to the 1d case

LEMMA 2

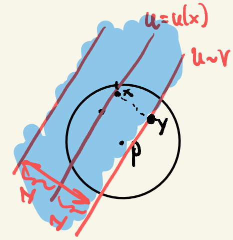
$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. the following holds, under the assumption of the theorem.

If $x \in B_2(p)$ and $v \in \mathbb{R}^k$ with $|v - u(x)| \leq 4$

Then $\exists \, y \in B_8(p)$ s.t.

$$\big| |u(x) - u(y)| - d(x,y) \big| \leq \varepsilon$$

$$|u(y) - \mathbf{v}| \leq \varepsilon$$



Pf. Let us give the proof for $k=1$ only.

LEMMA [Colding '97]

Let $p \in M$, $f \in C^\infty(M)$, $\ell, r > 0$, $0 \leq t \leq \ell$. Hence

unit speed geodesic with $\gamma_w(0) = q$, $\dot\gamma_w(0) = w$

$$\fint_{SB_r(p)} \left| (f \circ \gamma_w)'(t) - \frac{f(\gamma_w(\ell)) - f(\gamma_w(0))}{\ell} \right| d\mathcal{L}$$

$SB_r(p) = \{(q,w) : q \in B_r(p), \ w \in T_q^1 M\}$

$T_q^1 M$: norm 1 element in $T_q M$

Liouville measure. Preserved under the geodesic flow. It is the product measure of the volume measure and the measure on the surface of the unit sphere $S^1 \subset T_p M$ renormalized to have volume 1.

$$\leq \frac{2\ell}{\mathrm{Vol}(B_r(p))} \int_{B_{r+\ell}(p)} |\nabla^2 f| \, d\mathrm{vol}$$

Proof.

We have
$$\underbrace{(f \circ \gamma_w)'(t) - \frac{f(\gamma_w(\ell)) - f(\gamma_w(0))}{\ell}}_{(A)} = \int_0^t (f \circ \gamma_w)''(s) \, ds - \frac{\int_0^\ell \int_0^s (f \circ \gamma_w)''(\tau) \, d\tau \, ds}{\ell}$$

Hence $|A| \leq 2 \int_0^\ell |(f \circ \gamma_w)''(\tau)| \, d\tau$