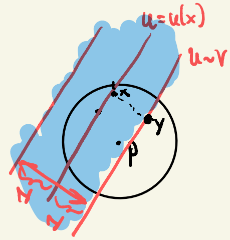
LEMMA 2

$\forall \varepsilon>0\ \exists \delta>0$ s.t. the following holds, under the assumption of the theorem.

If $x \in B_2(p)$ and $v \in \mathbb{R}^k$ with $|v-u(x)| \le 4$

Then $\exists\, y \in B_8(p)$ s.t.

$$\big||u(x)-u(y)| - d(x,y)\big| \le \varepsilon$$
$$|u(y)-v| \le \varepsilon$$



Pf. Let us give the proof for $k=1$ only.

LEMMA [Colding '97]

Let $p\in M$, $f\in C^\infty(M)$, $l, r>0$, $0\le t\le l$. Hence

unit speed geodesic with $\gamma_w(0)=q$, $\gamma_w'(0)=w$

$$⨍_{SB_r(p)} \left| (f\circ\gamma_w)'(t) - \frac{f(\gamma_w(l)) - f(\gamma_w(0))}{l} \right| d\lambda$$

$SB_r(p) = \{(q,w) : q\in B_r(p),\ w\in T^1_qM\}$, T^1_qM: norm 1 element in T_qM

Liouville measure. Preserved under the geodesic flow. It is the product measure of the volume measure and the measure on the surface of the unit sphere $S^1 \subseteq T_pM$ renormalized to have volume 1.

$$\le \frac{2l}{\mathrm{Vol}(B_r(p))} \int_{B_{r+l}(p)} |\nabla^2 f|\, d\mathrm{vol}$$

Proof.

We have
$$\overbrace{(f\circ\gamma_w)'(t) - \frac{f(\gamma_w(l)) - f(\gamma_w(0))}{l}}^{(A)} = \int_0^t (f\circ\gamma_w)''(s)\,ds - \frac{\int_0^l\int_0^s (f\circ\gamma_w)''(\tau)\,d\tau\, ds}{l}$$

Hence $|A| \le 2\int_0^l |(f\circ\gamma_w)''(\tau)|\,d\tau$

From which

$$\fint_{SB_r(p)} |A|\, d\lambda \leq \frac{2}{\lambda(SB_r(p))} \int_0^{\ell} \int_{SB_r(p)} |(f\circ\gamma_w)''(\tau)|\, d\lambda\, d\tau$$

$$\leq \frac{2}{\mathrm{Vol}(B_r(p))} \int_0^{\ell} \int_{SB_r(p)} |D^2 f|(\gamma_w(t))\, d\lambda\, d\tau$$

$$\leq \frac{2}{\mathrm{Vol}(B_r(p))} \int_0^{\ell} \int_{SB_{r+\ell}(p)} |D^2 f|(\gamma_w(0))\, d\lambda\, d\tau$$

$$= \frac{2}{\mathrm{Vol}(B_r(p))} \int_0^{\ell} \int_{B_{r+\ell}(p)} |D^2 f|\, d\mathrm{vol}\, d\tau$$

$$= \frac{2\ell}{\mathrm{Vol}(B_r(p))} \int_{B_{r+\ell}(p)} |D^2 f|\, d\mathrm{vol}$$

Let us come back to the proof of the LEMMA 2.

Let us apply Colding's lemma with $r = \Psi$ to determine, $\ell = |v - u(x)|$, $t = 0$, $f \equiv u$. Hence $\ell \leq 4$ and

$$\fint_{SB_4(x)} \left| \langle \nabla u, w\rangle - \frac{u(\gamma_w(\ell)) - u(\gamma_w(0))}{\ell} \right| d\lambda \leq \frac{2\ell\, \mathrm{Vol}(B_{\Psi+\ell}(x))}{\mathrm{Vol}(B_\Psi(x))} \fint_{B_{\Psi+\ell}(x)} |\mathrm{Hess}\, u|$$

Hence, by ([redacted]) and Bishop-Gromov, it follows

determination of Ψ small enough as explained in the argument about the segment inequality

$$\fint_{SB_\psi(x)} \left|\ell\langle \nabla u, w\rangle - \left(u(\gamma_w(\ell)) - u(\overbrace{\gamma_w(0)}^{x' \,=})\right)\right| d\lambda(x', w) \le \tilde{\Psi}$$

Hence $\exists\, x' \in B_\psi(x)$ s.t.

(*) $$\int_{T^1_{x'}M} \left|\ell\langle \nabla u, w\rangle - \left(u(\gamma_w(\ell)) - u(\overbrace{\gamma_w(0)}^{x' \,=})\right)\right| d\sigma(w) \le \tilde{\Psi}$$

surface measure on the unit sphere $T^1_x M$

and we can also ask

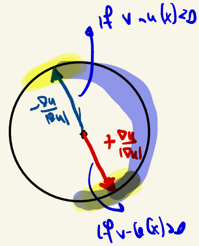
(**) $$\big|\, |\nabla u|(x') - 1 \,\big| \le \tilde{\Psi}$$

because u is $(1,\delta)$-splitting

By (*), (**) we can find a vector $w \in T^1_{x'}M$ s.t.

(*) $\rightsquigarrow$ $|\ell\langle \nabla u(x'), w\rangle - (u(\gamma_w(\ell)) - u(x'))| \le \tilde{\Psi}$

(**) $\rightsquigarrow$ $|\ell\langle \nabla u(x'), w\rangle - (v - u(x))| \le \tilde{\Psi}$



Set $y = \gamma_w(\ell) \in B_\delta(p)$. Moreover from the previous ones you have $|v - u(x) - (u(y) - u(x'))| \le \tilde{\psi}$. (D)
Moreover, since $|\nabla u| \le 1+\delta$, we have $|u(x) - u(x')| \le \tilde{\Psi}$ and hence $|v - u(y)| \le \tilde{\psi}$ from (D). This proves the IInd inequality of the statement

Finally $|u(y) - u(x)| \overset{\delta\text{-splitting}}{\le} (1+\psi)\, d(x,y) \le d(x,y) + \Psi$ and

$$|u(y) - u(x)| \ge |v - u(x)| - |v - u(y)| = d(y,x') - |v - u(y)|$$
$$\ge d(y,x') - \Psi \ge d(y,x) - \Psi$$

Thus, from the previous two inequalities, the Ist inequality of the statement follows.

CONCLUSION

For simplicity, let's give the proof with $k=1$

By contradiction $\exists\, \varepsilon > 0$, $\delta_J \to 0$ (M^n_J, g_J, p_J) s.t.

- $\mathrm{Ric}_{M^n_J} \geq -(n-1)\delta_J$
- $\exists\, u_J : B_{16}(p_J) \to \mathbb{R}$ that is $(1, \delta_J)$ splitting
- No metric space (Z, d^Z, z) can realize

$$d_{GH}\left(B_1(p_J), B_1((z,0))\right) < \varepsilon \quad \text{for some } J$$

Without loss of generality, $u_J(p_J) = 0$.

Then, up to ~~subsequences~~ subsequences and using Ascoli-Arzelà + Gromov pre-compactness we have

- $(M^n_J, g_J, p_J) \xrightarrow[pGH]{} (X, d, x)$
- $\exists\, u_\infty : B_{16}(x) \to \mathbb{R}$ 1-Lip. s.t. $u_\infty(x) = 0$

given $\tilde{x} \in B_2^X(x)$, for which $|u_\infty(\tilde{x})| \leq 2$

Now by using lemma 2 we have that given $|v| \leq 1$

for J large enough and up to sub. $\exists\, \tilde{p}_J \in B_2(p_J)$, $\tilde{p}_J \to \tilde{x}$ $\exists\, y_J \in B_8(p_J)$ s.t.

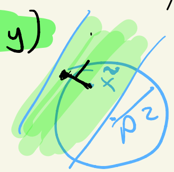
$$\left| |u_J(y_J) - u_J(\tilde{p}_J)| - d_J(\tilde{p}_J, y_J) \right| \longrightarrow 0$$

$$|u_J(y_J) - v| \longrightarrow 0 \qquad \text{as } J \to +\infty$$

Hence up to subsequence, $y_J \longrightarrow y \in B_8(x)$

$V = 0{,}5$

s.t. $u_\infty(y) = v$ and $|u_\infty(y) - u_\infty(\tilde{x})| = d(\tilde{x}, y)$



Let us see that such y is unique.

Assume $u_\infty(y_1) = u_\infty(y_2) = v$ and

$$|u_\infty(y_\ell) - u_\infty(\tilde{x})| = d(\tilde{x}, y_\ell) \qquad \ell = 1, 2.$$

Now approximate $y_{i,\ell} \xrightarrow[i \to +\infty]{} y_\ell \qquad \ell = 1, 2.$

Hence $\big|\, |u_J(\tilde{p}_J) - u_J(y_{J,\ell})| - d_J(\tilde{p}_J, y_{J,\ell}) \big| \xrightarrow[J \to +\infty]{} 0$

$$|u_J(y_{J,1}) - u_J(y_{J,2})| \xrightarrow[J \to +\infty]{} 0$$

Hence from <u>Lemma 1</u>, up to subsequences

$$|d_J^2(y_{J,1}, y_{J,2}) + d_J^2(y_{J,2}, \tilde{p}_J) - d_J^2(y_{J,1}, \tilde{p}_J)| \to 0$$

We have that $d_J^2(y_{J,\ell}, \tilde{p}_J) = |v - u_\infty(\tilde{x})|^2 + o(1)$ as $J \to +\infty$, and hence

$$d_J^2(y_{J,1}, y_{J,2}) \xrightarrow[J \to +\infty]{} 0$$

from which taking the limit we have $y_1 = y_2$.

Let us now define $Z := u_\infty^{-1}(0)$ and

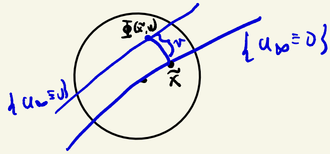
$$\Phi : (\bar{B}_2^X(x) \cap Z) \times \bar{B}_1^{\mathbb{R}}(0) \to X$$
$$(\tilde{x}, v) \to \Phi(\tilde{x}, v)$$

where $\bar\Phi(\tilde x, v)$ is the unique point (in $B_8(x) \subseteq X$) s.t.

$$d(\bar\Phi(\tilde x, v), \tilde x) = |u_\infty(\tilde x) - u_\infty(\bar\Phi(\tilde x, v))|$$

and

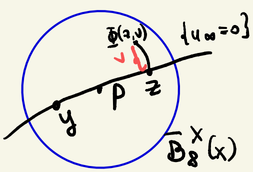
$$u_\infty(\bar\Phi(\tilde x, v)) = v$$



We have that $\bar\Phi$ surjective on $\bar B_1^X(x)$. Indeed take $w \in \bar B_1^X(x)$. We have $|u_\infty(w)| \le 1$ by 1-Lip. Apply lemma 2 with $v = 0$ and then, by approx., we get $\tilde x$ s.t. $u_\infty(\tilde x) = 0$ and $d(\tilde x, w) = |u_\infty(w)|$. [Hence $d(x, \tilde x) \le d(\tilde x, w) + d(x, w) \le 2$]
$\tilde x \in B_2(x)$

Let us now prove that $\bar\Phi$ is distance preserving.

CASE 1

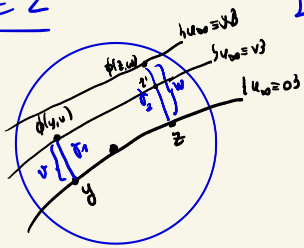


Take $(y, 0), (z, v) \in (\bar B_2^X(x) \cap Z) \times \bar B_1^{\mathbb{R}}(0)$

Hence $\bar\Phi(y, 0) = y$. By lemma 1 (Pythagoras) we have

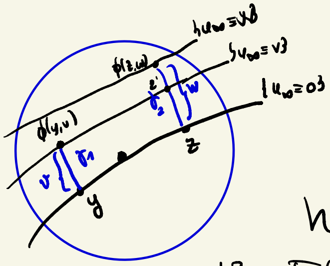
$$d(\bar\Phi(y,0), \bar\Phi(z,v))^2 = d^2(y,z) + v^2$$

CASE 2



[SKETCH] Wlog $w > v > 0$. Take γ_1, γ_2 geodesic as in figure, z' the intersection of γ_2 with $\{u_\infty = v\}$.

Hence lemma 1 implies



$$d^2(\Phi(y,v), z')^2 + (w-v)^2 =$$
$$= d^2(\Phi(y,v), \Phi(z,w)) \,. \qquad (\sim)$$

But applying Lemma 1 twice we have

$$d^2(\Phi(y,v), z') = -v^2 + d^2(z', y)$$
$$= -\cancel{v^2} + \cancel{v^2} + d^2(y,z)$$
$$= d^2(y,z)$$

Hence by the previous and $(\sim)$ we get the conclusion.

Hence $\Phi\big|_{\Phi^{-1}(\bar{B}_1^X(x))} : \Phi^{-1}(\bar{B}_1^X(x)) \longrightarrow \bar{B}_1^X(x)$

is an isometry. This is a contradiction with the absurd hypothesis since then $\bar{B}_1^X(x)$ is isometric to the ball of radius 1 in the product $\mathbb{R} \times Z$.