CONCLUSION USING THE LEMMAS

By contradiction $\exists \varepsilon > 0$, and $\delta_J \to 0$, (M^n_J, g_J) complete Riemannian manifolds with $Ric_{M^n_J} \geq -(n-1)\delta_J$, $Vol(B_1(p_J)) \geq v$,

$$\frac{Vol(B_{\delta_J}(p_J))}{v(n,-\delta_J,\delta_J)} - \frac{Vol(B_{16}(p_J))}{v(n,-\delta_J,16)} < \delta_J$$

and $d_{GH}((B_1(p_J),p_J),(B_1(z^*),z^*)) \geq \varepsilon$ (Contr)

for every cone $C(Z)$ with tip z^* and $\operatorname{diam} Z \leq \pi$.

First, up to passing to subsequences,

$$(M^n_J, g_J, p_J) \longrightarrow (X, d, x).$$

STEP 1 Take $y \in B^X_1(x)$. Hence $\exists$ unit speed geodesic $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(s) = y$ for some $s \in [0,1]$.

Indeed, approximate $x_J \to y$, with $x_J \in B_1(p_J) \setminus B_\varepsilon(p_J)$ with some ε suff. small. Hence by Lemma 2 we find $s_J \in [\varepsilon, 1]$, $\gamma_J : [0,1] \to M^n_J$, unit speed geodesics, such that $\gamma_J(0) = p_J$, $d(\gamma_J(s_J), x_J) \leq \Psi_J$. By Ascoli-Arzelà we get up to subsequence $\gamma_J \to \gamma : [0,1] \to X$, which is still a unit speed geodesic. Moreover $s_J \to s$ s.t. $\gamma(s) = y$, by continuity of distances under GH-convergence.

STEP 2

Let W be the set of all the unit-speed minimizing geodesics $\gamma : [0,1] \to X$ emanating from x.

Define the map

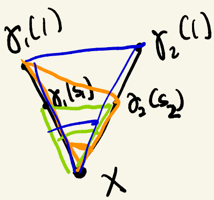
$$\Phi : W \times [0,1] \to X$$
$$(\gamma, s) \to \gamma(s)$$

Notice $\Phi(W \times [0,1]) \supset B_1(x)$ by STEP 1

For any $\gamma_1, \gamma_2 \in W$ define

$$\alpha(\gamma_1,\gamma_2) := \arccos\left(\frac{-d^2(\gamma_1(1),\gamma_2(1))+2}{2}\right)$$

This is between -1 and 1 by triangle inequality and so $0 \le \alpha \le \pi$



Notice α is a well defined pseudo-metric (*). Take $\tilde{W} := W/_{\{\alpha=0\}}$

(*) Not NEEDED SEE AFTER, actually it's a metric

Now notice that if you fix $s_1, s_2 \in [0,1]$ you have = = . This holds by approximation thanks to Lemma 1 and the fact that $d(x,\gamma_1(s_1)) + d(\gamma_1(s_1),\gamma_1(1)) = d(x,\gamma_1(1))$ and $d(x,\gamma_2(1)) = d(x,\gamma_2(s_2)) + d(\gamma_2(s_2),\gamma_2(1))$. Hence

$$\frac{d^2(\gamma_1(s_1),\gamma_2(s_2)) - s_1^2 - s_2^2}{2 s_1 s_2} = \frac{d^2(\gamma_1(1),\gamma_2(1)) - 2}{2}$$

that reduces to

$$d^2(\gamma_1(s_1), \gamma_2(s_2)) = s_1^2 + s_2^2 - 2 s_1 s_2 \cos\alpha(\gamma_1,\gamma_2).$$ (!!)

↳ By this we get that if $\alpha(\gamma_1,\gamma_2)=0 \Rightarrow \gamma_1 \equiv \gamma_2$ ⊕

↳ cone distance made with α

Hence if

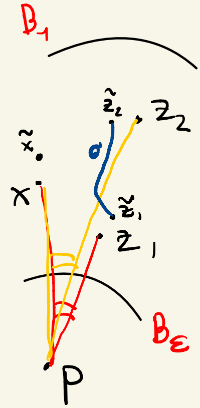
$$\tilde{\Phi} : \left(W \times [0,1] / W \times \{0\}, d_c \right) \to (X, d)$$
$$(\gamma, s) \to \gamma(s)$$

we get that $\tilde{\Phi}$ is a distance preserving map by (!!) and the domain is precisely a subset of the metric cone $(C(\tilde{W}), d_c)$.

Since $\tilde{\Phi}$ has image $\supset B_1(x)$, we get that $\tilde{\Phi}$ induces an isometry between $B_1^{C(\tilde{W})}(w^*)$ and $B_1(x)$, which is a contradiction with (Contr) passing to the limit.

Proof of LEMMA 1

[I am neglecting the possible issue that some geodesics in the construction cross $B_\varepsilon(x)$. This can be fixed by slightly adjusting the details below. Notice that we have the integral estimates of f outside B_ε]



Take

$$f = \overbrace{|\nabla^2 h - 2g|}^{(I)} + \overbrace{||\nabla h|^2 - 4h|}^{(II)}$$

and use the segment inequality to find $\tilde x, \tilde z_1, \tilde z_2$ s.t.

(i) $d(x,\tilde x) + d(z_1,\tilde z_1) + d(z_2,\tilde z_2) < \Psi$

(ii) $\exists!$ σ geodesic from $\tilde z_1$ to $\tilde z_2$ and

$$\int_0^{\tilde d := d(\tilde z_1,\tilde z_2)} f(\sigma(s))\,ds < \Psi$$

(iii) $\exists$ $U \subseteq [0, d(\tilde z_1,\tilde z_2)]$ full-measure such that being $\ell(s)$ the distance between $\tilde x, \sigma(s)$ measured along the geodesic $\tau_s(\ell)$

$$\int_U ds \int_0^{\ell(s)} f(\tau_s(\ell))\,d\ell < \Psi$$

Hence (ii) + (I) gives $|(h\circ\sigma)'(s) - (h\circ\sigma)'(0) - s| < \Psi$ (*)

and then $|(h\circ\sigma)(s) - (h\circ\sigma)(0) - (h\circ\sigma)'(0)s - s^2| < \Psi$ (∼)

$\forall s \in [0,\tilde d]$

Evaluating the previous (∼) at $s = \tilde d$, and using

$|h(\sigma(\tilde d)) - h(\sigma(0)) - \tilde d^2 - 2d(p,\tilde z_1)\tilde d| < \Psi$ due to the fact that $\|h - d_p^2\| < \Psi$ and $|d(p,z_1) + d(z_1,z_2) - d(p,z_2)| < \delta$

we obtain $|(h\circ\sigma)'(0) - 2d(p,\tilde z_1)| < \Psi$ (≈)

By the hypothesis $|(d(p,z_1)+d(z_1,z_2)) - d(p,z_2)| < \delta$ we get $|d(p,\sigma(s)) - d(p,\sigma(0)) - s| < \psi$ $(\approx)$

Now we want to estimate, calling $b := d(p,\tilde{z}_1)$ $a := d(p,\tilde{x})$

$$\frac{-d^2(\tilde{x},\sigma(s)) + d^2(\tilde{x},p) + d^2(\sigma(s),p)}{2d(\tilde{x},p)d(\sigma(s),p)} \overset{\text{using } (\approx)}{\sim_{\Psi}} \frac{a^2+(b+s)^2-\ell(s)^2}{2a(b+s)}$$

It suffices to prove that the values between $s=0$, $s=\tilde{d}$ are Ψ-apart, which would be exactly the final assertion up to using (i). then estimate the derivative

$$\frac{d}{ds}\frac{a^2+(b+s)^2-\ell(s)^2}{2a(b+s)} = \frac{(b+s)^2+\ell(s)^2-a^2-2(b+s)\ell\ell'}{2a(b+s)^2} \quad (\bullet)$$

Notice that $(*)+(\approx)$ gives $|(h\circ\sigma)'(s) - 2(b+s)| < \Psi$ hence, by using this, $(\approx)$, and $(ii)^{(II)}$ we have

$$\int_0^{\tilde{d}} |2(b+s)\dot{\sigma}(s) - \nabla h(\sigma(s))|^2\, ds \le \Psi$$

and hence, recalling $\ell'(s) = \langle \sigma'(s), \tau_s'(\ell(s))\rangle$ we have

$$\int_0^{\tilde{d}} |2(b+s)\ell'(s) - (h\circ\tau_s)'(\ell(s))|\, ds \le \Psi \quad (!)$$

Finally by using $\|h - d_p^2\|_{\infty} < \Psi$ $(\approx)$, we have

$$|(b+s)^2+\ell(s)^2-a^2-(h\circ\tau_s)'(\ell(s))\ell'| \le \Psi + \ell(s)\int_0^{\ell(s)} |D^2h - 2g|(\tau_s(r))\,dr$$

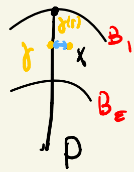
and then by (iii) $^{(\mathrm{I})}$, $(\S)$, and the previous, we get

$$\int_0^{\tilde{d}} |(b+s)^2 + \ell(s)^2 - a^2 - 2(b+s)\ell\ell'| \leq \underline{\Psi}$$

and thus $\displaystyle\int_0^{\tilde{d}} \left| \frac{d}{ds} \frac{a^2+(b+s)^2-\ell(s)^2}{2a(b+s)} \right| \leq \underline{\Psi}$ by $(\bullet)$

Now since the L^1-norm of the derivative is $\underline{\Psi}$-small the two endpoints of the fraction are $\underline{\Psi}$-apart and thus the proof is concluded.

Lemma 2 Let (M^n, g) be a complete Riemannian manifold s.t. (H0), (H1), (H2) holds for $\delta, v > 0$. Let $\varepsilon > 0$.
Hence for any $x \in B_1(p) \setminus B_\varepsilon(p)$ $\exists$ a minimizing geodesic $\gamma : [0,1] \to M$ parametrized by unit speed with $\gamma(0) = p$ and $d(x, \gamma(s)) \leq \psi(\delta | n, v, \varepsilon)$



Proof of Lemma 2

You can try to prove this by exercise, in the form I stated.
Or you can prove the following "quantitative" surjectivity property, which is what you really need to end the proof in the pages before.
Fix $x \in B_1(p) \setminus B_\varepsilon(p)$, call $d := d(p,x)$. Hence you can find y s.t.

$$|d(p,x) + d(x,y) - d(p,y)| + |d(x,y) - (1-d)| < \Psi$$

which is no ray $\overrightarrow{pxy}$ are Ψ-almost on a minimizing geodesic of length one

By using Colding's inequality as we argued some lectures ago, by exploiting

$$\int_{B_1(p)\setminus B_\varepsilon(p)} |\nabla(h - d_p^2)| \le \Psi \quad \& \quad \int_{B_1(p)\setminus B_\varepsilon(p)} |\nabla^2 h - 2g| \le \Psi$$

we find $\tilde{x}$, $v \in T^1_{\tilde{x}} M$ s.t. $\underbrace{d(x,\tilde{x})}_{(A1)} + \underbrace{|v - \nabla d_p(\tilde{x})|}_{(A2)} \le \Psi$ (A)

and

geodesic starting from $\tilde{x}$ with speed v

$$\underbrace{|\nabla h(\tilde{x}) - 2 d_p(\tilde{x}) v|}_{(B1)} + \underbrace{\left| \langle \nabla h(\tilde{x}), v \rangle - \frac{h(\gamma_v(t)) - h(\overbrace{\gamma_v(0)}^{= \tilde{x}})}{t} + t \right|}_{(B2)} \le \Psi \quad (B)$$

where $t := 1-d$. Let $y := \gamma_v(t)$, hence $d(\tilde{x},y) = 1-d$.

and then (A1) $|d(x,y) - (1-d)| \le \Psi$ (●)

Hence, by (A), (B), $|h - d_p^2| \le \Psi$,

$$|(d(p,\tilde{x}) + 1 - d)^2 - d(p,y)^2| \le \Psi \quad (●●)$$

from which also $|d(p,x) + d(x,y) - d(p,y)| \le \Psi$ by exploiting also (●).

Hence by (●) and (●●) we have the assertion.

gioacchino.antonelli@sns.it

END Thank you!!