References [See the references for the Ist lecture, and add the following ones]

IInd lecture

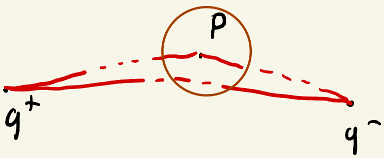
[GigliDePhilippis16] "From volume cone to metric cone in the nonsmooth setting" (2016) by G. De Philippis & N. Gigli

[Ketterer14] "Cones over metric measure spaces and the maximal diameter theorem" (2014) by C. Ketterer

THE ALMOST SPLITTING THEOREM

Thm [Cheeger-Colding '96]
$\forall \varepsilon \exists \delta$ s.t. the following holds

If (M,g) satisfies $Ric \geq -(n-1)\delta$ and



p, q^+, q^- are three points such that

(a) $\min\{d(p,q^+), d(p,q^-)\} \geq \delta^{-1}$

(b) $E(p) := d(p,q^+) + d(p,q^-) - d(q^+,q^-) \leq \delta$

Then

$$d_{GH}\left((B_1(p),p), (B_1(x,0'),(x,0')) \right) \leq \varepsilon$$

where $(x,0) \in X \times \mathbb{R}$, X being a metric boundedly compact metric space.

Let's prove it!

What we already proved...

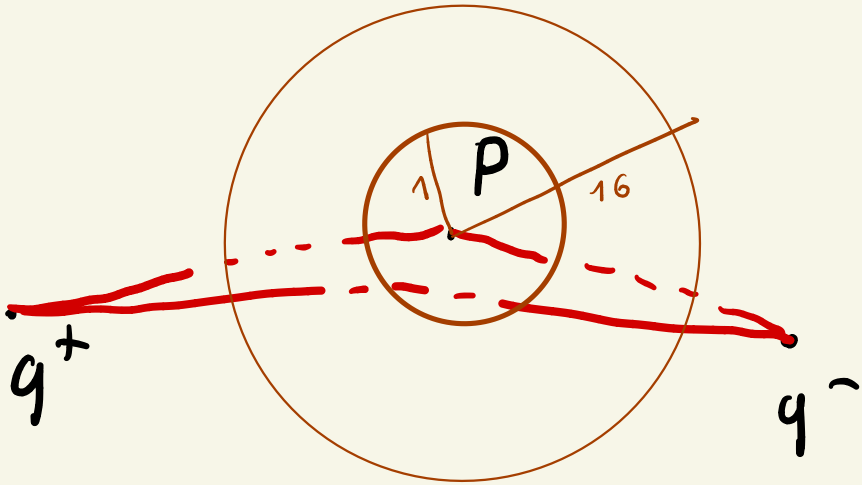
FROM SPLITTING MAP TO SPLITTING SPACES

Thm $\forall \varepsilon\ \exists \delta$ s.t the following holds.
If (M,g) satisfies $\mathrm{Ric} \geq -(n-1)\delta$ and $u : B_{16}(p) \to \mathbb{R}^k$ is a (k,δ)-splitting map, hence $\exists\ Z$ complete length metric space s.t.

$$d_{pGH}\Big((B_1(p), p) \,,\, (B_1(z,0^k), (z,0^k)) \Big) < \varepsilon$$

PROOF OF THE ALMOST SPLITTING THEOREM

Proof Idea: We will build a $(1, \Psi(\delta|n))$ splitting map on $B_1(p)$.



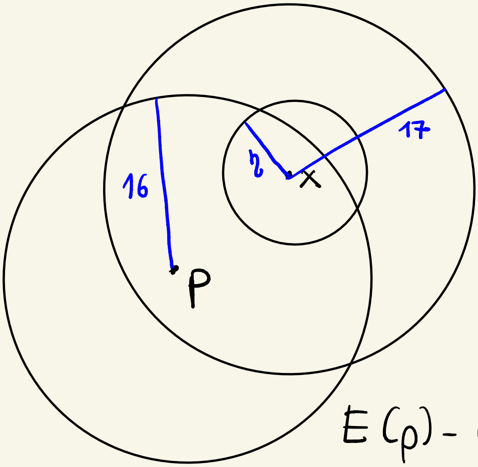
Let us work on the larger ball $B_{16}(p)$.

Let $E(\cdot) := d(\cdot, q^+) + d(\cdot, q^-) - d(q^+, q^-)$.

Step 1 We aim at showing $E \leq \Psi(\delta|n)$ on $B_{16}(p)$

From Laplacian comparison & the simple inequality $\sqrt{\delta}\, r \coth(\sqrt{\delta}\, r) \leq 1 + \sqrt{\delta}\, r$, we obtain

$$\Delta E \leq \underbrace{2(n-1)\left(\frac{1}{\delta^{-1} - 16} + \sqrt{\delta}\right)}_{C_{n,\delta}} \quad \text{on} \quad B_{16}(p)$$



Take any $x \in B_{16}(p)$.

Take

$$G := C_{n,\delta}\,\varphi_{-\delta}(d(x,\cdot), 17)$$

Hence

for δ small enough

$$E(p) - G(p) \overset{(\alpha 1)}{\underset{\substack{(b)+\\ G \text{ nonincreasing}\\ \text{in the distance}}}{\leq}} \delta - C_{n,\delta}\,\varphi_{-\delta}(16, 17) < 0$$

(α).

$$\underset{\substack{E \geq 0 \text{ by principle}\\ G(\bar{x})=0\ \forall \bar{x} \in \partial B_{17}(x)}}{\leq} E(\bar{x}) - G(\bar{x}) \qquad \forall \bar{x} \in \partial B_{17}(x)$$

Moreover, we have $\Delta(E-G) \leq 0$ on $B_{17}(x)$.

We have two cases

(β_1) • $\eta \geq d(p,x)$. Hence $E(x) \underset{\substack{E \text{ is}\\ 2\text{-Lip}}}{\leq} E(p) + 2d(p,x) \leq \delta + \eta$

(β_2) • $\eta \leq d(p,x)$. Hence $p \in B_{17}(x) \setminus B_\eta(x)$. By max. principle and (α) the minimum of $E-G$ is reached on $\tilde{x} \in \partial B_\eta(x)$

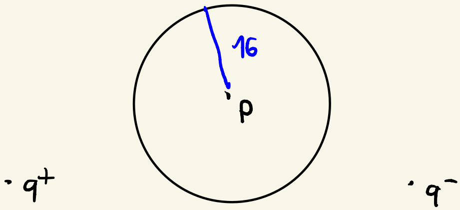
Hence $E(x) \underset{E \text{ is } 2\text{-Lip}}{\leq} E(\tilde{x}) + 2\eta = E(\tilde{x}) - G(\tilde{x}) + 2\eta + G(\tilde{x}) \leq$

$$\underset{\tilde{x}\text{ minimum}}{\leq} E(p) - G(p) + 2\eta + G(\tilde{x}) \underset{(\alpha 1)}{\leq} 2\eta + C_{n,\delta}\,\varphi_{\delta}(\eta, 17)$$

Take η s.t. $\eta = C_{n,\delta}\,\varphi_{-\delta}(\eta, 17)$, and notice that $\eta = \psi(\delta|n)$.

Hence joining (β_1), (β_2) we have $E \leq \psi(\delta|n)$ on $B_{16}(p)$.

Step 2 Let $b^\pm(x) := d(x,q^\pm) - d(p,q^\pm)$ and let



$$(\mathrm{I})\quad \begin{cases}\beta^\pm = b^\pm & \text{on } \partial B_{16}(p)\\ \boxed{\Delta\beta^\pm = 0} & \text{on } B_{16}(p)\end{cases}$$

Hence $|b^\pm - \beta^\pm| \le \Psi(\delta|n)$.

Indeed, by Laplace comparison

$$\Delta(b^\pm - \beta^\pm) \le (n-1)\left(\frac{1}{\delta^{-1}-16}+\sqrt{\delta}\right) \overset{(A)}{=:} k_{n,\delta} \quad \text{on } B_{16}(p).$$

Let $G^\pm = k_{n,\delta}\,\varphi_{-\delta}(d(\cdot,q^\pm),\, d(p,q^\pm)+16)$

Hence $\Delta(b^\pm-\beta^\pm-G^\pm)\le 0$ on $B_{16}(p)$

and then by max principle

$$b^\pm - \beta^\pm - G^\pm \ge -k_{n,\delta}\,\varphi_{-\delta}(d(p,q^\pm)-16,\, d(p,q^\pm)+16)$$

(γ) → $= -\Psi(\delta|n)$

$\varphi_{-\delta}(d(p,q^\pm)-16,\, d(p,q^\pm)+16)$
is uniformly bounded as $\delta\to 0$

(δ) → (b)

Now $-\delta \overset{(b)}{\le} -E(p) \le b^+ + b^- = E - E(p) \le E \overset{\text{Step 1}}{\le} \Psi$ on $B_{16}(p)$

Moreover, from the M.P., $\beta^+ + \beta^- \ge \min_{\partial B_{16}(p)}(\beta^+ + \beta^-) = \min_{\partial B_{16}(p)}(b^+ + b^-) \overset{(\delta)}{\ge} -\delta$ (ε)

Hence $b^\pm \overset{(\delta)}{\le} \Psi - b^\mp \overset{(\gamma)}{\le} \Psi - \beta^\mp \overset{(\varepsilon)}{\le} \Psi + \beta^\pm$ ← (ξ)

Then (γ)+(ξ) gives the conclusion

Step 3 We have $\boxed{\sup_{B_8(p)} |\nabla \beta^+| \leq C}$ (II)

Indeed notice that

$$\inf_{B_{16}(p)} \beta^+ = \inf_{B_{16}(p)} (\beta^+ - b^+ + b^+) \overset{\text{Step 2}}{\geq} -\psi - 16$$

and similarly $\sup_{B_{16}(p)} \beta^+ \leq \psi + 16.$

Hence let $\bar{\beta}^+ := \beta^+ + |\inf_{B_{16}(p)} \beta^+|$. From above we get $\bar{\beta}^+ \leq 2(\psi+16)$ on $B_{16}(p)$.

By Cheng-Yau estimate on $\bar{\beta}^+$ $\left(\begin{array}{ll} \Delta \bar{\beta}^+ \equiv 0 & \text{on } B_{16}(p) \\ \bar{\beta}^+ \geq 0 & \text{on } B_{16}(p) \end{array} \right)$

we have $\sup_{B_8(p)} |\nabla \beta^+| = \sup_{B_8(p)} |\nabla \bar{\beta}^+| \leq C_{n,\delta} \cdot 2(\psi+16) =: C$

Step 4 We have $\boxed{\int_{B_4(p)} \big| |\nabla \beta^+|^2 - 1 \big| \leq \Psi}$, (III)

Step 4a Let us first prove $\int_{B_4(p)} |\nabla \beta^+ - \nabla b^+|^2 \leq \Psi$ (!)

From Laplacian comparison, see (A), we have

$$\Delta(\beta^+ - b^+) \geq -\Psi \quad \text{on } B_{16}(p) \qquad \text{(A')}$$

Take $\varphi \in \mathrm{Lip}_c(M)$ such that $\varphi \equiv 1$ on $B_4(p)$, $\varphi \equiv 0$ on $M \setminus B_8(p)$, $|\nabla \varphi| \leq 20$ on M, $0 \leq \varphi \leq 1$ on M

Test (A') against $\varphi(\beta^+ - b^+ + \psi) \geq 0$ from (3).

Hence

$$-\int_{B_\delta(p)} \varphi |D\beta^+ - Db^+|^2 - \int_{B_\delta(p)} \langle D\varphi, D\beta^+ - Db^+ \rangle (\beta^+ - b^+ + \psi) \geq -\varphi \operatorname{Vol}(B_\delta(p))$$

From which

$$\int_{B_\delta(p)} \varphi |D\beta^+ - Db^+|^2 \leq \varphi \operatorname{Vol}(B_\delta(p)) + \int_{B_\delta(p)} |D\varphi| |D\beta^+ - Db^+| |\beta^+ - b^+ + \psi| \qquad (\oint)$$

From $(\gamma) + (\xi)$ we have $|\beta^+ - b^+ + \psi| \leq 2\Psi$.

From step 3 we have

$$\sup_{B_\delta(p)} |D\beta^+| \leq C$$

Hence from $(\oint)$ we conclude

$$\int_{B_4(p)} |D\beta^+ - Db^+|^2 \leq \varphi \operatorname{Vol}(B_\delta(p)) + 20\tilde{C}\Psi = \Psi$$

and thus we have the claim (∇)

$$\left| |\nabla\beta^+|^2 - 1 \right| \overset{\text{almos everywhere}}{=} \left| |D\beta^+| - |\nabla b^+| \right| \left| |D\beta^+| + |Db^+| \right|$$

$$\leq |\nabla\beta^+ - \nabla b^+| \, \left| |D\beta^+| + 1 \right|$$

Hence, since $|\nabla\beta^+|$ is uniformly bounded above by a constant in $B_\delta(p)$, see, we conclude that

$$\int_{B_4(p)} \left| |D\beta^+|^2 - 1 \right| \leq C \int_{B_4(p)} |D\beta^+ - Db^+|$$
$$\leq C \, \mathrm{Vol}(B_4(p))^{1/2} \left(\int_{B_4(p)} |D\beta^+ - Db^+|^2 \right)^{1/2}$$
$$\leq \Psi .$$

Thus the step 4 is proved

Step 5 We have $\boxed{\int_{B_1(p)} |D^2\beta^+|^2 \leq \Psi}$ (IV)

Take a good cut-off function such that

$$\begin{cases} \phi \equiv 1 & \text{on } B_1(p) \\ \phi \equiv 0 & \text{on } M \setminus B_4(p) \\ 0 \leq \phi \leq 1 & \text{on } M \\ |D\phi| + |\Delta\phi| \leq C(n) & \text{on } M \end{cases}$$

Here we are supposing $\delta < 1$

Since $Ric \geq -(n-1)\delta$, by also exploiting Bochner, we have (remember $\Delta\beta^+ \equiv 0$!)

$$\frac{1}{2}\Delta|D\beta^+|^2 + (n-1)\delta|D\beta^+|^2 = |D^2\beta^+|^2 + Ric(D\beta^+, D\beta^+) + (n-1)\delta|D\beta^+|^2$$
$$\geq |D^2\beta^+|^2$$

Hence testing against φ we obtain

$$\int_{B_4(p)} \varphi|D^2\beta^+|^2 \leq \int_{B_4(p)} \frac{1}{2}\varphi\,\Delta(|D\beta^+|^2 - 1) + (n-1)\delta \int_{B_4(p)} |D\beta^+|^2$$

$$= \int_{B_4(p)} \frac{1}{2} \Delta\varphi \cdot (|D\beta^+|^2 - 1) + (n-1)\delta \int_{B_4(p)} |D\beta^+|^2$$

$$\leq C \left[\int_{B_4(p)} \left| |D\beta^+|^2 - 1 \right| + (n-1)\delta \right]$$

where in the last inequality we exploited the fact that $|D\beta^+|$ is uniformly bounded above by a constant in $B_4(p)$ [see step 3] and $|\Delta\varphi| \leq C$.

Hence, by exploiting the previous inequality, we have that

$$\int_{B_1(p)} |D^2\beta^+|^2 \leq \int_{B_4(p)} \varphi |D^2\beta^+|^2 \leq \Psi$$

and thus the proof of the step is concluded.

<u>CONCLUSION</u> (I) + (II) + (III) + (IV) imply that β^+ is $(1, \Psi)$-splitting on $B_1(p)$.

Hence we use "from splitting maps to splitting spaces" that we already proved and the proof of the almost splitting theorem is finished! $\blacksquare$

ALMOST VOLUME IMPLIES ALMOST METRIC CONE

Thm. [Cheeger-Colding '96] $\forall \varepsilon, v > 0$ $\forall n \in \mathbb{N}$ $\exists \delta$ s.t. the following holds.

If (M^n, g) has $Ric \geq -(n-1)\delta$ (H0),

and $Vol(B_1(p)) \geq v$ (H1), and

$$\frac{Vol(B_\delta(p))}{v(n,-\delta,\delta)} - \frac{Vol(B_{16}(p))}{v(n,-\delta,16)} < \delta \quad \text{(H2)}$$

Hence

$$d_{GH}\left((B_1(p), p), (B_1(z^*), z^*)\right) \leq \varepsilon$$

where z^* is the tip of a metric cone $C(Z)$.

Proof. follows the lines in [online notes of UCSD seminars] with minor modifications

WHAT IS A METRIC CONE?

For references, see [BBI, Chapter 3]

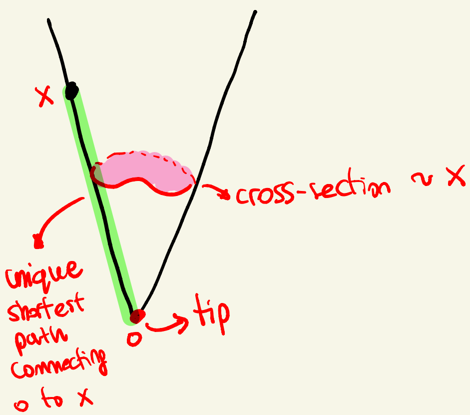
Def. Let (X,d) be a metric space.

Let $C(X) := [0,+\infty)\times X / \{0\}\times X$.

Given $a=(t,x)$, $b=(s,y)$ in $C(X)$, let us define

$$d_c(a,b) = \begin{cases} \sqrt{t^2+s^2-2ts\cos(d(x,y))} & \text{, if } d(x,y)\leq\pi \\ t+s & \text{, if } d(x,y)>\pi \end{cases}$$

Hence $(C(X),d_c)$ is a metric space called metric cone over X.



... and the measure? See [Ketterer 14]

Let
$$\mathrm{sn}_k(t) := \begin{cases} \frac{1}{\sqrt{k}} \sin(\sqrt{k}\, t) & k > 0 \\ t & k = 0 \\ \frac{1}{\sqrt{-k}} \sinh(\sqrt{-k}\, t) & k < 0 \end{cases}$$

$$\cos_k(t) := \begin{cases} \cos(\sqrt{k}\, t) & k > 0 \\ 1 & k = 0 \\ \cosh(\sqrt{-k}\, t) & k < 0 \end{cases}$$

$$C_k(X) := \begin{cases} [0, \pi/\sqrt{k}] \times X \,/\, \{0, \pi/\sqrt{k}\} \times X & k > 0 \\ [0, +\infty) \times X \,/\, \{0\} \times X & k \leq 0 \end{cases}$$

$$d_{C_k}((t,x),(s,y)) := \begin{cases} \cos_k^{-1}\left(\cos_k(s)\cos_k(t) + k\, \mathrm{sn}_k(s)\, \mathrm{sn}_k(t) \cos(d(x,y) \wedge \pi)\right) & k \neq 0 \\ \sqrt{s^2 + t^2 - 2ts \cos(d(x,y) \wedge \pi)} & k = 0 \end{cases}$$

$$m^n_{C_k} := \mathrm{sn}^n_k(t)\, dt \otimes m$$

Def. (X, d, m) m.m.s., hence $(C_k(X), d_{C_k}, m^n_{C_k})$ is the (k,n)-m.m. cone over X. [it's the warped product $[0, \pi_k) \times_{\mathrm{sn}_k} X$]

Thm [Ketterer14]

(i) Let (X,d,m) be an $RCD(N-1,N)$ space with $N\geq 1$ and $\operatorname{diam} X\leq\pi$. Let $k\geq 0$. Hence $(C_k(X), d_{C_k}, m_k^N)$ is an $RCD(kN, N+1)$ space.

(ii) Let (X,d,m) be a P.I. m.m.s. Let $k\in\mathbb{R}$ and $N\geq 0$. Suppose $(C_k(X), d_{C_k}, m_k^N)$ is an $RCD(kN, N+1)$ space. Hence

- if $N\geq 1$, (X,d,m) is an $RCD(N-1,N)$ space and $\operatorname{diam} X\leq\pi$
- if $N\in[0,1)$, X is a point or $N=0$ and X is exactly two points at distance π

For the version for sectional curvature bounds see [BBI, Theorem 4.7.1]