Lecture 4

VOLUME CONE IMPLIES METRIC CONE

Theorem [Gigli DePhilippis 16]

Let $N \in (0, +\infty)$, (X, d, m) $RCD(0, N)$ with $\mathrm{supp}(m) = X$.

Let $p \in X$, $R > r > 0$ such that

$$\frac{m(B_R(p))}{m(B_r(p))} = \left(\frac{R}{r}\right)^n .$$

Then

(i) If $\partial B_{R/2}(p) \equiv$ one point, $(X, d) \overset{\text{isom.}}{\simeq} [0, \mathrm{diam}\, X]$ with an isometry that sends p to 0 and $m|_{B_R(p)}$ to $N m(B_R(p))\, x^{n-1} dx$

(ii) If $\partial B_{R/2}(p) \equiv$ two points, $(X, d) \overset{\text{isom.}}{\simeq}$ 1-d Riemann. manifold (possibly with boundary) and there is a bijective local isometry from $B_R(p)$ to $(-R, R)$ that sends p to 0 and $m|_{B_R(p)}$ to $\frac{1}{2} N m(B_R(p)) |x|^{n-1} dx$

(iii) Otherwise $N \geq 2$, $\exists$ (Z, d_Z, m_Z) $RCD(N-2, N-1)$ with $\mathrm{diam}\, Z \leq \pi$ such that $B_R(p) \overset{\text{loc isom}}{\simeq} B_R(c)$ and such a loc. isom. is isom. on $\bar{B}_{\frac{R}{2}}(p)$.

$\to$ tip of the $(0, N)$-cone over Z

See [Gigli DePhilippis 16, Section 4] for $K \neq 0$, annulus
[A. Bruè Semola 19] for the "almost" result on ncRCD spaces.

A COROLLARY OF THE ALMOST VOLUME CONE IMPLIES ALMOST METRIC CONE THEOREM

Thm [Cheeger–Colding '97]

Let (X,d) be a non collapsed Ricci-limit space

i.e. $(M_i^n, d_{g_i}, p_i) \xrightarrow[pGH]{} (X,d,x)$

$\mathrm{Ric}_{M_i^n} \geq -1$

$\inf_i \mathrm{vol}(B_1(p_i)) > 0$

Hence, for every $x \in X$, every tangent cone at x

i.e. any limit point of $(X, r_i^{-1} d, x)$ with $r_i \downarrow 0$

is a metric cone over a metric space of diam $\leq \pi$.

See [GigliDePhilippis 18] for the nonsmooth version

ALMOST VOLUME IMPLIES ALMOST METRIC CONE

Thm. $\forall \varepsilon, v > 0$ $\forall n \in \mathbb{N}$ $\exists \delta$ s.t. the following holds.

If (M^n, g) has $Ric \geq -(n-1)\delta$ (H0),

and $Vol(B_1(p)) \geq v$ (H1), and

$$\frac{Vol(B_\delta(p))}{V(n,-\delta,\delta)} - \frac{Vol(B_{16}(p))}{V(n,-\delta,16)} < \delta \quad \text{(H2)}$$

Hence

$$d_{GH}\left((B_1(p), p), (B_1(z^*), z^*)\right) \leq \varepsilon$$

where z^* is the tip of a metric cone $C(Z)$, with $\operatorname{diam} Z \leq \pi$.

Let's start to prove it!

Strategy mainly elaborated from

- Some notes online from UCSD seminar on Cheeger-Colding Theory, Ricci flow, Einstein metrics and related topics (Fall 2020)

STEP 1 From (H2) to a control of boundries

By hypothesis and BG (ii a) we have

$$\frac{|B_{16}(p)|}{v(n,\delta,16)} \geq \frac{|B_s(p)|}{v(n,\delta,s)} - \delta \quad \forall s \in [\delta, 1]$$

Hence

$$|B_{16}(p) \setminus B_s(p)| \overset{(*)}{\geq} |B_s(p)| \left(\frac{v(n,\delta,16) - v(n,\delta,s)}{v(n,\delta,s)} \right) - \delta v(n,\delta,16) \quad \forall s \in [\delta, 1]$$

Thus

$$\frac{Per(B_s(p))}{s(n,\delta,s)} \underset{\substack{BG \\ (ii\,b)}}{\geq} \frac{|B_{16}(p) \setminus B_s(p)|}{v(n,\delta,16) - v(n,\delta,s)} \overset{\text{Using } (*)}{\geq}$$

$$\geq \frac{|B_s(p)|}{v(n,\delta,s)} - \frac{\delta v(n,\delta,16)}{v(n,\delta,16) - v(n,\delta,s)} =$$

$$= \frac{|B_s(p)|}{v(n,\delta,s)} \left[1 - \frac{\delta v(n,\delta,16)\, v(n,\delta,s)}{|B_s(p)|\, (v(n,\delta,16) - v(n,\delta,s))} \right]$$

$$\underset{\substack{BG \\ + |B_1| \geq v}}{\geq} \frac{|B_s(p)|}{v(n,\delta,s)} \left[1 - \frac{\delta v(n,\delta,16)\, v(n,\delta,1)}{v\, (v(n,\delta,16) - v(n,\delta,s))} \right].$$

Taking $s = \delta$ we have

$$(\alpha_0) \qquad \frac{Per(B_\delta(p))}{s(n,\delta,\delta)} \geq \frac{|B_\delta(p)|}{v(n,\delta,\delta)} \left(1 - \varphi(\delta \mid n, v) \right)$$

STEP 2 Existence of conical functions

Let $0<\varepsilon<1/2$ be fixed.

Proposition Let (M^n, g) be a smooth manifold such that

- $Ric \geq -(n-1)\delta$, $\delta>0$
- (H2) $\dfrac{Vol(B_\delta(p))}{v(n,-\delta,\delta)} - \dfrac{Vol(B_{16}(p))}{v(n,-\delta,16)} < \delta$ (H2)
- $|B_1(p)| \geq v$

Hence on $B_1(p)\setminus B_\varepsilon(p)$ $\exists h$ a "$\Psi(\delta|n,v,\varepsilon)$-conical function", i.e.

(a) $\displaystyle\int_{B_1(p)\setminus B_\varepsilon(p)} |\nabla^2 h - 2g| \leq \Psi$

(b) $\displaystyle\int_{B_1(p)\setminus B_\varepsilon(p)} |\,|\nabla h|^2 - 4h| \leq \Psi$

(c) $\displaystyle\int_{B_1(p)\setminus B_\varepsilon(p)} |\nabla(h - d_p^2)| \leq \Psi$

(d) $\displaystyle\sup_{B_1(p)\setminus B_\varepsilon(p)} |h - d_p^2| \leq \Psi$

Proof. Let us start with (c).

Solve $\begin{cases} \Delta h = 2n & B_2(p)\setminus B_{\frac{\varepsilon}{2}}(p) \\ h = d_p^2 & \partial(B_2(p)\setminus B_{\frac{\varepsilon}{2}}(p)) \end{cases}$ $h \in C^\infty(B_2(p)\setminus B_{\frac{\varepsilon}{2}}(p)) \cap Lip(\bar{B}_2(p)\setminus B_{\frac{\varepsilon}{2}}(p))$

By max. principle ($\geq$ ok, $\leq$ the quantitative one)

(α) $\displaystyle\sup_{B_2(p)\setminus B_{\frac{\varepsilon}{2}}(p)} |h| \leq C(n,\delta,\varepsilon)$

Now

$$\Delta(d_p^2) \overset{\text{Lapl. comp.}}{\le} 2(n-1)\,d_p\sqrt{\delta}\,\coth(\sqrt{\delta}\cdot d_p) + 2$$
$$\le 2n + C(n)\,\delta \qquad \text{on } B_2\setminus B_{\varepsilon/2}$$

Thus $\Delta(h-d_p^2) \overset{(I)}{\ge} -C(n)\,\delta$ on $B_2\setminus B_{\varepsilon/2}$ (distr.)

Moreover $\displaystyle\int_{B_2\setminus B_{\varepsilon/2}} \Delta(h-d_p^2) - 2n|B_2\setminus B_{\varepsilon/2}| - 4|\partial B_2| + \varepsilon|\partial B_{\frac{\varepsilon}{2}}|$

$$= 2\left(n|B_2| - 2\,\mathrm{Per}(B_2)\right) - 2\left(n|B_{\frac{\varepsilon}{2}}| - \tfrac{\varepsilon}{2}\mathrm{Per}(B_{\frac{\varepsilon}{2}})\right) \overset{(II)}{\underset{\text{use } (\alpha_0)}{\le}} \Psi(\delta|\varepsilon,n,v)$$

So from (I)+(II), $\displaystyle\int_{B_2\setminus B_{\varepsilon/2}} |\Delta(h-d_p^2)| \le \Psi \qquad (\beta)$

Hence i.b.p. $\displaystyle\int_{B_2\setminus B_{\varepsilon/2}} |\nabla(h-d_p^2)|^2 \underset{(\alpha)+(\beta)}{\le} \Psi$ which is (c)

Now by Poincaré

$$(\gamma)\quad \int_{B_2\setminus B_{\varepsilon/2}} |h-d_p^2|^2 \le \Psi \quad \text{using (c).}$$

Since $\Delta|\nabla h|^2 = 2\underbrace{|\nabla^2 h|^2}_{\ge 0} + 2\,Ric(\nabla h,\nabla h) \ge -2(n-1)\,\delta\,|\nabla h|^2$

And thus, using Moser iteration, $\displaystyle\sup_{B_{\frac32}\setminus B_{\frac34\varepsilon}} |\nabla h| \le C(n,v,\varepsilon)\left(\int_{B_2\setminus B_{\frac{\varepsilon}{2}}} |\nabla h|^2\right)^{\frac12}$

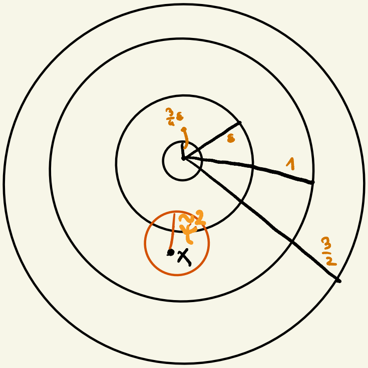
But $\||\nabla h|\|_{2,B_2\setminus B_{\frac{\varepsilon}{2}}} \le \||\nabla(h-d_p^2)|\|_{2,B_2\setminus B_{\frac{\varepsilon}{2}}} + \||\nabla d_p^2|\|_{2,B_2\setminus B_{\frac{\varepsilon}{2}}} \le C(n,v,\varepsilon)$

Hence $\displaystyle\sup_{B_{\frac32}\setminus B_{\frac34\varepsilon}} |\nabla h| \le C(n,v,\varepsilon)$ (γ') and then $\displaystyle\sup_{B_{\frac32}\setminus B_{\frac34\varepsilon}} |\nabla(h-d_p^2)| \le C(n,v,\varepsilon)$ (δ)

Now $(\gamma)+(\delta) \Rightarrow \displaystyle\sup_{B_1\setminus B_\varepsilon} |h-d_p^2| \le \Psi$, which is (d). Indeed, see the following lemma

Lemma. $|\nabla f| \le 1$ $\quad B_{\frac{3}{2}} \setminus B_{\frac{3}{4}\varepsilon}$

$$\int_{B_{\frac{3}{2}} \setminus B_{\frac{3}{4}\varepsilon}} |f| \le \psi$$



Hence $|f| \le \tilde{\psi}$ on $B_1 \setminus B_\varepsilon$.

Pf. By contr. $|f(x)| > \tilde{\psi} > \psi$ for $x \in B_1 \setminus B_\varepsilon$

Hence on $B_{\tilde{\psi}^2}(x)$ $|f| > \tilde{\psi}(1-\tilde{\psi})$

and integrating $\int_{B_{\tilde{\psi}^2}(x)} |f| > |B_{\tilde{\psi}^2}(x)| \, \tilde{\psi}(1-\tilde{\psi})$

$> c\, v \cdot v(n,-\delta,\tilde{\psi}^2)\, \tilde{\psi}(1-\tilde{\psi})$

BG+ (H1) $\quad = \Psi$ $\quad$ contr.

choose $\tilde{\psi}$ such that this holds

Now for (b)

$$\int_{B_1 \setminus B_\varepsilon} \left| |\nabla h|^2 - 4h \right| \le \int \left| |\nabla h|^2 - |\nabla d_p^2|^2 \right| + \int |4h - |\nabla d_p^2|^2|$$

$$\overset{(\gamma)}{\le} \int \left| |\nabla h| - |\nabla d_p^2| \right| \left| |\nabla h| + |\nabla d_p^2| \right| + \Psi$$

$$\overset{(\gamma'),(C)}{\le} C \int |\nabla (h - d_p^2)| + \Psi \le \Psi.$$

Up to inserting another annulus, this could have been obtained on $B_{\frac{3}{2}} \setminus B_{\frac{3}{4}\varepsilon}$ I skip these details!

Let us finish with (a)

By simple computations we have

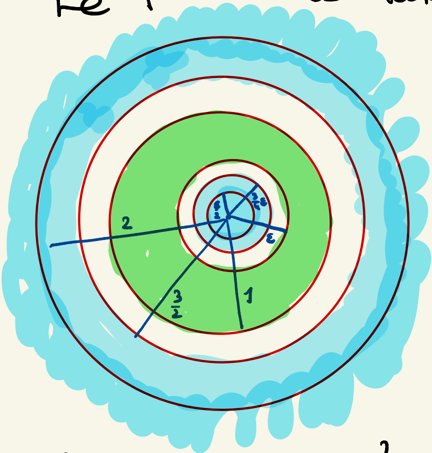
$$\Delta(|\nabla h|^2 - 4h) \overset{?}{=} 2|\nabla^2 h|^2 + 2Ric(\nabla h, \nabla h) - 8n$$

$\Delta h = 2n$

$$\ge 2|\nabla^2 h - 2g|^2 - 2(n-1)\delta |\nabla h|^2 \quad \text{on } B_2 \setminus B_{\frac{\varepsilon}{2}}$$

Hence $\quad |\nabla^2 h - 2g|^2 \le (n-1)\delta |\nabla h|^2 + \frac{1}{2}\Delta(|\nabla h|^2 - 4h) \quad$ on $B_2 \setminus B_{\frac{\varepsilon}{2}}$

Let: us take a cut-off φ s.t.



$$\begin{cases} \varphi \equiv 0 & \text{on} \\ \varphi \equiv 1 & \text{on} \\ |\nabla \phi| + |\Delta \phi| \le C(n, \varepsilon) \\ \varphi \le 1 \end{cases}$$

$$\int_{B_1 \setminus B_\varepsilon} |\nabla^2 h - 2g|^2 \le \int_{B_{3/2} \setminus B_{3/4\,\varepsilon}} \varphi\, |\nabla^2 h - 2g|^2$$

$$\le \int_{B_{\frac{3}{2}} \setminus B_{3/4\,\varepsilon}} \varphi\,(n-1)\delta |\nabla h|^2 + \frac{1}{2} \varphi\, \Delta (|\nabla h|^2 - 4h)$$

$$\le C\,\delta + \frac{1}{2} \int_{B_{\frac{3}{2}} \setminus B_{\frac{3}{4}\varepsilon}} \Delta \varphi\, (|\nabla h|^2 - 4h) \le \Psi$$

$\varphi \le 1$
$\sup_{B_{\frac{3}{2}} \setminus B_{\frac{3}{4}\varepsilon}} |\nabla h| \le C$ (ψ')

$|\Delta \varphi| \le C(n, \varepsilon)$ and (b)
on $B_{\frac{3}{2}} \setminus B_{\frac{3}{4}\varepsilon}$

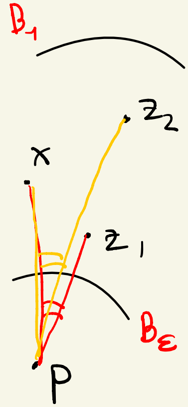
NEXT STEPS THROUGH THE CONCLUSION

Lemma 1 Let (M^n, g) be a complete Riemannian manifold s.t. (H0), (H1), (H2) holds for $\delta, v > 0$. Let $\varepsilon > 0$.
Let $x, z_1, z_2 \in B_1(p) \setminus B_\varepsilon(p)$ such that

$$\left| d(p,z_2) - \left(d(p,z_1) + d(z_1,z_2) \right) \right| < \delta$$

Hence

$$\left| \frac{d^2(x,z_1) - d^2(x,p) - d^2(z_1,p)}{2d(z_1,p)d(x,p)} - \frac{d^2(x,z_2) - d^2(x,p) - d^2(z_2,p)}{2d(z_2,p)d(x,p)} \right| \le \psi(\delta | n, v, \varepsilon)$$



Lema 2 Let (M^n, g) be a complete Riemannian manifold s.t. (H0), (H1), (H2) holds for $\delta, v > 0$. Let $\varepsilon > 0$.
Hence for any $x \in B_1(p) \setminus B_\varepsilon(p)$ $\exists$ a minimizing geodesic $\gamma : [0,1] \to M$ parametrized by unit speed with $\gamma(0) = p$ and $d(x, \gamma(s)) \le \psi(\delta | n, v, \varepsilon)$