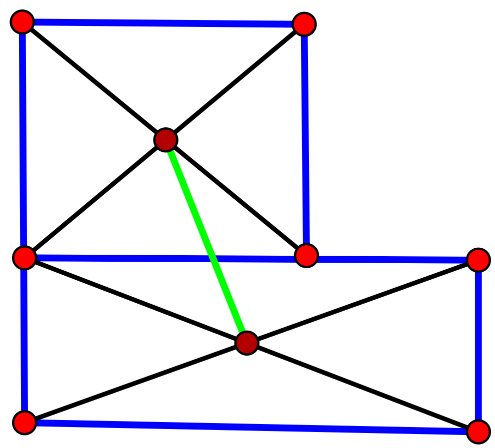
Elementary geometry is dead.

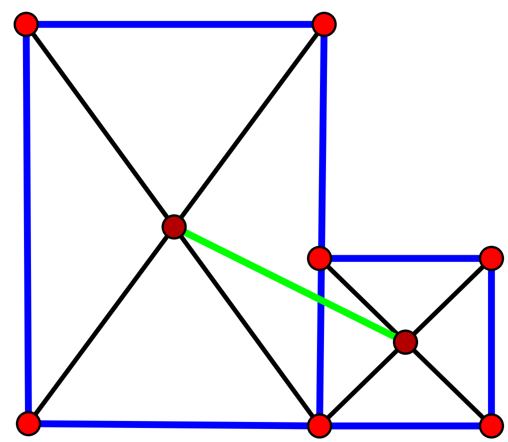
Long live (experimental) elementary geometry!

Seminario de Geometría diferencial y sistemas dinámicos,
CIMAT, November 2020

1. Circumcenter of mass

Center of mass satisfies the Archimedes Lemma: *if an object is divided into two smaller objects, then the center of mass of the compound object is the weighted sum of the centers of mass of the two smaller objects.*





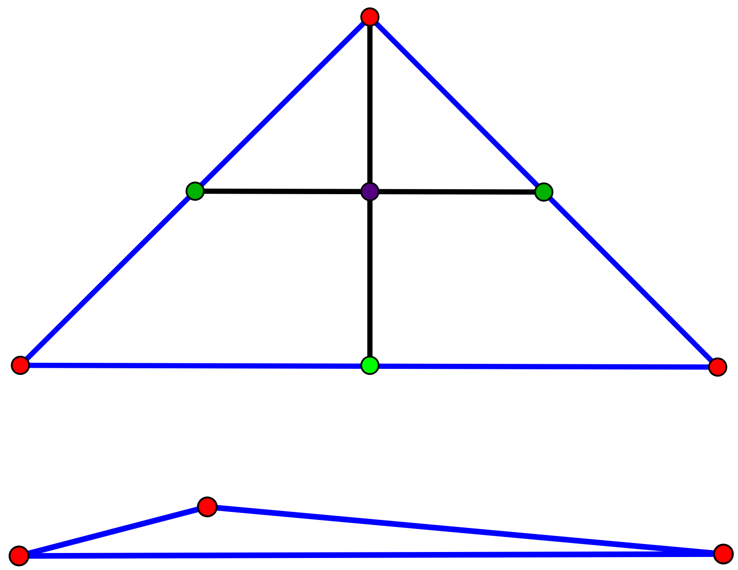
Circumcenter of mass: triangulate a polygon P, take the circumcenter of each triangle with the weight equal to its area, and take the center of mass, $CCM(P)$.

Theorem: $CCM(P)$ *is well defined and it satisfies the Archimedes Lemma.*

Formulas: If the vertices are (x_i, y_i), then

$$CCM(P) = \frac{1}{4A(P)} \Big(\sum_{i=0}^{n-1} y_i(x_{i-1}^2 + y_{i-1}^2 - x_{i+1}^2 - y_{i+1}^2), \\ \sum_{i=0}^{n-1} -x_i(x_{i-1}^2 + y_{i-1}^2 - x_{i+1}^2 - y_{i+1}^2) \Big).$$

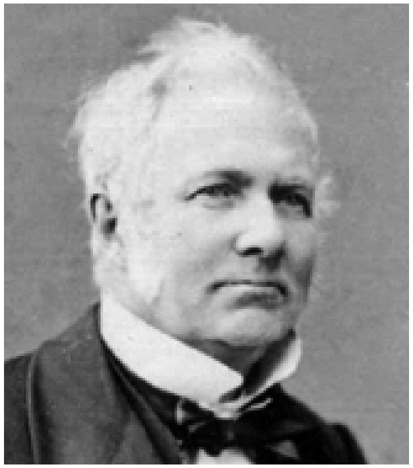
A subtlety:



Proposition: *If P is an equilateral polygon then*

$$CCM(P) = CM(P).$$

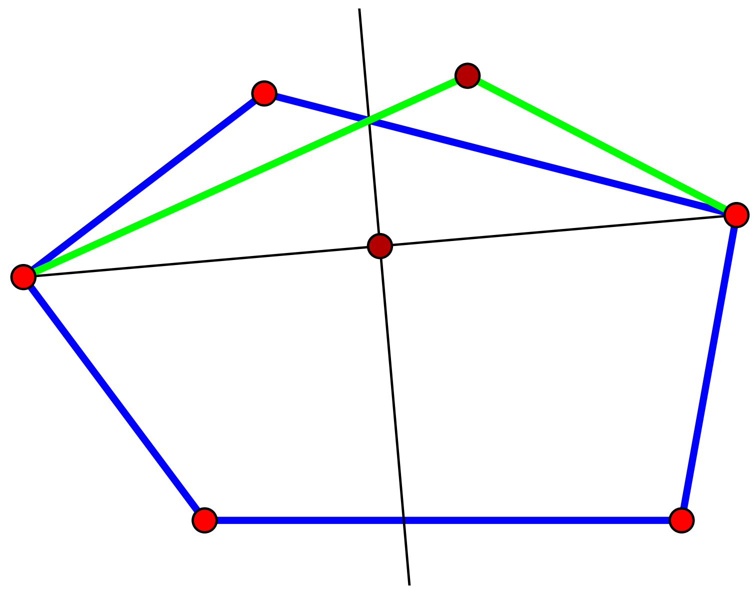
A historical remark (thanks to B. Grünbaum):
C.-A. Laisant, *Théorie et applications des équipollences*. Gauthier-Villars, Paris 1887. On pp. 150–151, the construction is described and attributed to Giusto Bellavitis (1803 –1880).



"*Nothing is ever discovered for the first time*".

Sir Michael Berry

Rediscovered by V. Adler (1993) as an invariant of the polygon recutting transformation:



In spherical and hyperbolic geometries: *the center of mass* of a system of mass-points $V_i \in S^2$ is the point

$$\frac{\sum m_i V_i}{|\sum m_i V_i|} \in S^2,$$

taken with mass $|\sum m_i V_i|$.

The circumcenter of a spherical triangle ABC is taken with the mass equal to the area of the *plane* triangle ABC:

$$CCM(ABC) = A \times B + B \times C + C \times A$$

(the direction gives the point, the magnitude gives the weight). Given a triangulation, consider the center of mass of the circumcenters of the triangles involved to obtain $CCM(P)$.

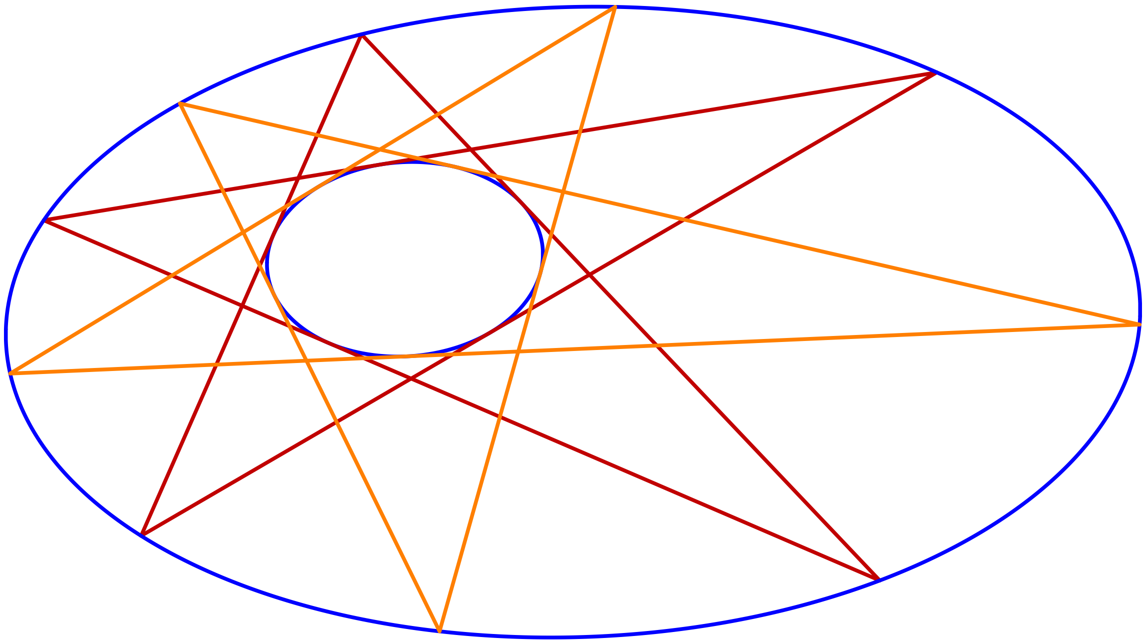
Likewise in H^2, in the hyperboloid model. One can also generalize to simplicial polytopes in all three geometries.

More in

S. T. & E. Tsukerman. *Circumcenter of Mass and generalized Euler line*. Discr. Comp. Geom., 51 (2014), 815–836.
Remarks on the the circumcenter of mass. Arnold Math. J., 1 (2015), 101–112.

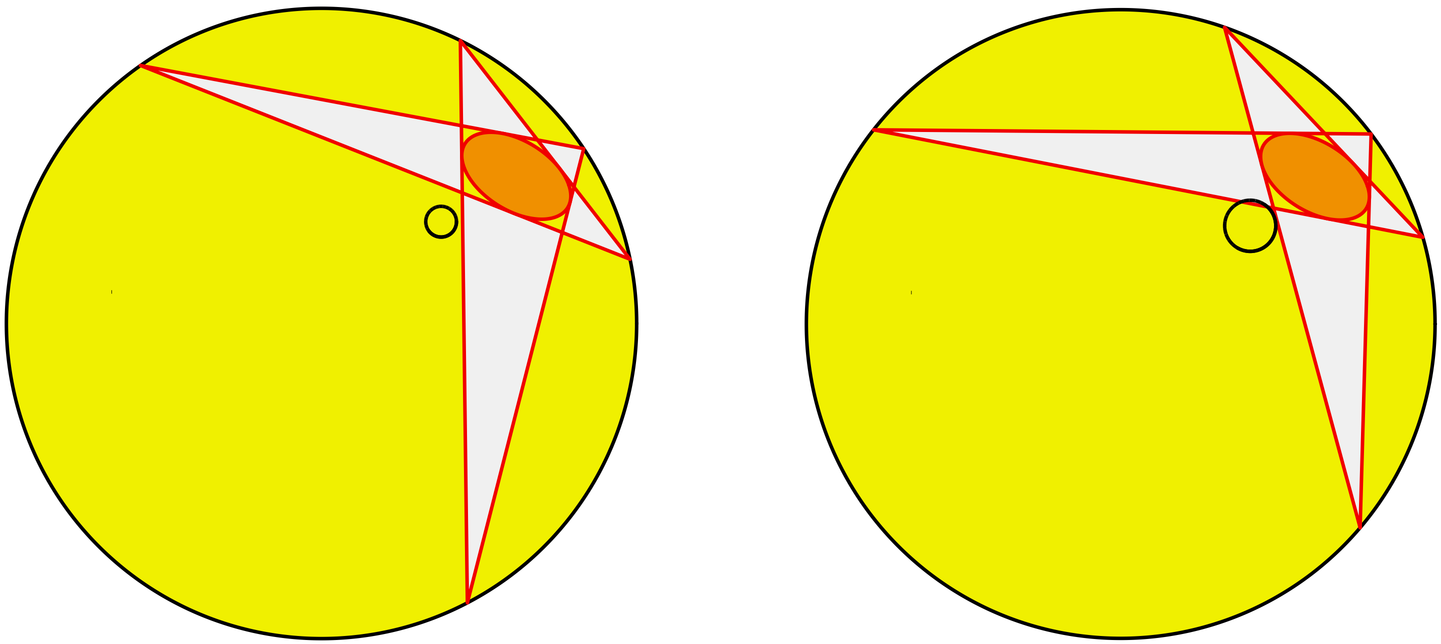
A. Akopyan. *Some remarks on the circumcenter of mass.* Discrete Comput. Geom. 51 (2014), 837–841.

2. Centroids of Poncelet polygons



Two centers of mass, CM_0 and CM_2.

Theorem: *The trajectories of* CM_0 *and* CM_2 *are homothetic to the outer ellipse.*

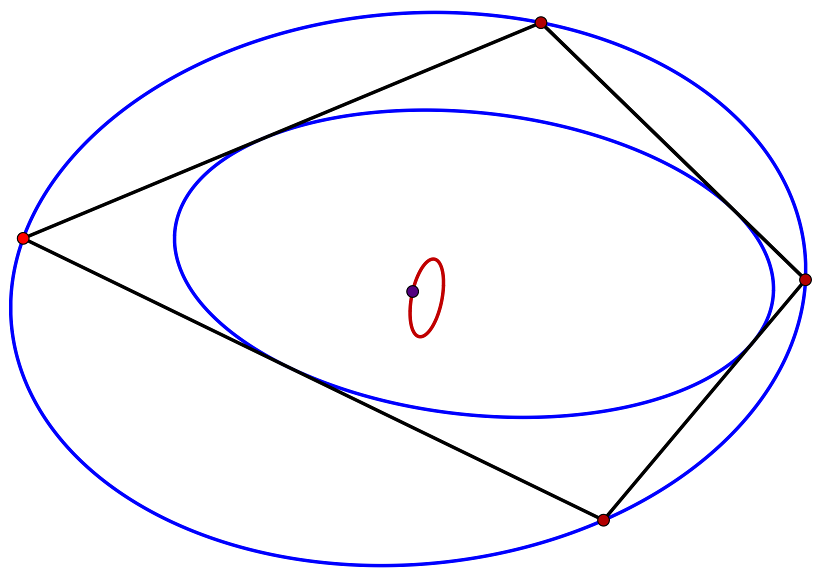


We don't know the relation between the three circles and the ellipse.

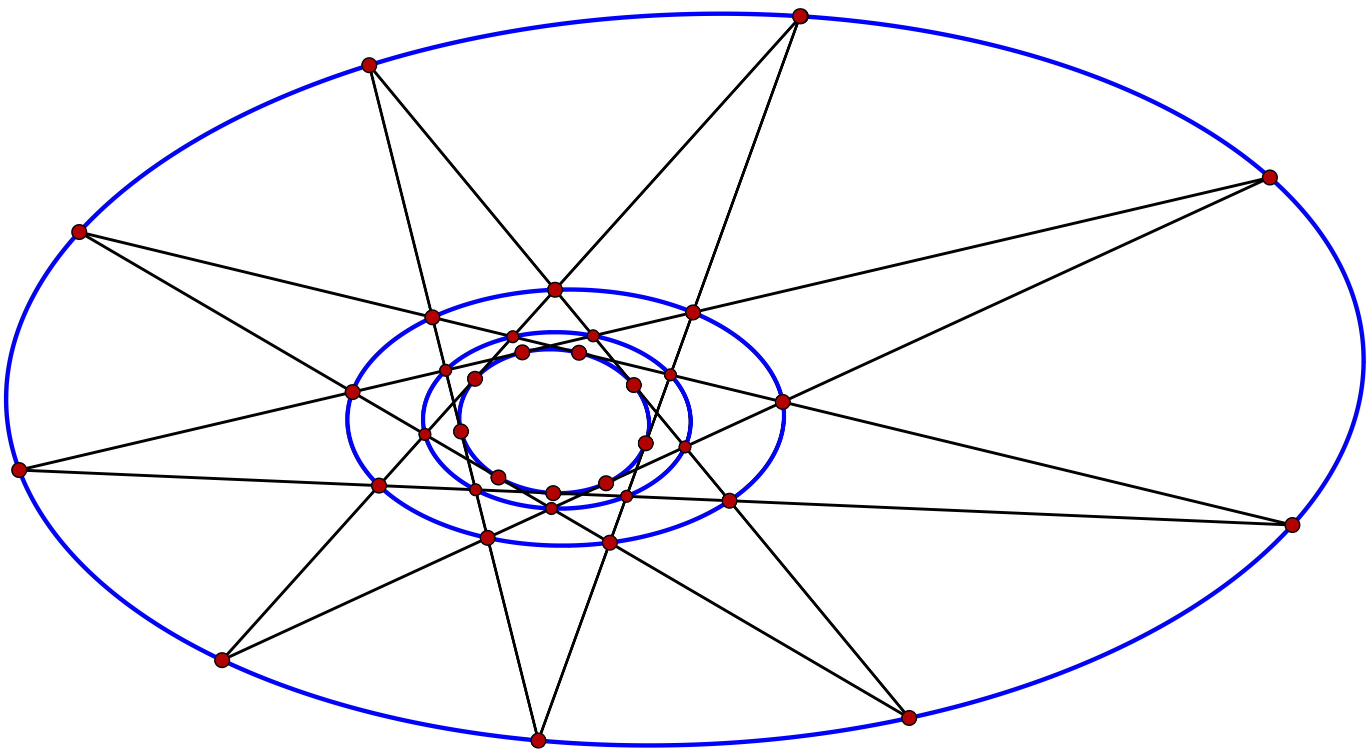
More in R. Schwartz, S.T. *Centers of mass of Poncelet polygons, 200 years after* Math. Intelligencer, v. 38, No 2 (2016), 29–34.

Ana Chavez. *More about areas and centers of Poncelet polygons*, arXiv:2004.05404, to appear in Arnold Math. J.

Theorem: *1) If a non-degenerate Poncelet polygon has zero area, then area is zero in the whole 1-parameter family.*
2) The locus of CCM of the Poncelet polygons is also a conic.



3. Poncelet grid



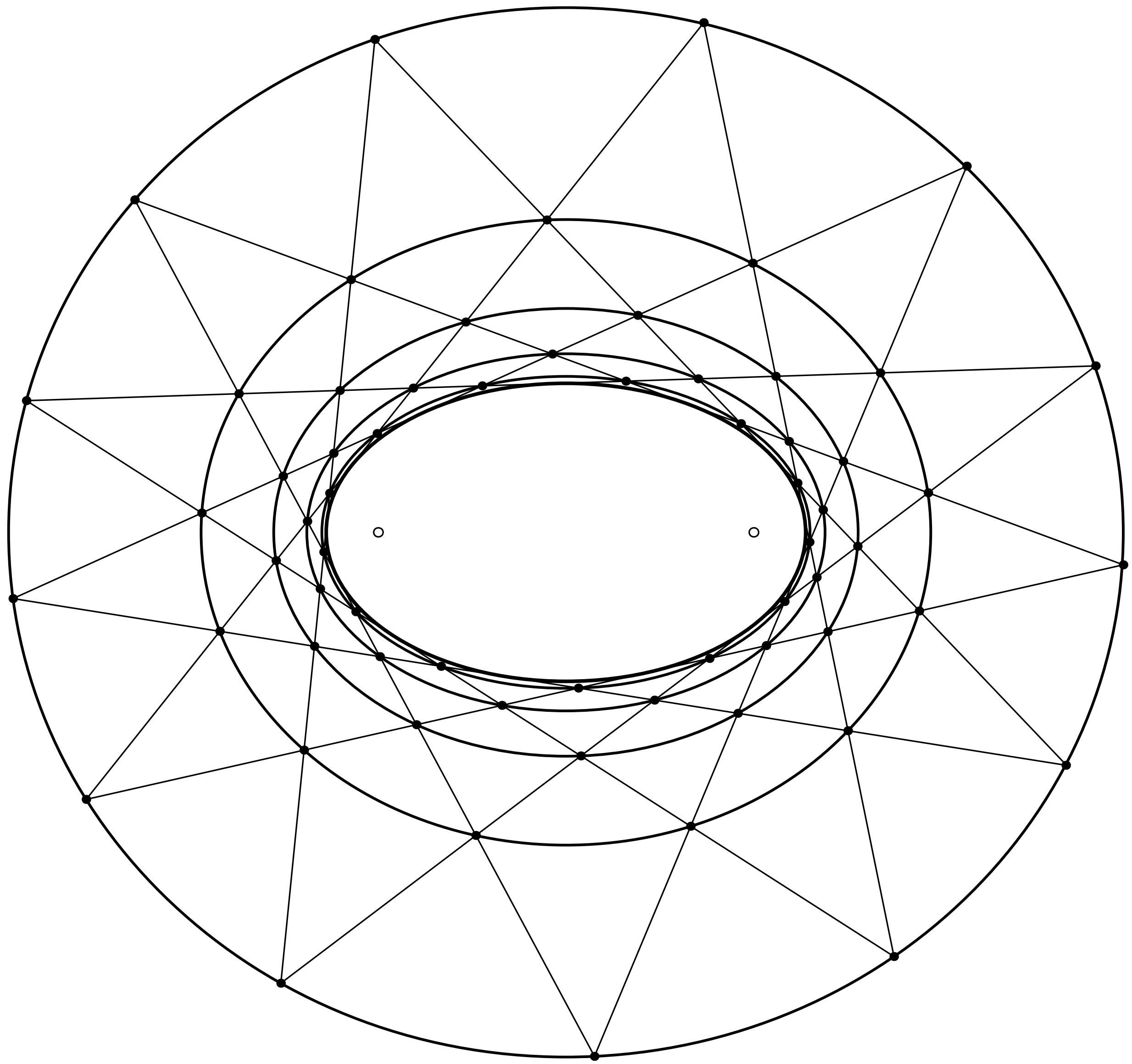
Concentric and radial sets: $P_k = \cup_{i-j=k} \ell_i \cap \ell_j, Q_k = \cup_{i+j=k} \ell_i \cap \ell_j$ (shown are P_0, P_2, P_3 and P_4).

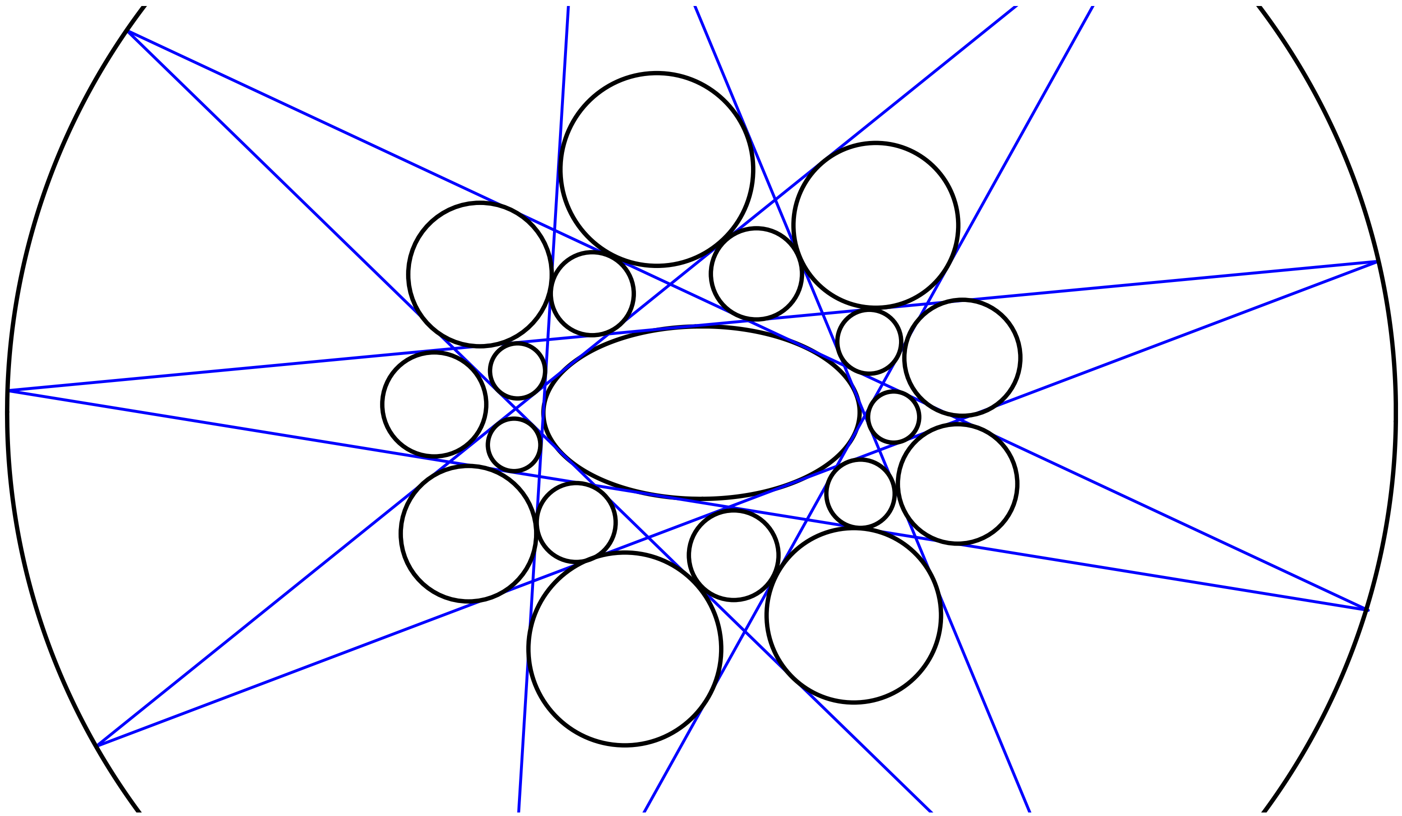
Theorem: *The concentric sets lie on nested ellipses, the radial sets on disjoint hyperbolas. These conics share four (complex) tangent lines. All the concentric sets are projectively equivalent to each other, and so are all the radial sets.*

R. Schwartz. *The Poncelet grid.* Adv. Geom. 7 (2007), 157–175.

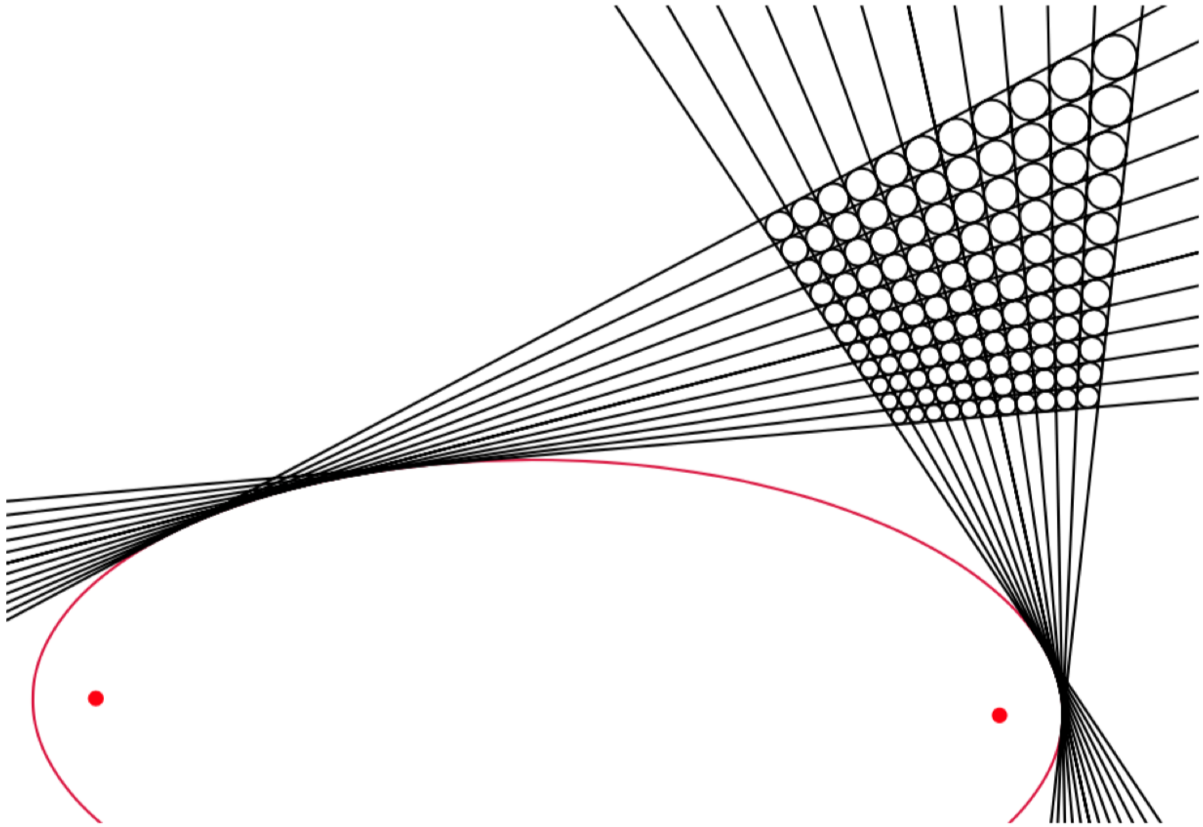
Theorem: *If the conics are confocal, the concentric sets are periodic billiard trajectories, and they are linearly equivalent.*

M. Levi, S. T. *The Poncelet grid and billiards in ellipses.* Amer. Math. Monthly 114 (2007), 895–908.





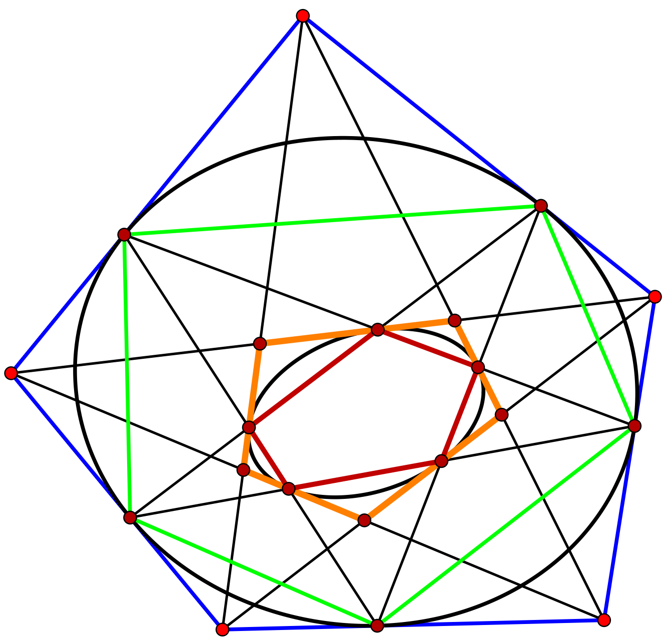
Reye-Chasles theorem.



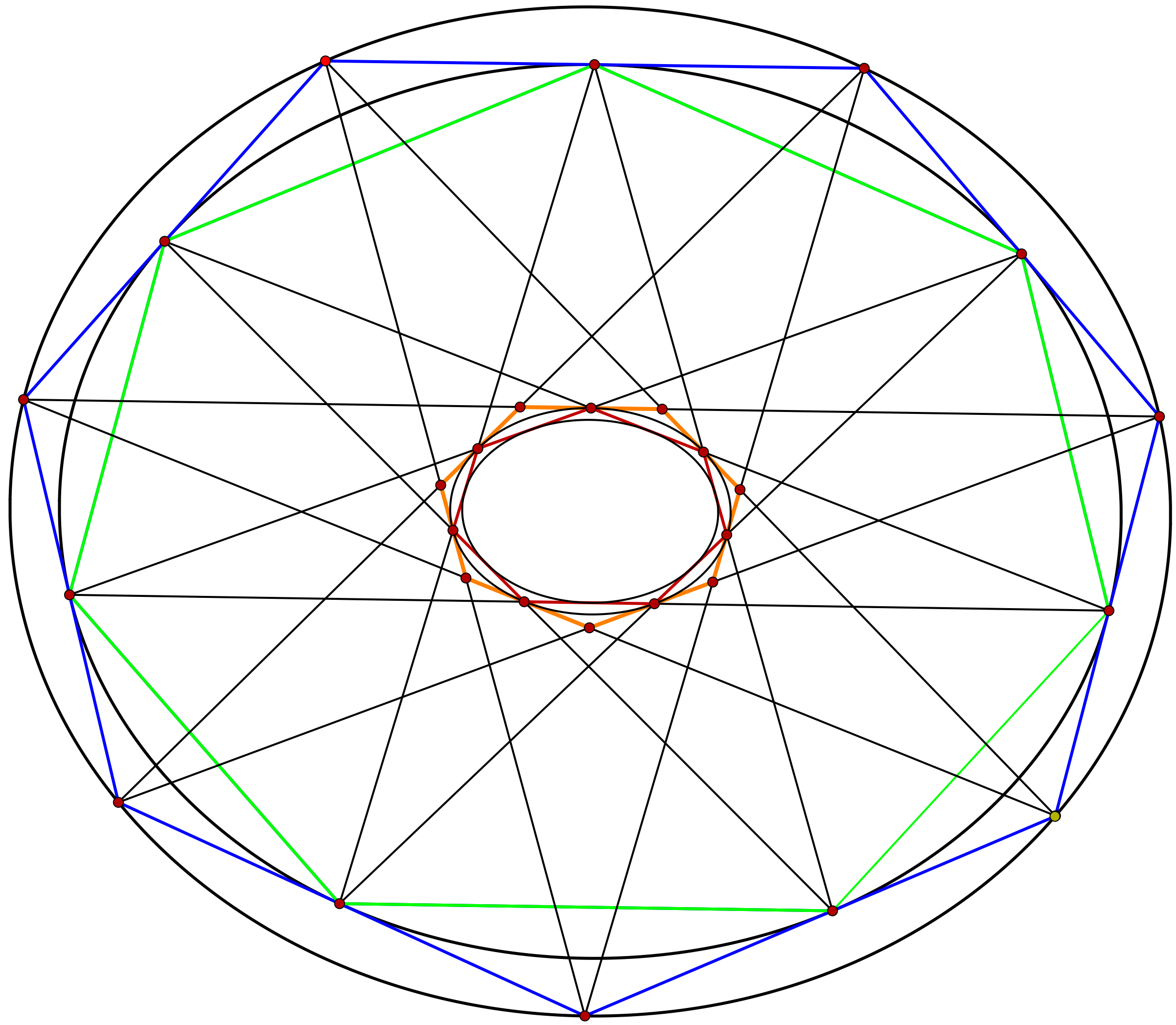
More in A. Akopyan, A. Bobenko. *Incircular nets and confocal conics*. Trans. Amer. Math. Soc. 370 (2018), 2825–2854.

4. Kasner's theorem and its extension

Theorem [Kasner]: *The two operations on pentagons commute:* $ID(P) = DI(P)$.



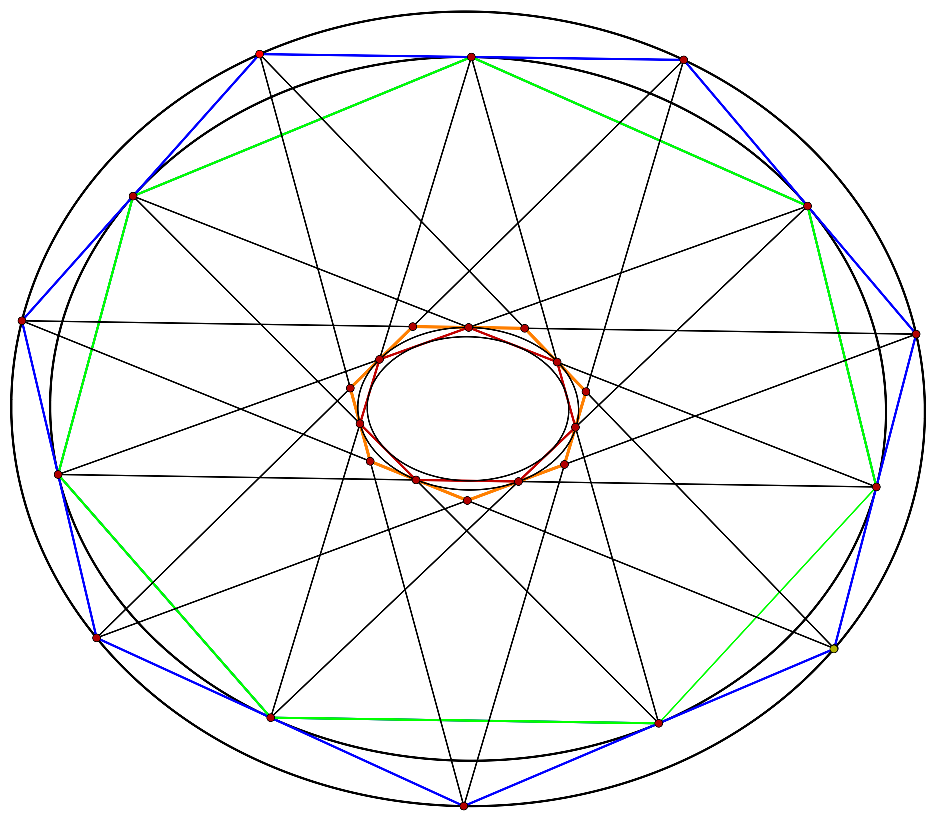
Theorem: *For a Poncelet polygon,* $ID_k(P) = D_kI(P)$.



Sketch of proof. Let (Γ, γ) be a pair of *confocal* ellipses, and P a Poncelet n-gon, the tangency points form the set Q_0, and the vertices form the set Q_1.

Consider the set Q_k. These n points lie on a confocal ellipse, say δ. Let A be the linear map that takes Q_0 to Q_1. This map takes the tangent lines ℓ_i to γ at points Q_0 to the tangent lines L_i to Γ at points Q_1. Therefore A takes $\ell_i \cap \ell_{i+k}$ to $L_i \cap L_{i+k}$, $i = 1, \dots, n$. The latter set S lies on the ellipse $\Delta = A(\delta)$.

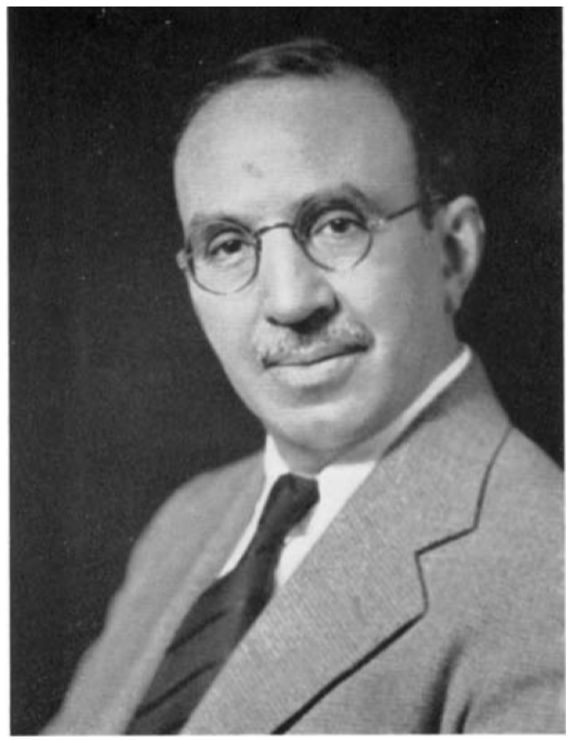
Let B be the linear map that takes Q_1 to Q_k. Then the map BA takes Q_0 to Q_k. These linear maps commute: $BA = AB$. The map AB takes Γ to Δ, and it takes the Poncelet n-gon P to a Poncelet n-gon on the ellipses (Δ, δ) with the vertex set S. QED.



γ and Γ are the two inner ellipses, δ and Δ are the outer ones:

$$A(\gamma) = \Gamma, A(\delta) = \Delta, B(\Gamma) = \delta \quad \Rightarrow \quad BA(\gamma) = \delta, AB(\Gamma) = \Delta.$$

A historical remark: Edward Kasner (1878–1955) was a prominent geometer at Columbia University, the advisor of Jesse Douglas.

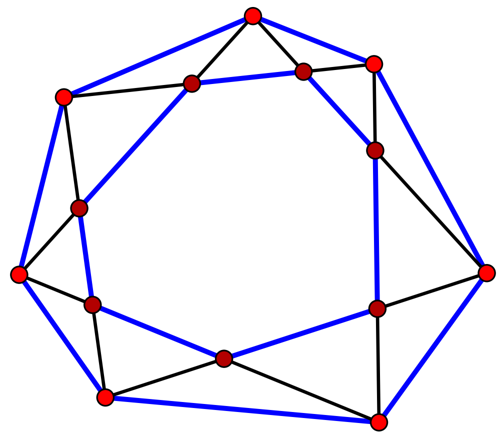


Kasner (or rather his nephew) introduced the terms “googol” and “googolplex” in his book “Mathematics and the Imagination” (with J. Newman).

More in S. T. *Kasner meets Poncelet.* Math. Intelligencer 41 (2019), no. 4, 56–59.

5. Projective configuration theorems

Pentagram map (on projective equivalence classes of polygons):



It is the identity for pentagons and an involution for hexagons (can you prove it?)

Two projective planes: $\mathbf{RP}^2$ and $(\mathbf{RP}^2)^*$, two spaces of n-gons: $\mathcal{C}_n$ and $\mathcal{C}_n^*$. The k-diagonal maps $T_k : \mathcal{C}_n \to \mathcal{C}_n^*$: for $P = \{p_1, ..., p_n\}$,

$$T_k(P) = \{(p_1 p_{k+1}), (p_2 p_{k+2}), \ldots, (p_n p_{k+n})\}.$$

Each map T_k is an involution.

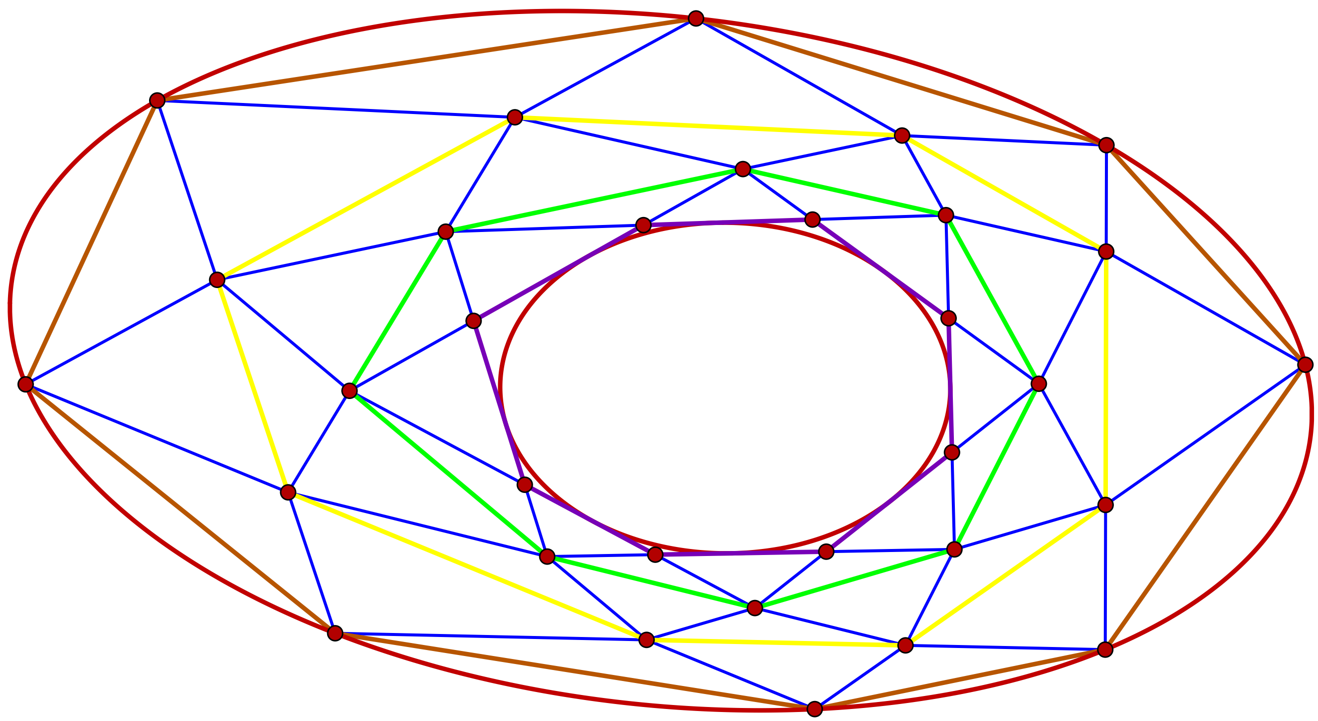
Extend the notation:

$$T_{ab} = T_a \circ T_b, T_{abc} = T_a \circ T_b \circ T_c,$$

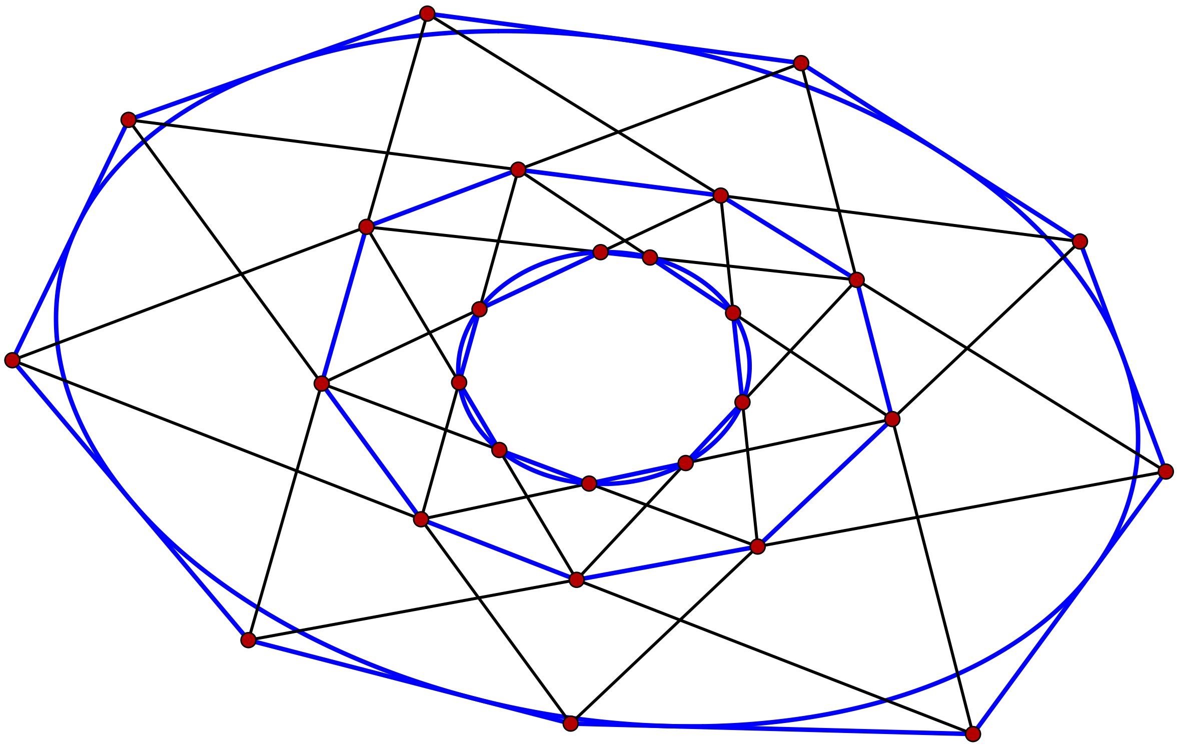
etc. For example, the pentagram map is T_{12}.

If P is a polygon in $\mathbf{RP}^2$ and Q a polygon in $(\mathbf{RP}^2)^*$, and there exists a projective transformation $\mathbf{RP}^2 \to (\mathbf{RP}^2)^*$ that takes P to Q, write: $P \sim Q$.

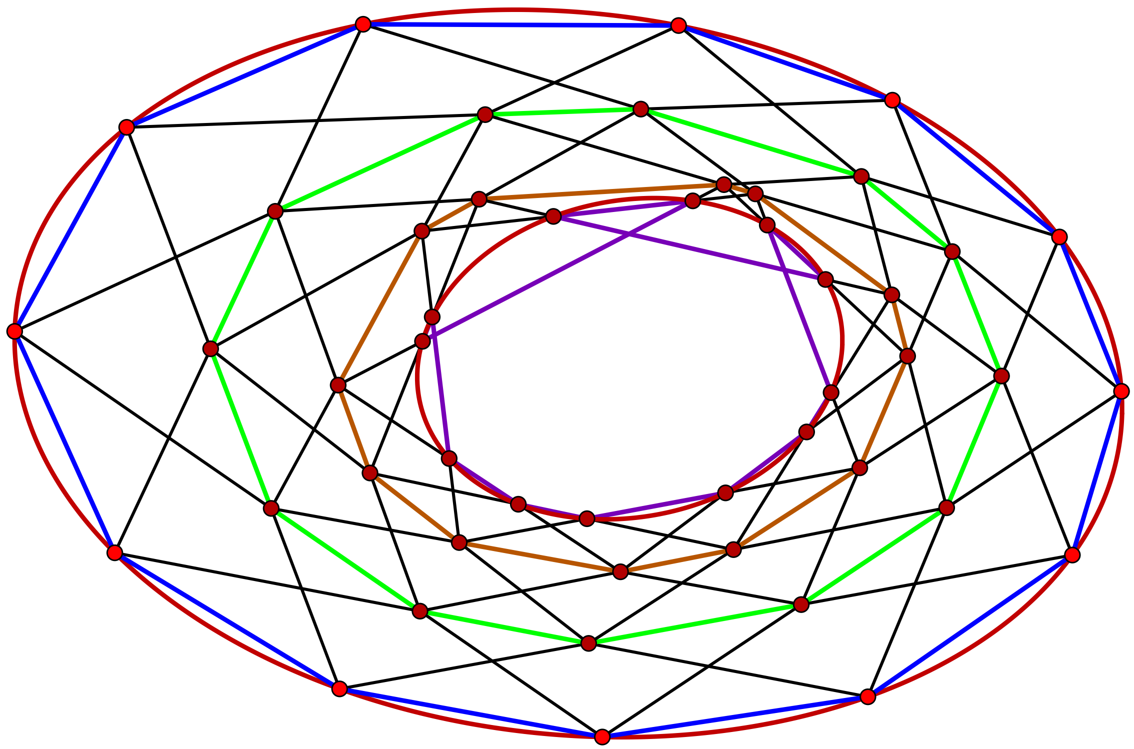
Theorem: *(i) If P is an inscribed 6-gon, then $P \sim T_2(P)$.*
(ii) If P is an inscribed 7-gon, then $P \sim T_{212}(P)$.
(iii) If P is an inscribed 8-gon, then $P \sim T_{21212}(P)$.



Theorem: *(i) If P is a circumscribed 9-gon, then $P \sim T_{313}(P)$.*
(ii) If P is an inscribed 12-gon, then $P \sim T_{3434343}(P)$.



Theorem: *(i) If* P *is an inscribed* 8*-gon, then* $T_3(P)$ *is circumscribed.*
(ii) If P *is an inscribed* 10*-gon, then* $T_{313}(P)$ *is circumscribed.*
(iii) If P *is an inscribed* 12*-gon, then* $T_{31313}(P)$ *is circumscribed.*

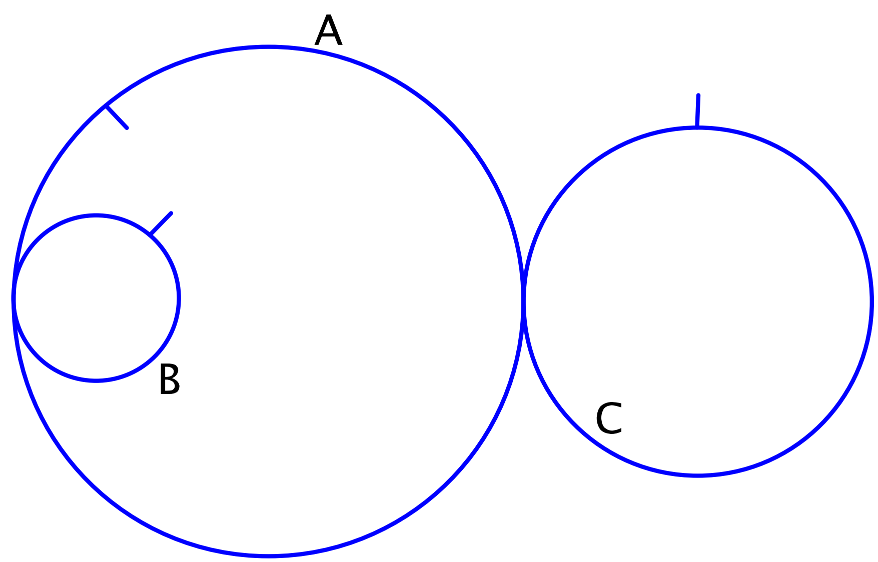


More in R. Schwartz, S.T. *Elementary surprises in projective geometry*. Math. Intelligencer, v. 32, No 3 (2010), 31–34.

6. Variations on Steiner's porism

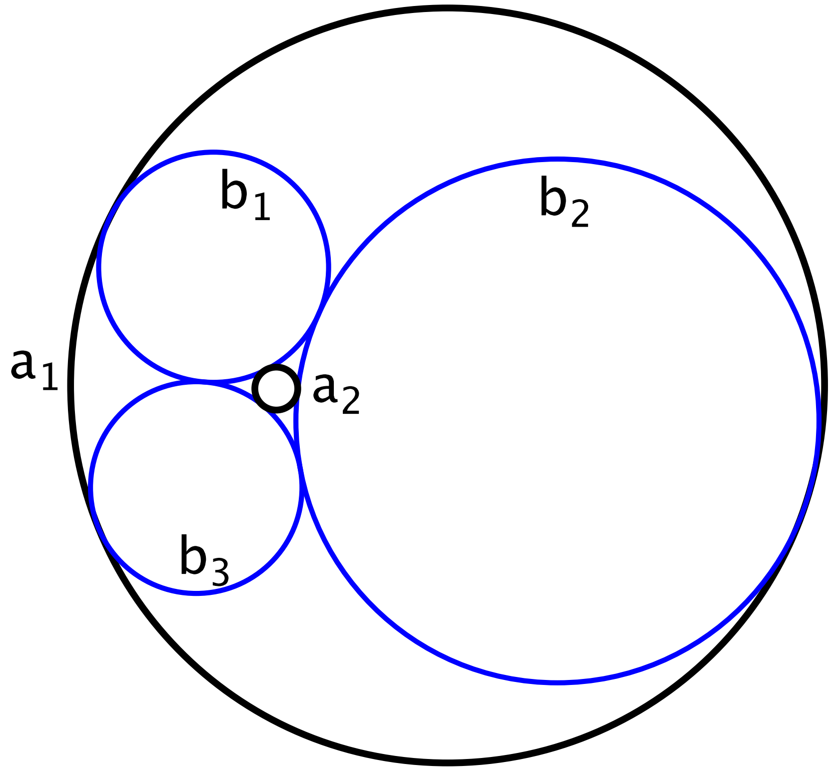
Descartes Circle Theorem.

Tangency of cooriented circles:



A and B are tangent, but A and C are not.

Signed curvature: $\pm 1/r$, with plus when the coorientation is inward and minus otherwise.

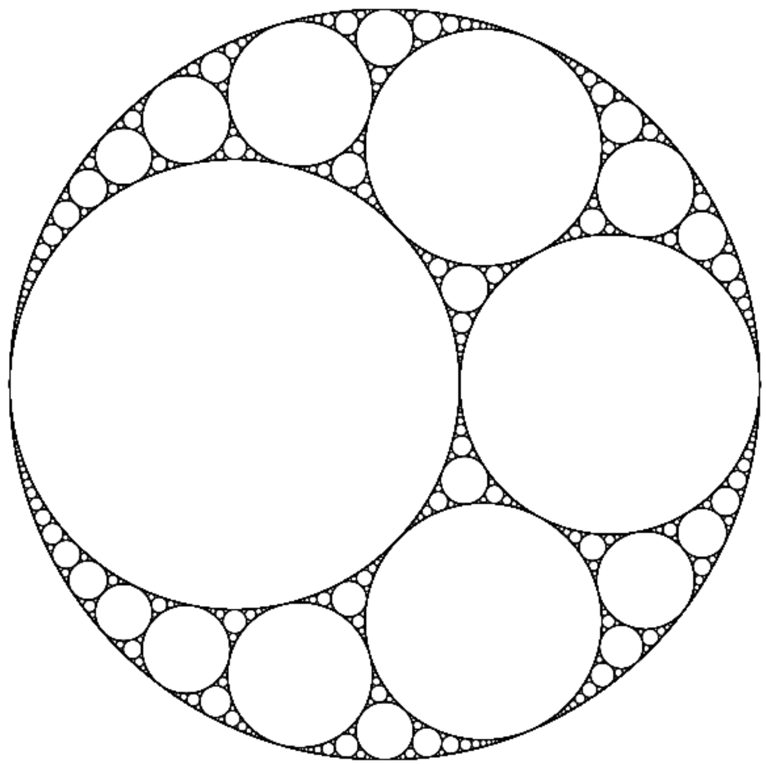


Theorem [Descartes]: $(a + b_1 + b_2 + b_3)^2 = 2(a^2 + b_1^2 + b_2^2 + b_3^2)$.

Hence

$$a_1+a_2 = 2(b_1+b_2+b_3), \;\; a_1a_2 = b_1^2+b_2^2+b_3^2-2(b_1b_2+b_2b_3+b_3b_1),$$

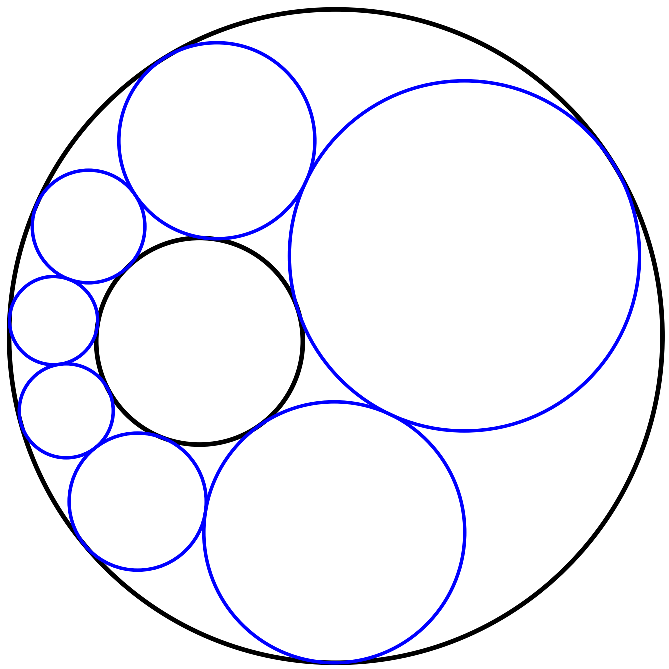
leading to Apollonian gasket:



Also

$$b_1 + b_2 + b_3 = \frac{a_1 + a_2}{2}, \quad b_1^2 + b_2^2 + b_3^2 = \frac{6a_1a_2 - a_1^2 - a_2^2}{4}$$

for a Steiner chain of length 3 with fixed *parent circles*.
Here is a chain of length 7:



Theorem: *For every* $m = 1, 2, \ldots, k-1$, *in the 1-parameter family of Steiner chains of length* k, *the moments*

$$I_m = \sum_{j=1}^{k} b_j^m$$

remains constant. These moments are symmetric polynomials of a_1 *and* a_2.

Complex Descartes Circle Theorem.

Theorem [Lagarias-Mallows-Wilks, 2002]:

$$(aw + b_1 z_1 + b_2 z_2 + b_3 z_3)^2 = 2(a^2 w^2 + b_1^2 z_1^2 + b_2^2 z_2^2 + b_3^2 z_3^2).$$

Hence

$$J_{1,1} = b_1 z_1 + b_2 z_2 + b_3 z_3 \quad \text{and} \quad J_{2,2} = b_1^2 z_1^2 + b_2^2 z_2^2 + b_3^2 z_3^2$$

remain constant in the 1-parameter family of Steiner chains of length 3 with fixed parent circles.

The invariance under parallel translation yields another conserved quantity:

$$J_{2,1} = b_1^2 z_1 + b_2^2 z_2 + b_3^2 z_3.$$

Theorem: *For all* $0 \leq n \leq m \leq k-1$*, in the 1-parameter family of Steiner chains of length* k*, the sum*

$$J_{m,n} = \sum_{j=1}^{k} b_j^m z_j^n$$

remains constant.

These integrals are not independent (there are too many of them). What are the syzygies?

In the Spherical and Hyperbolic Geometries

A circle in S^2 has two centers and, respectively, two radii, say α and $\pi - \alpha$. A coorientation is a choice of one center, and the signed curvature is $\cot \alpha$. In H^2, the curvature is $\coth \alpha$.

Theorem [Mauldon, 1962]: *The versions of the Descartes circle theorem are*

$$\sum_{i=1}^{4} (\cot \alpha_i)^2 = \frac{1}{2}(\sum_{i=1}^{4} \cot \alpha_i)^2 - 2$$

and

$$\sum_{i=1}^{4} (\coth \alpha_i)^2 = \frac{1}{2}(\sum_{i=1}^{4} \coth \alpha_i)^2 + 2,$$

The Steiner porism still holds.

Theorem: *Let* α_j *denote the radii of the circles in a Steiner chain of length* k *on the sphere* S^2. *Then, for* $m = 1, 2, \ldots, k-1$, *in the 1-parameter family of spherical Steiner chains of length* k,

$$\sum_{j=1}^{k} \cot^m \alpha_j$$

remain constant. Likewise, for Steiner chains in the hyperbolic plane,

$$\sum_{j=1}^{k} \coth^m \alpha_j$$

remain constant.

More in R. Schwartz, S.T. *Descartes Circle Theorem, Steiner Porism, and Spherical Designs*. Amer. Math. Monthly 127 (2020), 238–248.

Thank you!