Global Minimizers of Autonomous Lagrangians

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Contents

Chapter 1

Introduction.

1-1 Lagrangian Dynamics.

Let M be a boundaryless n-dimensional complete riemannian manifold. An (autonomous) Lagrangian on M is a smooth function $L : TM \to \mathbb{R}$ satisfying the following conditions:

(a) *Convexity:* The Hessian $\dfrac{\partial^2 L}{\partial v_i \partial v_j}(x,v)$, calculated in linear coordinates on the fiber T_xM, is uniformly positive definite for all $(x,v) \in TM$, i.e. there is $A > 0$ such that

$$w \cdot L_{vv}(x,v) \cdot w \geq A\,|w|^2 \quad \text{for all } (x,v) \in TM \text{ and } w \in T_xM.$$

(b) *Superlinearity:*

$$\lim_{|v| \to +\infty} \frac{L(x,v)}{|v|} = +\infty, \qquad \text{uniformly on } x \in M,$$

equivalently, for all $A \in \mathbb{R}$ there is $B \in \mathbb{R}$ such that

$$L(x,v) \geq A\,|v| - B \quad \text{for all } (x,v) \in TM.$$

(c) *Boundedness*[1]*:* For all $r \geq 0$,

$$\ell(r) = \sup_{\substack{(x,v)\in TM,\\ |v|\leq r}} L(x,v) < +\infty. \tag{1.1}$$

$$g(r) = \sup_{\substack{|w|=1\\ |(x,v)|\leq r}} w \cdot L_{vv}(x,v)\cdot w < +\infty. \tag{1.2}$$

The *Euler-Lagrange equation* associated to a lagrangian L is (in local coordinates)

$$\frac{d}{dt}\frac{\partial L}{\partial v}(x,\dot{x}) = \frac{\partial L}{\partial x}(x,\dot{x}). \tag{E-L}$$

The condition (c) implies that the Euler-Lagrange equation (E-L) defines a complete flow φ_t on TM (proposition 1-3.2), called the *Euler-Lagrange flow*, by setting $\varphi_t(x_0,v_0) = \big(x_v(t),\dot{x}_v(t)\big)$, where $x_v : \mathbb{R} \to M$ is the solution of (E-L) with $x_v(0) = x_0$ and $\dot{x}_v(0) = v_0$.

We shall be interested on coverings $p : N \to M$ of a compact manifold M and the lifted Lagrangian $\mathbb{L} = L \circ dp : TN \to \mathbb{R}$ of a convex superlinear lagrangian L on M. The lagrangian $\mathbb{L}$ then satisfies (a)–(c) and its flow ψ_t is the lift of φ_t.

Observe that when we add a *closed* 1-form ω to the lagrangian L, the new lagrangian $L + \omega$ also satisfies the hypothesis (a)-(c) and has the same Euler-Lagrange equation as L. This can also be seen using the variational interpretation of the Euler-Lagrange equation (see 1-2.3).

[1]The Boundedness condition (c) is equivalent to the condition that the associated hamiltonian H is convex and superlinear, see remark 1-4.2. This condition is immediate when the manifold M is compact.

1-2 The Euler-Lagrange equation.

The action of a differential curve $\gamma : [0, T] \to M$ is defined by

$$A_L(\gamma) = \int_0^T L(\gamma(t), \dot{\gamma}(t))\, dt$$

One of the main problems of the calculus of variations is to find and to study the curves that minimize the action. Denote by $C^k(q_1, q_2; T)$ the set of C^k-differentiable curves $\gamma : [0, T] \to M$ such that $\gamma(0) = q_1$ and $\gamma(T) = q_2$.

1-2.1 Proposition. *If a curve $x(t)$ in the space $C^k(q_1, q_2; T)$ is a critical point of the action functional on $C^k(q_1, q_2; T)$, then x satisfies the Euler-Lagrange equation*

$$\frac{d}{dt} L_v(x(t), \dot{x}(t)) = L_x(x(t), \dot{x}(t)) \tag{E-L}$$

in local coordinates. Consequently, this equation does not depend on the coordinate system.

Proof: Choose a coordinate system $(x_1, \dots, x_n)$ about $x(t)$. Let $h(t)$ a differentiable curve such that $h(0) = h(T) = 0$. Then for every ε, sufficiently small the curve $y_\varepsilon = x + \varepsilon h$ is on $C^k(q_1, q_2; T)$ and contained in the coordinate system. Define

$$g(\varepsilon) = A_L(y_\varepsilon)$$

Then g has a minimum in zero and

$$\begin{aligned}
\lim_{\varepsilon\to 0}\frac{g(\varepsilon)-g(0)}{\varepsilon} &= \lim_{\varepsilon\to 0}\int_0^T \frac{L(x+\varepsilon h,\dot{x}+\varepsilon\dot{h})-L(x,\dot{x})}{\varepsilon}\,dt\\
&= \int_0^T \lim_{\varepsilon\to 0}\frac{\varepsilon L_x h+\varepsilon L_v\dot{h}+o(\varepsilon)}{\varepsilon}\,dt\\
&= \int_0^T L_x h + L_v\dot{h}\;dt\\
&= \int_0^T (L_x-\tfrac{d}{dt}L_v)\,h\;dt + L_v\,h\Big|_0^T\\
&= \int_0^T (L_x-\tfrac{d}{dt}L_v)\,h\;dt.
\end{aligned}$$

Hence

$$0=\int_0^T\left[L_x(x(t),\dot{x}(t))-\frac{d}{dt}L_v(x(t),\dot{x}(t))\right]\,h\;dt,$$

for any function $h\in C^k(0,0;T)$. This implies that $x(t)$ satisfies the Euler Lagrange equation (E-L). □

The Euler Lagrange equation is a second order differential equation on M, but the convexity hypothesis (L_{vv} invertible) implies that this equation can also be seen as a first order differential equation on TM:

$$\begin{aligned}
\dot{x} &= v,\\
\dot{v} &= (L_{vv})^{-1}(L_x - L_{vx}v).
\end{aligned}$$

The associated vector field X on TM is called the lagrangian vector field and its flow φ_t the lagrangian flow. Observe that X is of the form

$$X(x,v)=(v,\cdot).$$

1-2.2 Remark. It is possible to do the same thing in the space $\mathcal{C}_T(p,q)$, the set of absolutely continuous curves $\gamma:[0,T]\to M$ such that $\gamma(0)=p$ and $\gamma(T)=q$. A priori minimizers do not have to be differentiable and there are examples where they are not, see Ball & Mizel [4]. However when the lagrangian flow is complete (cf. proposition 1-3.2), every absolutely continuous minimizers is C^2 and satisfies the Euler-Lagrange equation. See Mather [46].

1-2.3 Remark. If we add a closed 1-form ω to the lagrangian L, the lagrangian $L + \omega$ also satisfies the hypothesis (a)-(c). Moreover, the action functional $A_{L+\omega}$ on a neighbourbood of a curve $\gamma \in C^k(x_1, x_2, T)$ satisfies

$$A_{L+\omega}(\eta) = A_L(\eta) + \oint_\gamma \omega,$$

because the curve η is homologous to γ. Therefore, the critical points for $A_{L+\omega}$ and for A_L are the same. This implies that the Euler-Lagrange equations for L and $L + \omega$ are the same. But since the values of $A_{L+\omega}$ and A_L are different, minimizers of these two actions may be different.

1-3 The Energy function.

The *energy function* of the lagrangian L is $E : TM \to \mathbb{R}$, defined by

$$E(x,v) = \frac{\partial L}{\partial v}(x,v)\cdot v - L(x,v). \tag{1.3}$$

Observe that if $x(t)$ is a solution of the Euler-Lagrange equation (E-L), then

$$\tfrac{d}{dt}\, E(x,\dot{x}) = \big(\tfrac{d}{dt}L_v - L_x\big)\cdot \dot{x} = 0.$$

Hence $E : TM \to \mathbb{R}$ is an *integral* (i.e. invariant function[2]) for the lagrangian flow φ_t and its level sets, called *energy levels* are invariant under φ_t. Moreover, the convexity implies that

$$\tfrac{d}{ds}\, E(x,sv)\big|_{s=1} = v\cdot L_{vv}(x,v)\cdot v > 0.$$

Thus

$$\min_{v\in T_xM} E(x,v) = E(x,0) = -L(x,0).$$

Write

$$e_0 := \max_{x\in M} E(x,0) = -\min_{x\in M} L(x,0) > -\infty, \tag{1.4}$$

by the superlinearity $e_0 > -\infty$, then

$$e_0 = \min\{\, k\in\mathbb{R} \mid \pi : E^{-1}\{k\} \to M \text{ is surjective }\}. \tag{1.5}$$

By the uniform convexity, and the boundedness condition,

$$A := \inf_{\substack{(x,v)\in TM\\ |w|=1}} w\cdot L_{vv}(x,v)\cdot w > 0,$$

and then using (1.1) and (1.2),

$$\begin{aligned} E(x,v) &= E(x,0) + \int_0^{|v|} \tfrac{d}{ds}\, E\big(x, s\tfrac{v}{|v|}\big)\, ds \\ &\geq -\ell(0) + \tfrac{1}{2}\, A\, |v|^2. \end{aligned} \tag{1.6}$$

[2]The energy is invariant only for *autonomous* (i.e. time-independent) lagrangians.

Similarly, using (1.2),

$$E(x,v) \le e_0 + g(|v|)\,|v|. \tag{1.7}$$

Hence

1-3.1 Remark.
If $k \in \mathbb{R}$ and $K \subseteq M$ is compact, then $E^{-1}\{k\} \cap T_K M$ is compact.

1-3.2 Proposition. *The Euler-Lagrange flow is complete.*

Proof: Suppose that $]\alpha, \beta[$ is the maximal interval of definition of $t \mapsto \varphi_t(v)$, and $-\infty < \alpha$ or $\beta < +\infty$. Let $k = E(v)$. Since $E(\varphi_t(v)) \equiv k$, by (1.6), there is $a > 0$ such that $0 \le |\varphi_t(v)| \le a$ for $\alpha \le t \le \beta$. Since $\varphi_t(v)$ is of the form $(\gamma(t), \dot\gamma(t))$, then $\varphi_t(v)$ remains in the interior of the compact set

$$Q := \left\{\, (y,w) \in TM \;\middle|\; d(y,x) \le a\,\big[|\beta - \alpha| + 1\big],\;\; |w| \le a + 1 \,\right\},$$

where $x = \pi(v)$. The Euler-Lagrange vector field is uniformly Lipschitz on Q. Then by the theory of ordinary differential equations, we can extend the interval of definition $]\alpha, \beta[$ of $t \mapsto \varphi_t(v)$. $\square$

1-4 Hamiltonian Systems.

Let T^*M be the cotangent bundle of M. Define the *Liouville's 1-form* Θ on T^*M as

$$\Theta_p(\xi) = p\,(d\pi\,\xi) \qquad \text{for } \xi \in T_p(T^*M),$$

where $\pi : T^*M \to M$ is the projection. The *canonical symplectic form* on T^*M is defined as $\omega = d\Theta$.

A local chart $\mathbf{x} = (x_1, \ldots, x_n)$ of M induces a local chart $(\mathbf{x}, \mathbf{p}) = (x_1, \ldots, x_n; p_1, \ldots, p_n)$ of T^*M writing $\mathbf{p} \in T^*M$ as $\mathbf{p} = \Sigma_i\, p_i\, dx_i$. In these coordinates the forms Θ and ω are written

$$\Theta = \mathbf{p} \cdot d\mathbf{x} = \sum_i \; p_i\; dx_i \,,$$
$$\omega = d\mathbf{p} \wedge d\mathbf{x} = \sum_i \; dp_i \wedge dx_i \,.$$

A *hamiltonian* is a smooth function $H : T^*M \to \mathbb{R}$. The *hamiltonian vector field* X_H associated to H is defined by

$$\omega(X_H, \cdot\,) = dH. \tag{1.8}$$

In local charts, the hamiltonian vector field defines the differential equation

$$\begin{aligned} \dot{x} &= \;\;\; H_p \,, \\ \dot{p} &= -\; H_x \,, \end{aligned} \tag{1.9}$$

where H_x and H_p are the partial derivatives of H with respect to x and p. Let ψ_t be the *hamiltonian flow*. Observe that it preserves H, because

$$\tfrac{d}{dt} H = H_x\, \dot{x} + H_p\, \dot{p} = 0.$$

Moreover, it preserves the symplectic form ω, because[3]

$$\tfrac{d}{dt}(\psi_t^* \omega) = \mathcal{L}_{X_H} \omega = d\, i_{X_H} \omega + i_{X_H} \omega = d(dH) + i_{X_H}(0) = 0.$$

[3] $\mathcal{L}_X$ is the *Lie derivative*, defined on forms η by $\mathcal{L}_X \eta = d i_X \eta + i_X d\eta$, where $i_X \eta = \eta(X, \cdot)$ is the *contraction* by X. The Lie derivative satisfies $\mathcal{L}_X \eta = \frac{d}{dt} \psi_t^* \eta|_{t=0}$, where ψ_t is the flow of X.

We shall be specially interested in hamiltonians obtained by the Fenchel transform of a lagrangian:

$$H(x,p) = \max_{v \in T_x M} \; p\, v - L(x,v).$$

Observe that $H = E \circ \mathcal{L}^{-1}$, where E is the energy function (1.3) and $\mathcal{L}(x,v) = (x, L_v(x,v))$ is the Legendre transform of L. Moreover

1-4.1 Proposition. *The Legendre transform* $\mathcal{L} : TM \to T^*M$, $\mathcal{L}(x,v) = (x, L_v(x,v))$ *is a conjugacy between the lagrangian flow and the hamiltonian flow.*

Proof: By corollary D.2, the convexity and superlinearity hypothesis imply that $L = L^{**} = H^*$. So if $p = L_v(x,v)$ then $v = H_p(x,p)$. With this notation:

$$\begin{aligned} H(x,p) &= v \cdot L_v(x,v) - L(x,v) = E \circ \mathcal{L}^{-1} \\ &= p \cdot H_p(x,p) - L(x, H_p(x,p)). \end{aligned}$$

Thus $H_x = -L_x$, and the Euler-Lagrange equation

$$\begin{aligned} \dot{x} &= \tfrac{d}{dt}\, x \;\; = v \;\; = \;\; H_p\,, \\ \dot{p} &= \tfrac{d}{dt}\, L_v = L_x = -H_x\,, \end{aligned}$$

is the same as the hamiltonian equations. $\square$

1-4.2 Remark. Using that $L^* = H$ and $H^* = L$, from proposition D.2 in the appendix we obtain that the boundedness condition is equivalent to

(c) *Boundedness:* $H = L^*$ is convex and superlinear.

We say that an energy level $H^{-1}(k)$ is *regular*, if k is a regular value of H, i.e. $dH(x,p) \neq 0$ whenever $H(x,p) = k$.

1-4.3 Proposition. *Two hamiltonian flows restricted to a same regular energy level are reparametrizations of each other.*

Proof: Suppose that $H, G : T^*M \to \mathbb{R}$ are two hamiltonians with $H^{-1}(k) = G^{-1}(\ell)$ and k, ℓ are regular values for H and G respectively. Then, if $H(x,p) = k$,

$$\ker d_{(x,p)}H = T_{(x,p)}H^{-1}(k) = T_{(x,p)}H^{-1}(k) = \ker d_{(x,p)}G.$$

Thus there exists $\lambda(x,p) > 0$ such that $d_{(x,p)}H = \lambda(x,p)\, d_{(x,p)}G$. Equation (1.8) implies that $X_H = \lambda(x,p)\, X_G$ when $H(x,p) = k$. □

We shall need the following estimate on the norm of the partial derivative $L_v(x,v)$.

1-4.4 Lemma. *There is a function $f : [0,\infty[\to \mathbb{R}^+$ such that $\|L_v(x,v)\| \le f(|v|)$ for all $(x,v) \in TM$.*

Proof: The convexity condition implies that the maximum in

$$H(x,p) = \max_{w\in T_xM} p\cdot w - L(x,w)$$

is attained at $w = v_0$ with $p = L_v(x,v_0)$. Since $H(x, L_v(x,v)) = E(x,v)$,

$$L_v(x,v)\cdot w \le E(x,v) + L(x,w), \qquad \forall v,w \in T_xM,\ \forall x \in M.$$

Applying this inequality to $-w$, we get that

$$L_v(x,v)\cdot w \ge -E(x,v) - L(x,-w).$$

Thus using (1.6), (1.7) and (1.1), for $|v| \le r$, we have

$$\begin{aligned}\|L_v(x,v)\| &\le |E(x,v)| + \max\{\,|L(x,v)|, |L(x,-v)|\,\}\\ &\le \max\{|\ell(0)| + \tfrac{1}{2}Ar^2,\ e_0 + g(r)r\} + |\ell(r)| =: f(r).\end{aligned}$$

□

1-5 Examples.

We give here some basic examples of lagrangians.

Riemannian Lagrangians:

Given a riemannian metric $g = \langle \cdot, \cdot \rangle_x$ on TM, the riemannian lagrangian on M is given by the *kinetic energy*

$$L(x, v) = \tfrac{1}{2}\ \|v\|_x^2 . \tag{1.10}$$

Its Euler-Lagrange equation (E-L) is the equation of the geodesics of g:

$$\tfrac{D}{dt}\, \dot{x} \equiv 0, \tag{1.11}$$

and its Euler-Lagrange flow is the geodesic flow. Its corresponding hamiltonian is

$$H(x, p) = \tfrac{1}{2}\, \|p\|_x^2 .$$

Analogous to the riemannian lagrangian is the *Finsler lagrangian*, given also by formula (1.10), but where $\|\cdot\|_x$ is a *Finsler metric*, i.e. $\|\cdot\|_x$ is a (non necessarily symmetric[4]) norm on T_xM which varies smoothly on $x \in M$. The Euler-Lagrange flow of a Finsler lagrangian is called the *geodesic flow* of the Finsler metric $\|\cdot\|_x$.

Mechanic Lagrangians:

The *mechanic lagrangian*, also called *natural lagrangian*, is given by the kinetic energy minus the *potential energy* $U : M \to \mathbb{R}$,

$$L(x, v) = \tfrac{1}{2}\, \|v\|_x^2 - U(x). \tag{1.12}$$

Its Euler-Lagrange equation is

$$\tfrac{D}{dt}\, \dot{x} = -\nabla U(x),$$

[4] i.e. $\|\lambda\, v\|_x = \lambda\ \|v\|_x$ only for $\lambda \geq 0$

where $\frac{D}{dt}$ is the covariant derivative and ∇U is the gradient of U with respect to the riemannian metric g, i.e.

$$d_x U(v) = \langle \nabla U(x), v \rangle_x \quad \text{for all } (x,v) \in TM.$$

Its energy function and its hamiltonian are given by the kinetic energy plus potential energy:

$$E(x,v) = \tfrac{1}{2}\,\|v\|_x^2 + U(x),$$
$$H(x,p) = \tfrac{1}{2}\,\|p\|_x^2 + U(x).$$

Symmetric Lagrangians.

The symmetric lagrangians is a class of lagrangian systems which includes the riemannian and mechanic lagrangians. These are the lagrangians which satisfy

$$L(x,v) = L(x,-v) \qquad \text{for all } (x,v) \in TM. \tag{1.13}$$

Their Euler-Lagrange flow is *reversible* in the sense that $\varphi_{-t}(v) = -\varphi_t(-v)$.

Magnetic Lagrangians.

If one adds a closed 1-form ω to a lagrangian, $\mathbb{L}(x,v) = L(x,v) + \omega_x(v)$, the Euler-Lagrange flow does not change. This can be seen by first observing that the solutions of the Euler-Lagrange equation are the critical points of the action functional on curves on $\mathcal{C}(x,y,T)$ (with fixed time interval and fixed endpoints). Since ω is closed, the action functional of $\mathbb{L}$ and L on $\mathcal{C}(x,y,T)$ differ by a constant and hence they have the same critical points.

But adding a non-closed 1-form to a lagrangian does change the Euler-Lagrange flow. We call a *magnetic lagrangian* a lagrangian of the form

$$L(x,v) = \tfrac{1}{2}\ \|v\|_x + \eta_x(v) - U(x), \tag{1.14}$$

where $\|\cdot\|_x$ is a riemannian metric, η is a 1-form on M with $d\eta \neq 0$, and $U : M \to \mathbb{R}$ a smooth function. If $Y : TM \to TM$ is the bundle map such that

$$d\eta(u, v) = \langle Y(u), v\rangle$$

then the Euler-Lagrange equation of (1.14) is

$$\tfrac{D}{dt}\,\dot{x} = Y_x(\dot{x}) - \nabla U(x). \tag{1.15}$$

This models the motion of a particle with unit mass and unit charge under the effect of a magnetic field with Lorentz force Y and potential energy $U(x)$. The energy functional is the same as that of the mechanical lagrangian but its hamiltonian changes because of the change in the Legendre transform:

$$\begin{aligned} E(x, v) &= \tfrac{1}{2}\ \|v\|_x^2 + U(x), \\ H(x, p) &= \tfrac{1}{2}\ \|p - A(x)\|_x^2 + U(x), \end{aligned}$$

where $A : M \to TM$ is the vector field given by $\eta_x(v) = \langle A(x), v\rangle_x$.

Twisted geodesic flows.

The twisted geodesic flows correspond to the motion of a particle under the effect of a magnetic field with no potential energy. This can be modeled as the Euler-Lagrange flow of a lagrangian of the form $L(x, v) = \frac{1}{2}\,\|v\|_x^2 + \eta_x(v)$, where $d\eta \not\equiv 0$. But the Euler-Lagrange equations depend only on the riemannian metric and $d\eta$. A generalization of these flows can be made using a non-zero 2-form Ω instead of $d\eta$ and not requiring Ω to be exact. This is better presented in the hamiltonian setting.

Fix a riemannian metric $\langle\ ,\ \rangle$ and a 2-form Ω on M. Let $K : TTM \to TM$ be the connection map $K\xi = \nabla_{\dot{x}} v$, where $\xi = \frac{d}{dt}\big(x(t), v(t)\big)$. Let $\pi : TM \to M$ be the canonical projection. Let ω_0 be the symplectic form in TM obtained by pulling back the canonical symplectic form via the Legendre transform associated to the riemannian metric, i.e.

$$\omega_0(\xi, \zeta) = \langle d\pi\,\xi\,,\, K\zeta\rangle - \langle d\pi\,\zeta\,,\, K\xi\rangle\,.$$

The coordinates $T_\theta TM \ni \xi \longleftrightarrow (d\pi\,\xi, K\xi) \in T_{\pi(\theta)}M \oplus T_\pi(\theta)M = H(\theta)\oplus V(\theta)$ are the standard way of writing the horizontal and vertical components of a vector $\xi \in T_\theta TM$ for a riemannian manifold M (see Klingenberg [31]).

Define a new symplectic form ω_Ω on TM by

$$\omega_\Omega = \omega_0 + \pi^*\Omega\,.$$

This is called a *twisted symplectic structure* on TM. Let $H : TM \to \mathbb{R}$ be the hamiltonian

$$H(x,v) = \tfrac{1}{2}\,\|v\|_x^2\ .$$

Consider the hamiltonian vector field X_F corresponding to (H, ω_Ω), i.e.

$$\omega_\Omega\big(X_\Omega(\theta),\cdot\big) = dH\,. \tag{1.16}$$

Define $Y : TM \to TM$ as the bundle map such that

$$\Omega_x(u,v) = \langle Y(u), v\rangle_x\,. \tag{1.17}$$

The hamiltonian vector field $X_\Omega(\theta) \in T_\theta TM$ is given by $X_\Omega(\theta) = (\theta, Y(\theta)) \in H(\theta)\oplus V(\theta)$. Hence the hamiltonian equation is

$$\tfrac{D}{dt}\,\dot{x} = Y_x(\dot{x}),$$

recovering equation (1.15) with $U \equiv 0$, but where Ω doesn't need to be exact.

If $H^1(M,\mathbb{R}) = 0$, both approaches coincide, and any twisted geodesic flow is the lagrangian flow of a magnetic lagrangian of the form $L(x,v) = \frac{1}{2}\,\|v\|_x^2 + \eta_x(v)$, with $d\eta = \Omega$. For example if N is a compact manifold Ω is a 2-form in N and M is the abelian cover or the universal cover of N; if Ω is not exact, then the corresponding twisted geodesic flow is a lagrangian flow on M but not on N (where it is locally a lagrangian flow). This lagrangian flow on M is actually the lift of the twisted geodesic flow on N.

Embedding flows:

There is a way to embed the flow of any bounded vector field on a lagrangian system. Given a smooth bounded vector field $F : M \to TM$, let

$$L(x,v) = \tfrac{1}{2}\, \|v - F(x)\|_x^2 \,. \tag{1.18}$$

Since $F(x)$ is bounded, then the lagrangian L is convex, superlinear and satisfies the boundedness condition. The lagrangian L on a fiber T_xM is minimized at $(x, F(x))$, hence the integral curves of the vector field, $\dot{x} = F(x)$, are solutions to the Euler-Lagrange equation.

Chapter 2

Mañé's critical value.

2-1 The action potential and the critical value.

We shall be interested on action minimizing curves with *free* time interval. Unless otherwise stated, *all the curves will be assumed to be absolutely continuous.* For $x, y \in M$, let

$$\mathcal{C}(x,y) = \{\, \gamma : [0,T] \to M \,|\, T > 0,\ \gamma(0) = x,\ \gamma(T) = y \,\}.$$

For $k \in \mathbb{R}$ define the *action potential* $\Phi_k : M \times M \to \mathbb{R} \cup \{-\infty\}$, by

$$\Phi_k(x,y) = \inf_{\gamma \in \mathcal{C}(x,y)} A_{L+k}(\gamma).$$

Observe that if there exists a closed curve γ on N with negative $L + k$ action, then $\Phi_k(x,y) = -\infty$ for all $x, y \in N$, by going round γ many times.

Define the *critical level* $c = c(L)$ as

$$c(L) = \sup\{\, k \in \mathbb{R} \,|\, \exists \text{ closed curve } \gamma \text{ with } A_{L+k}(\gamma) < 0 \,\}.$$

Observe that the function $k \mapsto \Phi_k(x,y)$ is increasing. The superlinearity implies that L is bounded below. Hence there is $k \in \mathbb{R}$ such that

$L+k \geq 0$. Thus $c(L) < +\infty$. Since $k \mapsto A_{L+k}(\gamma)$ is increasing for any γ, we have that

$$c(L) = \inf\{\, k \in \mathbb{R} \mid A_{L+k}(\gamma) \geq 0 \ \forall \text{ closed curve } \gamma \,\}.$$

2-1.1 Proposition.

1. *(a) For $k < c(L)$, $\Phi_k(x,y) = -\infty$ for all $x, y \in M$.*
   (b) For $k \geq c(L)$, $\Phi_k(x,y) \in \mathbb{R}$ for all $x, y \in M$.
2. *For $k \geq c(L)$, $\Phi_k(x,z) \leq \Phi_k(x,y) + \Phi_k(y,z)$, $\forall\, x, y, z \in M$.*
3. *$\Phi_k(x,x) = 0$, $\forall\, x \in M$.*
4. *$\Phi_k(x,y) + \Phi_k(y,x) \geq 0 \quad \forall\, x, y \in M$.*
   For $k > c(L)$, $\Phi_k(x,y) + \Phi_k(y,x) > 0$ if $x \neq y$.
5. *For $k \geq c(L)$ the action potential Φ_k is Lipschitz.*

2-1.2 Remark. The action potential Φ_k is not symmetric in general, but items 2, 3, 4 imply that

$$d_k(x,y) = \Phi_k(x,y) + \Phi_k(y,x)$$

is a metric for $k > c(L)$ and a pseudo-metric for $k = c(L)$ [i.e. perhaps $d_c(x,y) = 0$ for some $x \neq y$ and $c = c(L)$].

Proof:

2. We first prove 2 for all $k \in \mathbb{R}$. Since $\Phi_k(x,y) \in \mathbb{R} \cup \{-\infty\}$, the inequality in item 2 makes sense for all $k \in \mathbb{R}$. If $\gamma \in \mathcal{C}(x,y)$, $\eta \in \mathcal{C}(y,z)$, then $\gamma * \eta \in \mathcal{C}(x,z)$ and hence

$$\Phi_k(x,z) \leq A_{L+k}(\gamma * \eta) \leq A_{L+k}(\gamma) + A_{L+k}(\eta).$$

   Taking the infima on $\gamma \in \mathcal{C}(x,y)$ and $\eta \in \mathcal{C}(y,z)$, we obtain 2.

1. (a) If γ is a closed curve with $A_{L+k}(\gamma) < 0$ and $\gamma(0) = z$, then

$$\Phi_k(z,z) \le \lim_{N\to\infty} A_{L+k}(\gamma * \overset{N}{\cdots} * \gamma) = \lim_N N\, A_{L+k}(\gamma) = -\infty.$$

For $x, y \in M$, item 2 implies that

$$\Phi_k(x,y) \le \Phi_k(x,z) + \Phi_k(z,z) + \Phi_k(z,y) = -\infty.$$

Since the function $k \mapsto \Phi_k(x,y)$ is increasing, then item 1(a) follows.

(b) Conversely, if $\Phi_k(x,y) = -\infty$ for some $k \in \mathbb{R}$ and $x, y \in M$, then

$$\Phi_k(x,x) \le \Phi_k(x,y) + \Phi_k(y,x) = -\infty.$$

Thus there is $\gamma \in \mathcal{C}(x,x)$ with $A_{L+k}(\gamma) < 0$. Then $k \le c(L)$. Observe that the set $\{\, k \in \mathbb{R} \mid A_{L+k}(\gamma) < 0$ for some closed curve $\gamma\,\}$ is open. Hence $\Phi_k(x,y) = -\infty$ actually implies that $k < c(L)$. This proves item 1(b).

3. Let $k \in \mathbb{R}$ by the boundedness condition there exists $Q > 0$ be such that

$$|L(x,v) + k| \le Q \quad \text{for } |v| \le 2. \tag{2.1}$$

Now let $\gamma : [0,\varepsilon] \to M$ be a differentiable curve with $|\dot\gamma| \equiv 1$ and $\gamma(0) = x$. Then

$$\begin{aligned}\Phi_k(x,x) &\le \Phi_k(x,\gamma(\varepsilon)) + \Phi_k(\gamma(\varepsilon),x)\\ &\le A_{L+k}\left(\gamma|_{[0,\varepsilon]}\right) + A_{L+k}\left(\gamma(t-\varepsilon)|_{[0,\varepsilon]}\right)\\ &\le 2\,Q\,\varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0$ we get that $\Phi_k(x,x) \le 0$. But the definition of $c(L)$ and the monotonicity of $k \mapsto \Phi_k(x,x)$ imply that $\Phi_k(x,x) \ge 0$ for all $k \ge c(L)$.

5. Let $k \ge c(L)$. Given $x_1, x_2 \in M$ we have that

$$\Phi_k(x_1,x_2) \le A_{L+k}(\gamma) \le Q\, d_M(x_1,x_2),$$

where $\gamma : [0, d(x_1, x_2)] \to N$ is a unit speed minimizing geodesic joining x_1 to x_2 and $Q > 0$ is from (2.1). If $y_1, y_2 \in M$, then the triangle inequality implies that

$$\begin{aligned}\Phi_k(x_1, y_1) - \Phi_k(x_2, y_2) &\le \Phi_k(x_1, x_2) + \Phi_k(y_2, y_1)\\ &\le Q\,[d_M(x_1, x_2) + d_M(y_1, y_2)].\end{aligned}$$

Changing the roles of (x_1, y_1) and (x_2, y_2) we get item 5.

4. The first part of item 4 follows from items 2 and 3. Now suppose that $k > c(L)$, $x \neq y$ and $d_k(x, y) = 0$. Let $\gamma_n : [0, T_n] \to M$, $\gamma_n \in \mathcal{C}(x, y)$ be such that $\Phi_k(x, y) = \lim_n A_{L+k}(\gamma_n)$. We claim that T_n is bounded below.

   Indeed, suppose that $\lim_n T_n = 0$. Let $A > 0$, from the superlinearity there is $B > 0$ such that $L(x, v) \ge A\,|v| - B$, $\forall\ (x, v) \in TM$. Then

$$\begin{aligned}\Phi_k(x, y) &= \lim_n \int_0^{T_N} L(\gamma_n, \dot\gamma_n) + k\\ &\ge \lim_n\ A \textstyle\int |\dot\gamma| + (k - B)\,T_n\\ &= A\ d_M(x, y)\end{aligned}$$

   Letting $A \to +\infty$ we get that $\Phi_k(x, y) = +\infty$ which is false.

   Now let $\eta_n : [0, S_n] \to M$, $\eta_n \in \mathcal{C}(y, x)$ with $\lim_n A_{L+k}(\eta_n) = \Phi_k(y, x)$. Choose $0 < T < \liminf_n T_n$ and $0 < S < \liminf_n S_n$. Then for $c = c(L) < k$,

$$\begin{aligned}\Phi_c(x, x) &\le \lim_n A_{L+c}(\gamma_n * \eta_n)\\ &\le \lim_n A_{L+k}(\gamma_n) + (c - k)T + A_{L+k}(\eta_n) + (c - k)S\\ &\le \lim_n \Phi_k(x, y) + \Phi_k(y, x) + (c - k)(T + S)\\ &\le (c - k)(T + S) < 0,\end{aligned}$$

   which contradicts item 3. □

2-2 Continuity of the critical value.

2-2.1 Lemma. *The function* $C^\infty(M,\mathbb{R}) \ni \psi \mapsto c(L+\psi)$ *is continuous in the topology induced by the supremum norm.*

Proof: Suppose that $\psi_n \to \psi$ and let $c_n := c(L+\psi_n)$ and $c := c(L+\psi)$. We will prove that $c_n \to c$.

Fix $\varepsilon > 0$. Since $c - \varepsilon < c$, by the definition of critical value there exists a closed curve $\gamma : [0,T] \to M$ such that $A_{L+\psi+c-\varepsilon}(\gamma) < 0$, hence for all n sufficiently large

$$A_{L+\psi_n+c-\varepsilon}(\gamma) < 0.$$

Therefore for n sufficiently large $c-\varepsilon < c_n$, and thus $c-\varepsilon \le \liminf_n c_n$. Since ε was arbitrary we have that $c \le \liminf_n c_n$.

We show now that $\limsup_n c_n \le c$. Suppose that $c < \limsup_n c_n$. Take ε such that

$$c < c + \varepsilon < \limsup_n c_n. \tag{2.2}$$

Since $\psi_n \to \psi$, there exists n_0 such that for all $n \ge n_0$,

$$-\varepsilon \le \psi - \psi_n \le \varepsilon. \tag{2.3}$$

By (2.2), there exists $m \ge n_0$ such that

$$c < c + \varepsilon < c_m.$$

By the definition of critical value there exists a closed curve $\gamma : [0,T] \to M$ such that

$$A_{L+\psi_m+c+\varepsilon}(\gamma) < 0,$$

and hence using (2.3) we have

$$A_{L+\psi+c}(\gamma) \le A_{L+\psi_m+c+\varepsilon}(\gamma) < 0,$$

which yields a contradiction to the definition of the critical value c.

$\square$

This proof also shows that $L \mapsto c(L)$ is continuous if we endow the set of lagrangians L with the topology induced by the supremum norm on compact subsets of TM.

2-3 Holonomic measures.

Let C^0_ℓ be the set of continuous functions $f : TM \to \mathbb{R}$ having linear growth, i.e.

$$\|f\|_\ell := \sup_{(x,v)\in TM} \frac{|f(x,v)|}{1+\|v\|} < +\infty.$$

Let $\mathcal{M}_\ell$ be the set of Borel probabilities μ on TM such that

$$\int_{TM} \|v\| \; d\mu < +\infty,$$

endowed with the topology such that $\lim_n \mu_n = \mu$ if and only if

$$\lim_n \int f \; d\mu_n = \int f \; d\mu \tag{2.4}$$

for all $f \in C^0_\ell$.

Let $(C^0_\ell)'$ the dual of C^0_ℓ. Then $\mathcal{M}_\ell$ is naturally embedded in $(C^0_\ell)'$ and its topology coincides with that induced by the weak* topology on $(C^0_\ell)'$.

We shall see that this topology is metrizable. Let $\{f_n\}$ be a sequence of functions with compact support on C^0_ℓ which is dense on C^0_ℓ in the topology of uniform convergence on compact sets of TM. Define a metric $d(\cdot,\cdot)$ on $\mathcal{M}_\ell$ by

$$d(\mu_1,\mu_2) = \left| \int |v| \; d\mu_1 - \int |v| \; d\mu_2 \right| + \sum_n \frac{1}{2^n}\frac{1}{c_n} \left| \int f_n \; d\mu_1 - \int f_n \; d\mu_2 \right| \tag{2.5}$$

where $c_n = \sup_{(x,v)} |f_n(x,v)|$.

2-3.1 Exercises:

1. Construct $\mu \in \mathcal{M}_\ell$ such that $\int |v|^2 \; d\mu = +\infty$.
2. Show that the first term in (2.5) is necessary.

2-3.2 Proposition.

The metric $d(\cdot,\cdot)$ induces the weak topology on $\mathcal{M}_\ell \subset (C^0_\ell)'$.*

Proof: We prove that $d(\cdot,\cdot)$ generates the weak* topology on $\mathcal{M}_\ell$. Suppose that

$$\int f\, d\mu_n \to \int f\, d\mu, \qquad \forall f \in C^0_\ell.$$

Given $\varepsilon > 0$, choose $M > 0$ such that $\sum_{m\geq M} \frac{1}{2^m}\cdot 2 < \varepsilon$, and choose $N > 0$ such that

$$\left|\int f_m\, d\mu_n - \int f_m\, d\mu\right| < \varepsilon, \qquad \text{for } 0 \leq m \leq M,\ \ n \geq N;$$

$$\left|\int |v|\, d\mu_n - \int |v|\, d\mu\right| < \varepsilon, \qquad \text{for } n \geq N.$$

Since $\frac{\|f_n\|_\infty}{c_n} = 1$, then for $n > N$ we have that

$$d(\mu_n,\mu) \leq \varepsilon + \sum_{m=1}^{M} \tfrac{1}{2^m}\cdot\varepsilon + \sum_{m\geq M+1} \tfrac{1}{2^m}\cdot 2\cdot \tfrac{\|f_m\|}{c_m} = 3\,\varepsilon.$$

Thus $d(\mu_n,\mu)\to 0$.

Now suppose that $d(\mu_n,\mu)\to 0$. Let K_m be compact sets such that $K_m \subset K_{m+1}$ and that $TM = \cup K_m$. Then

$$\int_{K_m} f\, d\mu_n \longrightarrow \int_{K_m} f\, d\mu, \qquad \forall f\in C^0_\ell, \quad \forall m;$$

$$\int |v|\, d\mu_n \longrightarrow \int |v|\, d\mu.$$

This implies that

$$\lim_{n\to\infty} \int_{TM-K_m} |v|\, d\mu_n = \int_{TM-K_m} |v|\, d\mu, \qquad \forall\, m. \tag{2.6}$$

Given $\varepsilon > 0$, choose $m(\varepsilon) > 0$ such that

$$\int_{TM-K_{m(\varepsilon)}} (1+|v|)\, d\mu < \frac{\varepsilon}{4},$$

and N such that

$$\int_{TM-K_{m(\varepsilon)}} (1+|v|)\, d\mu_n < \frac{\varepsilon}{2}, \qquad \forall n > N.$$

Fix $f \in C^0_\ell$. Choose $N > 0$ such that

$$\left| \int_{K_{m(\varepsilon)}} f\, d\mu_n - \int_{K_{m(\varepsilon)}} f\, d\mu \right| < \varepsilon, \qquad \forall\, n > N.$$

Then

$$\int_{TM-K_{m(\varepsilon)}} |f|\; d\mu_n \leq \|f\|_\ell \int_{TM-K_{m(\varepsilon)}} (1+|v|)\, d\mu_n \leq \|f\|_\ell\ \frac{\varepsilon}{2}, \quad \forall\, n > N.$$

Using a similar estimate for μ we obtain that

$$\left| \int f\; d\mu_n - \int f\; d\mu \right| \leq \varepsilon + \|f\|_\ell\ \left(\tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{4}\right).$$

□

If $\gamma : [0,T] \to M$ is a closed absolutely continuous curve, let $\mu_\gamma \in \mathcal{M}_\ell$ be defined by

$$\int f\; d\mu_\gamma = \frac{1}{T} \int_0^T f\big(\gamma(t), \dot\gamma(t)\big)\; dt$$

for all $f \in C^0_\ell$. Observe that $\mu_\gamma \in \mathcal{M}_\ell$ because if γ is absolutely continuous then $\int |\dot\gamma(t)|\; dt < +\infty$. Let $\mathcal{C}(M)$ be the set of such μ_γ's and let $\overline{\mathcal{C}(M)}$ be its closure in $\mathcal{M}_\ell$. Observe that the set $\overline{\mathcal{C}(M)}$ is convex. We call $\overline{\mathcal{C}(M)}$ the set of *holonomic measures* on M.

2-4 Invariance of minimizing measures.

Given a Borel probability measure μ in TM define its *action* by

$$A_L(\mu) = \int_{TM} L \, d\mu.$$

Since by the superlinearity the lagrangian L is bounded below, this action is well defined. Observe that $L \notin C^0_\ell$ and that $A_L(\mu) = +\infty$ for some $\mu \in \mathcal{M}_\ell$ (see exercise 2-3.1).

Let $\mathcal{M}(L)$ be the set of φ_t-invariant probabilities on TM.

2-4.1 Theorem (Mañé [38], prop. 1.1, 1.3, 1.2)**.**

1. *$\mathcal{M}(L) \subseteq \overline{\mathcal{C}(M)} \subseteq \mathcal{M}_\ell$.*
2. *If $\mu \in \overline{\mathcal{C}(M)}$ satisfies*

$$A_L(\mu) = \min\{\, A_L(\nu) \,|\, \nu \in \overline{\mathcal{C}(M)} \,\},$$

   then $\mu \in \mathcal{M}(L)$.
3. *If M is compact and $a \in \mathbb{R}$, then the set $\{\, \mu \in \overline{\mathcal{C}(M)} \,|\, A_L(\mu) \leq a \,\}$ is compact.*

Observe that item 3 implies the existence of a minimizer as in item 2.

The inclusion $\mathcal{M}(L) \subseteq \overline{\mathcal{C}(M)}$ follows from Birkhoff's ergodic theorem and the fact that $\overline{\mathcal{C}(M)}$ is convex. Taking $f = \|v\|$ in equation (2.4) we see that $\mathcal{M}_\ell$ is closed, so that $\overline{\mathcal{C}(M)} \subseteq \mathcal{M}_\ell$.

Proof of item 2-4.1.3:

Since $\overline{\mathcal{C}(M)}$ is closed, it is enough to prove that the set

$$\mathcal{A}(a) := \big\{\, \mu \in \mathcal{M}_\ell \,|\, A_L(\mu) \leq a \,\big\}$$

is compact in $\mathcal{M}_\ell$. First we prove that $\mathcal{A}(a)$ is closed. Let $k > 0$ and define $L_k := \min\{L, k\}$. Let

$$\mathcal{B}_k := \big\{\, \mu \in \mathcal{M}_\ell \,|\, \textstyle\int L_k \, d\mu \leq a \,\big\}.$$

Since $L_k \in \mathcal{C}^0_\ell$, then $\mathcal{B}_k$ is closed in $\mathcal{M}_\ell$. Since $\mathcal{A}(a) = \cap_{k>0}\mathcal{B}_k$, then $\mathcal{A}(a)$ is closed.

In order to prove the compactness, consider a sequence $\{\mu_n\} \subset \mathcal{A}(a)$. Applying the Riesz' theorem B.1, taking a subsequence we can assume that there exists a measure μ on the Borel σ-algebra of TM such that

$$\int f_i \, d\mu_n \longrightarrow \int f_i \, d\mu, \tag{2.7}$$

for every f_i in the sequence used for the definition of $d(\cdot,\cdot)$. Approximating the function 1 by the functions f_i we see that μ is a probability.

Approximating L_k by functions f_i we have that

$$\int L_k \, d\mu = \lim_n \int L_k \, d\mu_n \leq \liminf_n \int L \, d\mu_n \leq a.$$

Letting $k \uparrow +\infty$, by the monotone convergence theorem, we get that

$$A_L(\mu) \leq a. \tag{2.8}$$

Let $B > 0$ be such that $|v| < L(x,v) + B$ for all $(x,v) \in TM$. Then

$$\int |v| \, d\mu \leq A_L(\mu) + B \leq a + B < +\infty. \tag{2.9}$$

So that $\mu \in \mathcal{M}_\ell$.

We now prove that $\lim_n \int |v| \; d\mu_n \longrightarrow \int |v| \; d\mu$. Let $\varepsilon > 0$. By adding a constant we may assume that $L > 0$. Choose $r > 0$ such that $L(x,v) > a\,\varepsilon^{-1}\,|v|$ for all $|v| > r$. Then

$$\int_{|v|>r} |v| \, d\mu_n \leq \frac{\varepsilon}{a} \int_{|v|>r} L \, d\mu_n \leq \frac{\varepsilon}{a} \int L \, d\mu_n \leq \varepsilon.$$

Similarly, by (2.8),

$$\int_{|v|>r} |v| \, d\mu \leq \varepsilon.$$

From (2.7) we obtain that there is $N > 0$ such that

$$\left| \int_{|v|\le r} |v|\, d\mu - \int_{|v|\le r} |v|\, d\mu_n \right| < \varepsilon, \quad \text{for } n > N.$$

Adding these inequalities we get that

$$\left| \int |v|\, d\mu_n - \int |v|\, d\mu \right| \le 3\,\varepsilon.$$

□

The proof of item 2-4.1.2 requires some preliminary results which we present now. Item 2-4.1.2 is proved at the end of the section.

The following proposition is needed to show that the minimum of the action in $\overline{\mathcal{C}(M)}$ is the same as the minimum on $\mathcal{C}(M)$.

2-4.2 Proposition.
Given $\mu \in \overline{\mathcal{C}(M)}$, there are $\mu_{\eta_n} \in \mathcal{C}(M)$ such that $\mu_{\eta_n} \to \mu$ and

$$\lim_n \int L\, d\mu_{\eta_n} = \int L\, d\mu.$$

2-4.3 Remark. The statement of proposition 2-4.2 is not trivial. It is easy to see that the function $A_L : \overline{\mathcal{C}(M)} \to \mathbb{R}$ is always lower semicontinuous (see the last argument of the proof of 2-4.2), but in general it is not continuous. It is possible to give a sequence $\mu_{\gamma_n} \in \mathcal{C}(M)$ such that $\mu_{\gamma_n} \to \mu$ in $\overline{\mathcal{C}(M)}$ but $\liminf_n A_L(\mu_{\gamma_n}) > A_L(\mu)$ for a quadratic lagrangian L.

This can be made by calibrating the high speeds in γ_n so that $\int_{[|v|>R]} |v|\, d\mu_{\gamma_n} \to 0$ but $a := \liminf_n \int_{[|v|>R]} L\, d\mu_{\gamma_n} > 0$. Then the limit measure μ will have support on $[|v| \le R]$ and "will not see" the remnant a of the action.

Proof: Let $A > 1$ and let $\gamma : [0,T] \to M$ be a closed absolutely continuous curve. We reparametrize γ to a curve $\eta : [0,S] \to M$ such that $\dot\eta = \dot\gamma$ when $|\dot\gamma| < A$ and $\dot\eta = \frac{\dot\gamma}{|\dot\gamma|} A$ when $|\dot\gamma| > A$. So that $|\dot\eta| \le A$. Write $\eta(s(t)) = \gamma(t)$, $w(s) = |\dot\eta(s)|$ and $v(t) = |\dot\gamma(t)|$. We want

$$\int_0^{s(t)} w(s)\ ds = \int_0^t v(t)\ dt,$$

so that

$$s'(t) = \frac{v(t)}{w(s(t))} = \begin{cases} 1 & \text{when } v(t) \le A. \\ \frac{v(t)}{A} & \text{when } v(t) \ge A. \end{cases}$$

Then

$$\begin{aligned} S(T) &= \int_{[v(t)\le A]} dt + \int_{[v(t)\ge A]} \frac{v(t)}{A}\ dt, \\ \frac{S(T)}{T} &= \mu_\gamma([|v| \le A]) + \int_{[|v|\ge A]} \frac{|v|}{A}\ d\mu_\gamma, \\ \left|\frac{S(T)}{T} - 1\right| &\le \int_{[|v|>A]} d\mu_\gamma + \int_{[|v|\ge A]} \frac{|v|}{A}\ d\mu_\gamma \\ &\le 2\int_{[|v|>A]} |v|\ d\mu_\gamma. \end{aligned} \tag{2.10}$$

Suppose that $f : TM \to \mathbb{R}$ is μ_η-integrable. Since $\frac{ds}{dt} = \frac{v(t)}{A}$ when $v(t) > A$ then

$$\int_{[|\dot\gamma(t(s))|>A]} f(\eta(s), \tfrac{\dot\gamma(t(s))}{|\dot\gamma(t(s))|}A)\ ds = \int_{[|\dot\gamma(t)>A]} f\big(\gamma(t), \tfrac{\dot\gamma(t)}{|\dot\gamma(t)|}A\big)\ \frac{|\dot\gamma(t)|}{A}\ dt.$$

Then

$$\begin{aligned}
\int f\, d\mu_\eta &= \frac{1}{S(T)} \int f(\eta(s), \dot\eta(s))\, ds \\
&= \frac{1}{S(T)} \left[\int_{[|\dot\gamma| \le A]} f(\gamma(t), \dot\gamma(t))\, dt + \int_{[|\dot\gamma(t(s))| > A]} f\big(\eta(s), \tfrac{\dot\gamma(t(s))}{|\dot\gamma(t(s))|} A\big)\, ds \right] \\
&= \frac{T}{S(T)} \left[\int_{[|v| \le A]} f(v)\, d\mu_\gamma(v) + \int_{[|v|>A]} f\big(\tfrac{v}{|v|} A\big)\, \frac{|v|}{A}\, d\mu_\gamma \right]
\end{aligned}$$

For $A > 1$ big enough,

$$\int_{[|v|>A]} |v|\, d\mu_\gamma < \varepsilon < \tfrac{1}{4}. \tag{2.11}$$

Define

$$f_A(v) := \begin{cases} f(v) & \text{if } |v| \le A, \\ f\big(\tfrac{v}{|v|} A\big)\, \tfrac{|v|}{A} & \text{if } |v| > A. \end{cases}$$

Then

$$\int f\, d\mu_\eta = \tfrac{T}{S(T)} \int f_A\, d\mu_\gamma. \tag{2.12}$$

Observe that from (2.10) and (2.11), we have that

$$\left| \tfrac{T}{S(T)} - 1 \right| \le 4\,\varepsilon. \tag{2.13}$$

Then

$$\left| \int f\, d\mu_\eta - \int f_A\, d\mu_\gamma \right| \le \left| \tfrac{T}{S(T)} - 1 \right| \int |f_A|\, d\mu_\gamma \le 4\,\varepsilon \int |f_A|\, d\mu_\gamma. \tag{2.14}$$

If $\|f\|_\infty \le 1$, then

$$\begin{aligned}
\int |f_A|\, d\mu_\gamma &\le \int_{[|v|\le A]} |f|\, d\mu_\gamma + \int_{[|v|>A]} |f - f_A|\, d\mu_\gamma \\
&\le 1 + \int_{[|v|>A]} \frac{|v|}{A}\, d\mu_\gamma \le 1 + \varepsilon.
\end{aligned}$$

$$\begin{aligned}\left|\int f\,d\mu_\eta-\int f\,d\mu_\gamma\right| &= \left|\tfrac{T}{S(T)}\int f_A\,d\mu_\gamma-\int f\,d\mu_\gamma\right|\\ &\le \left|\tfrac{T}{S(T)}-1\right|\int |f_A|\,d\mu_\gamma+\int|f-f_A|\,d\mu_\gamma\\ &\le 4\,\varepsilon\,(1+\varepsilon)+\varepsilon\le 6\,\varepsilon.\end{aligned}$$

Also, using (2.14),

$$\left|\int |v|\,d\mu_\eta-\int |v|\,d\mu_\gamma\right| = \left|\tfrac{T}{S(T)}-1\right|\int |v|\,d\mu_\gamma\le 4\,\varepsilon\int |v|\,d\mu_\gamma.$$

Hence

$$d_{\mathcal{M}_\ell}(\mu_\eta,\mu_\gamma)\le 6\,\varepsilon\int(|v|+1)\,d\mu_\gamma. \tag{2.15}$$

Now let $\mu\in\overline{\mathcal{C}(M)}$. Let

$$K:=1+\int(|v|+1)\,d\mu. \tag{2.16}$$

For $R>0$, define

$$L_R(v):=\begin{cases}L(v) & \text{if } |v|\le R.\\ L\big(\frac{v}{|v|}R\big)\,\frac{|v|}{R} & \text{if } |v|>R.\end{cases}$$

Claim: If $E(v)>0$ for all $|v|\ge R$, then

$$L_R(v)\le L(v)\qquad\text{for all } v\in TM.$$

Proof:

If $|v|\le R$ then $L_R(v)=L(v)$. Suppose that $|v|=R$. For $s\ge 1$ let

$$f(s):=L(sv)-L_R(sv)=L(sv)-sL(v).$$

It is enough to prove that $f(s)\ge 0$ for all $s\ge 1$. We have

$$\begin{aligned}f'(s)&=v\cdot L_v(sv)-L(v)\\ f''(s)&=v\cdot L_{vv}(sv)\cdot v>0.\end{aligned}$$

We have that $f(1)=0$, $f'(1)=E(v)>0$, $f''(s)>0$ for all $s\geq 1$. This implies that $f(s)\geq 0$ for all $s\geq 1$.

$\diamondsuit$

Given $N>0$, choose $R=R(N)>1$ such that

$$E(v)>0 \quad \text{if } |v|\geq R \qquad \text{and} \qquad \int_{|v|>R} |v|\, d\mu < \tfrac{1}{N}. \tag{2.17}$$

Observe $L_{R(N)}$ has linear growth. Choose $\mu_{\gamma_N}\in\mathcal{C}(M)$ such that

$$d_{\mathcal{M}_\ell}(\mu_{\gamma_N},\mu) < \tfrac{1}{N}, \tag{2.18}$$
$$\int L_{R(N)}\, d\mu_{\gamma_N} \leq \int L_{R(N)}\, d\mu + \tfrac{1}{N},$$

and

$$\int_{[|v|\leq R(N)]} |v|\, d\mu_{\gamma_N} \geq \int_{[|v|\leq R(N)]} |v|\, d\mu - \tfrac{1}{N}. \tag{2.19}$$

Then

$$\int (|v|+1)\, d\mu_{\gamma_N} \leq K \qquad \text{from (2.16) and (2.18),} \tag{2.20}$$
$$\int_{[|v|>R(N)]} |v|\, d\mu_{\gamma_N} < \tfrac{3}{N} \qquad \text{from (2.17), (2.18) and (2.19).} \tag{2.21}$$

Construct η_N as above for γ_N and $A=R(N)$. Then from (2.11), (2.15), (2.20) and (2.21), $d_{\mathcal{M}_\ell}(\mu_{\eta_N},\mu_{\gamma_N}) < \frac{18}{N}K$. From (2.18),

$$d_{\mathcal{M}_\ell}(\mu_{\eta_N},\mu) < \tfrac{18}{N}K + \tfrac{1}{N}.$$

Thus $\mu_{\eta_N} \xrightarrow{N} \mu$ in $\overline{\mathcal{C}(M)}$. Moreover, from (2.12), (2.13), (2.11) and the claim,

$$\begin{aligned}\int L\, d\mu_{\eta_N} &= \tfrac{T_N}{S(T_N)} \int L_{R(N)}\, d\mu_{\gamma_N} \leq \tfrac{T_N}{S(T_N)} \left[\int L_{R(N)}\, d\mu + \tfrac{1}{N}\right] \\ &\leq \left(1+\tfrac{12}{N}\right) \left[\int L\, d\mu + \tfrac{1}{N}\right].\end{aligned}$$

Hence

$$\limsup_N \int L\, d\mu_{\eta_N} \le \int L\, d\mu.$$

Fix $R > 0$ such that $E > 0$ on $|v| > R$. Then L_R has linear growth and by the claim $L_R \le L$. Therefore

$$\liminf_N \int L\, d\mu_{\eta_N} \ge \lim_N \int L_R\, d\mu_{\eta_N} = \int L_R\, d\mu.$$

Letting $R \uparrow +\infty$, by the dominated convergence theorem we get that

$$\liminf_N \int L\, d\mu_{\eta_N} \ge \int L\, d\mu.$$

□

Given x, $y \in M$, define

$$S(x, y; T) := \inf_{\gamma \in \mathcal{C}_T(x,y)} A_L(\gamma).$$

Observe that $S(x, y; T) > -\infty$ because L is bounded below. If $\gamma \in \mathcal{C}^{ac}([0, T], M)$, define

$$S^+(\gamma) := A_L(\gamma) - S(\gamma(0), \gamma(T); T).$$

The absolutely continuous curves γ with $S^+(\gamma) = 0$ are called *Tonelli minimizers*. Observe that a Tonelli minimizer is a solution of (E-L). Given γ_1, $\gamma_2 \in \mathcal{C}^{ac}([0, T], M)$, the *absolutely continuous distance* $d_1(\gamma_1, \gamma_2)$ is defined by

$$d_1(\gamma_1, \gamma_2) := \sup_{t \in [0,T]} d\big(\gamma_1(t), \gamma_2(t)\big) + \int_0^T d_{TM}\big([\gamma_1(t), \dot\gamma_1(t)], [\gamma_2(t), \dot\gamma_2(t)]\big)\, dt.$$

2-4.4 Proposition. *Given a compact subset $K \subseteq M$ and given C, $\varepsilon > 0$ there exist $\delta > 0$ such that if $\gamma : [0, T] \to M$ is absolutely continuous and satisfies* *i.* $1 \le T \le C$.

ii. $A_L(\gamma) \le C$.

iii. $S^+(\gamma) \le \delta$.

Then either $\gamma([0,T]) \cap K = \varnothing$ *or there exists a Tonelli minimizer* $\gamma_0 : [0,T] \to M$ *such that* $d_1(\gamma_0, \gamma) \le \varepsilon$.

Proof: If such δ does not exists then there is a sequence $\gamma_n \in \mathcal{C}^{ac}([0,T_n],M)$ such that $\gamma_n([0,T_n]) \cap K \neq \varnothing$, $1 \le T_n \le C$, $S^+(\gamma_n) \to 0$, $A_L(\gamma_n) \le C$ and $d_1(\gamma_n, \eta) \ge \varepsilon$ for any Tonelli minimizer η.

Adding a constant we can assume that $L > 0$. Let $B > 0$ be such that $L(x,v) > |v| - B$ for all $(x,v) \in TM$. Choose $s_0 \in [0,T_n]$ such that $\gamma_n(s_0) \in K$. Then

$$\begin{aligned} d(K, \gamma_n(t)) \le d(\gamma_n(s_0), \gamma_n(t)) &\le \int_{[s_0,t]} |\dot\gamma_n| \\ &\le \int_{[s_0,t]} \big[\, L(\gamma_n, \dot\gamma_n) + B \,\big] \le C + B\,C. \end{aligned}$$

Let $Q := \{\, y \in M \mid d(y,K) \le C + B\,C \,\}$. Then we have that $\gamma_n([0,T_n]) \subseteq Q$.

We can assume that $T_n \to T$, $\gamma_n(0) \to x \in Q$ and $\gamma_n(T_n) \to y \in Q$. Moreover, we can assume that $T_n \equiv T$, $\gamma_n(0) \equiv x$ and $\gamma_n(T) \equiv y$. By theorem 3-1.2, the set $\mathcal{A}(b) = \big\{ \gamma \in \mathcal{C}_T(x,y) \mid A_L(\gamma) \le b \big\}$ is compact in the C^0-topology. Then we can assume that there is $\gamma_0 \in \mathcal{C}_T(x,y)$ such that $\gamma_n \to \gamma_0$ in the C^0-topology. Since the action functional is lower semicontinuous, then $A_L(\gamma_0) \le \liminf_n A_L(\gamma_n) = S(x,y;T)$, because $S^+(\gamma_n) \to 0$. Thus γ_0 is a Tonelli minimizer. Moreover, we have that $A_L(\gamma_n) \to A_L(\gamma_0)$. By proposition 3-1.3, $\gamma_n \to \gamma_0$ in the d_1-topology.

□

Let

$$\mathcal{H} := \Big\{\, h : TM \to \mathbb{R} \,\Big|\, \|f\|_\infty \le 1,\ [h]_{\text{Lip}} \le 1,\ h \text{ with compact support} \,\Big\},$$

where

$$[h]_{\mathrm{Lip}} = \sup_{(x,v)\neq(y,w)} \frac{|h(x,v)-h(y,w)|}{d_{TM}\big((x,v),(y,w)\big)}$$

is the smallest Lipschitz constant for h.

2-4.5 Corollary.

Given $h \in \mathcal{H}$ and $C > 0$ there exist $\delta = \delta(C,h) > 0$ such that if $\gamma : [0,T] \to M$ satisfies conditions 2-4.4.i, 2-4.4.ii, 2-4.4.iii then

$$\left| \oint_\gamma h - \oint_\gamma h \circ \varphi_1 \right| \leq 5. \tag{2.22}$$

Proof: Let $K = \pi\big(\operatorname{supp}(h) \cup \varphi_{-1}(\operatorname{supp}(h))\big)$. Given $C > 0$ and $\varepsilon > 0$ let $\delta = \delta(C,\varepsilon) > 0$ and $A > 0$ be given by proposition 2-4.4 then if $\gamma : [0,T] \to M$ satisfies conditions 2-4.4.i, 2-4.4.ii, 2-4.4.iii we have that either $\gamma([0,T]) \cap K = \varnothing$, or we can take γ_0 minimizing such that $d_1(\gamma_0,\gamma) \leq \varepsilon$.

Observe that if $\gamma([0,T]) \cap K = \varnothing$, then $h(\gamma,\dot\gamma) \equiv 0$ and $h\circ\varphi_1(\gamma,\dot\gamma) \equiv 0$. This implies (2.22). Suppose then that $d_1(\gamma_0,\gamma) \leq \varepsilon$.

We have that

$$\left| \oint_\gamma h - \oint_{\gamma_0} h \right| \leq [h]_{\mathrm{Lip}}\, d_1(\gamma,\gamma_0) \leq 1 \cdot 1 \cdot \varepsilon,$$

where $[h]_{\mathrm{Lip}}$ is the smallest Lipschitz constant of h. Let $Q(h) := \varphi_{-1}(\operatorname{supp}(h))$, then

$$\left| \oint_\gamma h \circ \varphi_1 - \oint_{\gamma_0} h \circ \varphi_1 \right| \leq [h]_{\mathrm{Lip}}\, [\varphi_1|_{Q(h)}]_{\mathrm{Lip}}\, d_1(\gamma,\gamma_0) \leq 1 \cdot [\varphi_1|_{Q(h)}]_{\mathrm{Lip}} \cdot \varepsilon.$$

Since γ_0 is a solution of (E-L), we have that

$$\begin{aligned} \left| \oint_{\gamma_0} h - \oint_{\gamma_0} h \circ \varphi_1 \right| &= \left| \int_0^T h\big(\gamma_0(t),\dot\gamma_0(t)\big) - h\big(\gamma_0(t+1),\dot\gamma_0(t+1)\big)\, dt \right| \\ &\leq \int_0^1 |h(\gamma_0,\dot\gamma_0)|\, dt + \int_T^{T+1} |h(\gamma_0,\dot\gamma_0)|\, dt \leq 2. \end{aligned}$$

Hence
$$\left|\oint_\gamma h-\oint_\gamma h\circ\varphi_1\right|\le\varepsilon\big(1+[\varphi_1|_{Q(h)}]_{\rm Lip}\big)+2.$$
□

Proof of item 2-4.1.2:

Observe that to prove that μ is invariant it is enough to prove that
$$\int h\,d\mu=\int h\,d(\varphi_1^*\,\mu)\qquad\text{for all } h\in\mathcal{H}. \tag{2.23}$$

By proposition 2-4.2, there exists a sequence $\mu_{\gamma_n}\in\mathcal{C}(M)$ such that $\mu_{\gamma_n}\to\mu$ and
$$\lim_n A_L(\mu_{\gamma_n})=A_L(\mu)=\min\big\{\,A_L(\nu)\,|\,\nu\in\overline{\mathcal{C}(M)}\,\big\}=:k. \tag{2.24}$$

Let T_n be a period of the curve $\gamma_n:\mathbb{R}\to M$. Take an integer $N>0$. By joining a constant curve if necessary, we can assume that every T_n is a multiple of N and that $\lim_{n\to\infty}T_n=+\infty$. Given $C>0$ let
$$\mathcal{B}_n(C):=\big\{\,j\in\mathbb{N}\,|\,1\le j\le\tfrac{T_n}{N},\ A_L(\gamma_{n,j})\ge C\,\big\},$$
where
$$\gamma_{n,j}:=\gamma_n|_{[jN,(j+1)N]}.$$
By the superlinearity L is bounded below, adding a constant we can assume that $L>0$. Then we can assume that
$$A_L(\mu_{\gamma_n})=\frac{1}{T_n}\int_0^{T_n}L(\gamma_n,\dot\gamma_n)\,dt\le 2\,k\qquad\forall n.$$
Hence
$$2\,k\,T_n\ge\sum_{j\in\mathcal{B}_n(C)}A_L(\gamma_{n,j})\ge C\,\#\mathcal{B}_n(C).$$
Thus
$$\frac{\#\mathcal{B}_n(C)}{T_n}\le\frac{2k}{C}. \tag{2.25}$$

Given $\delta > 0$, let

$$\mathcal{B}'_n(\delta) := \big\{\, j \in \mathbb{N} \,|\, 1 \le j \le \tfrac{T_n}{N} - 1, \;\; S^+(\gamma_{n,j}) > \delta \,\big\}.$$

Then

$$S^+(\gamma_n) \ge \sum_{j=1}^{(T_n/N)-1} S^+(\gamma_{n,j}) \ge \delta \; \#\mathcal{B}'_n(\delta).$$

Moreover,

$$k \le \frac{1}{T_n} \, S(\gamma_n(0), \gamma(T_n); T_n) = A_L(\mu_{\gamma_n}) - \frac{1}{T_n} \, S^+(\gamma_n).$$

Hence

$$S^+(\gamma_n) \le T_n \big(A_L(\mu_{\gamma_n}) - k \big).$$

Therefore

$$\frac{\#\mathcal{B}'_n(\delta)}{T_n} \le \frac{1}{\delta} \, \big(A_L(\mu_{\gamma_n}) - k \big). \tag{2.26}$$

Now fix $h \in \mathcal{H}$. Then

$$\left| \int h \; d\mu_{\gamma_n} - \int h \; d(\varphi_1^* \, \mu_{\gamma_n}) \right| \le \frac{1}{T_n} \sum_{j=0}^{(T_n/N)-1} \left| \oint_{\gamma_{n,j}} h - \oint_{\gamma_{n,j}} h \circ \varphi_1 \right|.$$

Denote $\mathcal{B}''_n := \mathcal{B}_n(C) \cup \mathcal{B}'_n(\delta)$. Since $\sup |h| \le 1$, then

$$\left| \int h \; d\mu_{\gamma_n} - \int h \; d(\varphi_1^* \, \mu_{\gamma_n}) \right| \le \frac{1}{T_n} \sum_{j \notin \mathcal{B}''_n} \left| \oint_{\gamma_{n,j}} h - \oint_{\gamma_{n,j}} h \circ \varphi_1 \right| + \frac{1}{T_n} \, 2N \, \#\mathcal{B}''_n.$$

Now choose $C \ge N^2$ and $\delta = \delta(C, h) > 0$ from corollary 2-4.5. Using equations (2.25), (2.26) and corollary 2-4.5 we obtain that

$$\begin{aligned} \left| \int h \; d\mu_{\gamma_n} - \int h \; d(\varphi_1^* \, \mu_{\gamma_n}) \right| &\le \frac{5}{T_n} \, \left(\frac{T_n}{N} - \#\mathcal{B}''_n \right) + \frac{1}{T_n} \, 2N \, \#\mathcal{B}''_n \\ &\le \frac{5}{N} + 2N \left(\frac{2k}{C} + \frac{1}{\delta} \Big(A_L(\mu_{\gamma_n}) - k \Big) \right). \end{aligned}$$

Now let $n \to \infty$. Using equation (2.24) and that $C \geq N^2$, $\mu_{\gamma_n} \to \mu$ and h, $h \circ \varphi_1 \in C^0_\ell$ (because they have compact support), we obtain that

$$\left| \int h \, d\mu - \int h \, d(\varphi_1^* \, \mu) \right| \leq \frac{5}{N} + \frac{4k}{N}.$$

Since N is arbitrary, this difference is zero and we get (2.23). $\square$

2-5 Ergodic characterization of the critical value.

Given a Borel probability measure μ in TM define its *action* by

$$A_L(\mu) = \int_{TM} L \, d\mu.$$

Since by the superlinearity the lagrangian L is bounded below, this action is well defined.

Let $\mathcal{M}(L)$ be the set of φ_t-invariant probabilities on TM.

2-5.1 Theorem (Mañé [35]). *If M is compact, then*

$$c(L) = -\min\{\, A_L(\mu) \,|\, \mu \in \mathcal{M}(L) \,\}.$$

We will obtain theorem 2-5.1 from theorem 2-5.2 below, which also applies to the non-compact case. If M is non-compact, theorem 2-5.1 may not hold, as seen in example 5-7.

Recall that if $\gamma : [0, T] \to M$ is a closed absolutely continuous curve, the measure $\mu_\gamma \in \mathcal{M}_\ell$ is defined by

$$\int f \, d\mu_\gamma = \frac{1}{T} \int_0^T f\big(\gamma(t), \dot\gamma(t)\big) \, dt$$

for all $f \in C^0_\ell$, and that $\overline{\mathcal{C}(M)}$ is the closure of the set of such μ_γ's in $\mathcal{M}_\ell$.

2-5.2 Theorem.

$$\begin{aligned} c(L) &= -\inf\{\, A_L(\mu) \,|\, \mu \in \overline{\mathcal{C}(M)} \,\} \\ &= -\inf\{\, A_L(\mu) \,|\, \mu \in \mathcal{C}(M) \,\}. \end{aligned} \tag{2.27}$$

2-5.3 Definition. We say that a holonomic measure $\mu \in \mathcal{C}(M)$ is (globally) *minimizing* if $A_L(\mu) = -c(L)$.

2-5.4 Remarks.

1. Recall that by theorem 2-4.1, if a minimizing measure exists, then it is invariant under the lagrangian flow.

2. The equality between the two infima in theorem 2-5.2 is non-trivial and follows from proposition 2-4.2.

3. Theorems 2-5.2 and 2-4.1 imply theorem 2-5.1.

4. If $p : N \to M$ is a covering, M is compact and $\mathbb{L} = L \circ dp$ is the lifted lagrangian, then theorems 2-5.2 and 2-4.1 imply that

$$c(\mathbb{L}) = -\min\{\, A_L(\mu) \,|\, \mu \in \mathcal{M}(L) \cap \overline{dp_*\mathcal{C}(N)}\,\}, \qquad (2.28)$$

   by noticing that $A_L(dp_*\,\nu) = A_{\mathbb{L}}(\nu)$ for $\nu \in \overline{\mathcal{C}(N)}$. Here $dp_*\mathcal{C}(N)$ is the set of probabilities μ_γ on TM where γ is a curve on M whose lifts to N are closed. The compactness property on theorem 2-4.1(3) allows to obtain a minimum on (2.28) instead of the infimum on (2.27) which may not be attained in the non-compact case.

5. The statement for coverings in equation (2.28) allows to obtain minimizing measures which don't appear in the Mather's theory. For example if c_u is the critical value of the universal cover $\widetilde{M}$ of M and c_0 is the critical value of the abelian cover $\widehat{M}$ of M; the minimizing measures on a fixed homology class (corresponding to Mather's theory) all have action $A(\mu) \geq -c_0$, (see equation (2.31) and proposition 2-7.3), while the minimizing measures for $\widetilde{M}$ have action $c_u < c_0$.

   The measures for $\widetilde{M}$ correspond to "minimizing in the zero homotopy class" while the measures for $\widehat{M}$ are minimizing in the zero homology class.

   The drawback of this approach is that we obtain honest minimizing invariant measures on TM which may not lift to finite measures on the covering TN.

Proof of theorem 2-5.2:

If $\mu_\gamma \in \mathcal{C}(M)$, then $A_{L+c(L)}(\mu_\gamma) \geq 0$. Hence $A_L(\mu_\gamma) \geq -c(L)$. Thus

$$-c(L) \leq \inf\{\, A_L(\mu) \,|\, \mu \in \mathcal{C}(M) \,\} = \inf\{\, A_L(\mu) \,|\, \mu \in \overline{\mathcal{C}(M)} \,\},$$

where the last equality follows from proposition 2-4.2.

If $k < c(L)$ then there is a closed absolutely continuous curve γ on M such that $A_{L+k}(\gamma) < 0$. Thus $\mu_\gamma \in \mathcal{C}(M)$ and

$$-k > A_L(\mu_\gamma) \geq \inf\{\, A_L(\mu) \,|\, \mu \in \overline{\mathcal{C}(M)}\}.$$

Now let $k \uparrow c(L)$. $\square$

2-6 The Aubry-Mather Theory.

Through this section we shall assume that M is compact.

2-6.a Homology of measures.

Observe that since M is compact, any 1-form is in C^0_ℓ. By definition, an holonomic probability $\mu \in \overline{\mathcal{C}(M)}$ satisfies $\int_{TM} |v|\, d\mu < +\infty$ and

$$\int_{TM} df\ d\mu = 0 \qquad \text{for all } f \in C^\infty(M,\mathbb{R}).$$

Then we can define its *homology class* as $\rho(\mu) \in H_1(M,\mathbb{R}) \approx H^1(M,\mathbb{R})^*$ by

$$\langle\, \rho(\mu)\,, [\omega]\,\rangle = \int_{TM} \omega\ d\mu\,, \tag{2.29}$$

for any closed 1-form ω on M, where $[\omega] \in H^1(M,\mathbb{R})$ is the cohomology class of ω. Here we have used the identification[1] $H_1(M,\mathbb{R}) \approx H^1(M,\mathbb{R})^*$ and equation (2.29) shows how the homology class $\rho(\mu)$ acts on $H^1(M,\mathbb{R})$. Since μ is holonomic, the integral in (2.29) depends only on the cohomology class of ω. The class $\rho(\mu)$ is called the *homology* of μ or the *rotation* of μ by analogy to the twist map theory.

Using a finite basis $\{\,[\omega_1],\ldots,[\omega_k]\,\}$ for $H^1(M,\mathbb{R})$ and the topology of $\overline{\mathcal{C}(M)}$, we have that

2-6.1 Lemma. *The map $\rho : \overline{\mathcal{C}(M)} \to H_1(M,\mathbb{R})$ is continuous.*

2-6.b The asymptotic cycle.

Given a differentiable flow φ_t on a compact manifold N and a φ_t-invariant probability μ, the Schwartzman's [65] *asymptotic cycle* of an

[1] In fact, $H^1(M,\mathbb{R}) \approx \hom\big(H_1(M,\mathbb{R}),\mathbb{R}\big) = H_1(M,\mathbb{R})^*$ by the universal coefficient theorem. Since M is compact, then $H_1(M,\mathbb{R})$ is a finite dimensional vector space and hence it is naturally isomorphic to its double dual $H^1(M,\mathbb{R})^*$.

invariant probability μ is defined to be the homology class $A(\mu) \in H_1(N,\mathbb{R}) \approx H^1(N,\mathbb{R})^*$ such that

$$\langle\, A(\mu)\,,[\omega]\,\rangle = \int_N \omega(X)\ \ d\mu\,,$$

for any closed 1-form ω, where $[\omega] \in H^1(N,\mathbb{R})$ is the cohomology class of ω and X is the vector field of φ_t. This integral depends only on the cohomology class of ω because the integral of a coboundary by an invariant measure is zero: in fact, if *df* is an exact 1-form, then define

$$F(y) := \lim_{T\to+\infty} \frac{1}{T}\int_0^T df\big(X(\varphi_t\, y)\big)\ dt = \lim_{T\to+\infty} \tfrac{1}{T}\ \big[\, f(\varphi_t\, y) - f(y)\,\big] = 0,$$

by Birkhoff's theorem,

$$\int_N df(X)\ d\mu = \int_N F\ d\mu = 0.$$

If μ is ergodic and $x \in N$ is a *generic point* [2] for μ, then

$$\langle\, A(\mu)\,,[\omega]\,\rangle = \lim_{T\to+\infty} \frac{1}{T}\int_0^T \omega\big(X(\varphi_t\, x)\big)\ dt\,.$$

Applying this to a basis $\{\omega_1,\dots,\omega_k\}$ for $H^1(N,\mathbb{R})$, we get that

$$A(\mu) = \lim_{T\to+\infty} \tfrac{1}{T}\ [\,\gamma_T * \delta_T\,] \in H_1(N,\mathbb{R}),$$

where $\gamma_T(t) = \varphi_t(x)$, $t \in [0,T]$, the curve δ_T is a unit speed geodesic from $\varphi_T(x)$ to x, and the limit is on the finite dimensional vector space $H_1(N,\mathbb{R})$.

In the case of a lagrangian flow, the phase space $N = TM$ is not compact, but it has the same homotopy type as the configuration space M because M is a deformation retract of TM (contracting TM along the

[2] i.e. $\lim_{T\to+\infty} \frac{1}{T}\int_0^T f(\varphi_t\, x)\ dt = \int f\ d\mu$ for all $f \in C^0(N,\mathbb{R})$.

fibers to the zero section $M \times 0$). Moreover, the ergodic components of an invariant measure of a lagrangian flow are contained in a unique energy level, which is a compact submanifold of TM by remark 1-3.1.

We see that the homology of an invariant probability and its asymptotic cycle coincide under the identification $H_1(TM,\mathbb{R}) \overset{\pi_*}{\approx} H_1(M,\mathbb{R})$.

2-6.2 Proposition.

$$\pi_*\big(A(\mu)\big) = \rho(\mu) \qquad \textit{for all } \mu \in \mathcal{M}(L),$$

where $\pi_* : H_1(TM,\mathbb{R}) \to H_1(M,\mathbb{R})$ *is the map induced by the projection* $TM \xrightarrow{\pi} M$.

Proof: If ω is a closed 1-form on M, then

$$(\pi^* \omega)(X(x,v)) = \omega\big[d\pi\big(X(x,v)\big)\big] = \omega_x(v)\,,$$

because the lagrangian vector field X has the form $X(x,v) = (v,*)$. Then

$$\begin{aligned}\big\langle\, \pi_*\, A(\mu)\,,[\omega]\,\big\rangle = \big\langle\, A(\mu)\,,\pi^*\,[\omega]\,\big\rangle &= \int_{TM} (\pi^*\,\omega)(X)\;\; d\mu \\ &= \int_{TM} \omega\; d\mu = \big\langle\, \rho(\mu)\,,[\omega]\,\big\rangle\,.\end{aligned}$$

□

2-6.3 Lemma. *The map* $\rho : \mathcal{M}(L) \to H_1(M,\mathbb{R})$ *is surjective.*

Proof: Let $h \in H_1(M,\mathbb{Z})$ be an integer homology class. Let $\eta : [0,1] \to M$ be a closed curve with homology class h. Let γ be a minimizer of the action of L among the set of absolutely continuous curves $[0,1] \to M$ with the same homotopy class as η. Then by remark 1-2.2, γ is a periodic orbit for the lagrangian flow with period 1. The invariant measure μ_γ satisfies $\rho(\mu_\gamma) = h$.

The map ρ is affine and $\mathcal{M}(L)$ is convex; hence $\rho\big(\mathcal{M}(L)\big)$ is convex and, in particular, it contains the convex hull of $H_1(M,\mathbb{Z})$. Thus, $H_1(M,\mathbb{R}) \subseteq \rho\big(\mathcal{M}(L)\big)$. □

2-6.c The alpha and beta functions.

The action functional $A_L : \mathcal{M}(L) \to \mathbb{R}$ is lower semicontinuous[3] and the sets

$$\mathcal{M}(h) := \{\, \mu \in \mathcal{M}(L) \,|\, \rho(\mu) = h \,\}$$

are closed. Hence we can define the *Mather's beta function* $\beta : H_1(M,\mathbb{R}) \to \mathbb{R}$, as

$$\beta(h) := \min_{\mu\in\mathcal{M}(h)} A_L(\mu).$$

We shall prove below that the β-function is convex. The *Mather's alpha function* $\alpha = \beta^* : H^1(M,\mathbb{R}) \to \mathbb{R}$ is the convex dual of the β-function:

$$\begin{aligned}\alpha([\omega]) &= \max_{h\in H_1(M,\mathbb{R})} \{\, \langle [\omega]\,, h\rangle - \beta(h) \,\} \\ &= -\min_{\mu\in\mathcal{M}(L)} \{\, A_L(\mu) - \langle [\omega]\,, \rho(\mu)\rangle \,\} \qquad \text{using 2-6.3,} \\ &= -\min_{\mu\in\mathcal{M}(L)} A_{L-\omega}(\mu) \\ &= c(L-\omega)\,, \qquad \text{by 2-5.1.} \end{aligned} \tag{2.30}$$

Observe that since $L - \omega$ is also a convex superlinear lagrangian, then $\alpha([\omega])$ is finite.

2-6.4 Theorem. *The α and β functions are convex and superlinear.*

Proof: We first prove that β is convex. Let $h_1,\, h_2 \in H_1(M,\mathbb{R})$ and $0 \le \lambda \le 1$. Let $\mu_1,\, \mu_2 \in \mathcal{M}(L)$ be such that $\rho(\mu_i) = h_i$ and $A_L(\mu_i) = \beta(h_i)$ for $i = 1, 2$. The probability $\nu = \lambda\,\mu_1 + (1-\lambda)\,\mu_2$ satisfies $\rho(\nu) = \lambda\, h_1 + (1-\lambda)\, h_2$. Hence

$$\beta\big(\lambda\, h_1 + (1-\lambda)\, h_2\big) \le A_L\big(\lambda\,\mu_1 + (1-\lambda)\,\mu_2\big) = \lambda\,\beta(h_1) + (1-\lambda)\,\beta(h_2).$$

By proposition 2-6.3, ρ is surjective, and hence β is finite. By proposition D.1 on the appendix, α is superlinear. By D.1, α and β are

[3] A_L is lower semicontinuous iff $\liminf_n A_L(\nu_n) \ge A_L(\mu)$ when $\nu_n \to \mu$.

convex. Formula (2.30), implies that α is finite and then by D.1, β is superlinear. $\square$

For $h \in H_1(M,\mathbb{R})$ and $\omega \in H^1(M,\mathbb{R})$, write

$$\mathcal{M}_h(L) := \{\, \mu \in \mathcal{M}(L) \mid \rho(\mu) = h,\ A_L(\mu) = \beta(h)\,\},$$
$$\mathcal{M}^\omega(L) := \{\, \mu \in \mathcal{M}(L) \mid A_{L-\omega}(\mu) = -c(L-\omega)\,\}.$$

Since the β-function has a supporting hyperplane at each homology class h, if $\omega \in \partial\beta(h)$, then $\mathcal{M}_h(L) \subseteq \mathcal{M}^\omega(L)$. Conversely, since by corollary D.2 $\alpha^* = \beta$, then $\mathcal{M}^\omega(L) \subseteq \mathcal{M}_h(L)$ if $h \in \partial\alpha(\omega)$. Thus

$$\bigcup_{h\in H_1(M,\mathbb{R})} \mathcal{M}_h(L) \;=\; \bigcup_{\omega\in H^1(M,\mathbb{R})} \mathcal{M}^\omega(L).$$

We call these measures *Mather minimizing* measures and the set

$$\mathcal{M} := \mathcal{M}^0(L) = \{\mu \in \mathcal{M}(L) \,|\, A_L(\mu) = -c(L)\,\},$$

the *Mather set*.

Define the *strict critical value* as

$$\begin{aligned} c_0(L) :&= \min_{\omega\in H^1(M,\mathbb{R})} c(L-\omega) = \min_{\omega\in H^1(M,\mathbb{R})} \alpha(\omega) \\ &= -\beta(0). \end{aligned} \tag{2.31}$$

By corollary 3-6.3 the strict critical value is the lowest energy level which supports Mather minimizing measures and since $c_0(L) = -\beta(0)$, these minimal energy Mather minimizing measures have trivial homology.

2-7 Coverings.

We shall deal mainly with compact manifolds M, but there are some important non-compact cases, for example the coverings of M. Particularly interesting are the abelian cover $\widehat{M}$, the universal cover $\widetilde{M}$ and the finite coverings.

The *abelian cover* $\widehat{M}$ of M is the covering whose fundamental group is the kernel of the Hurewicz homomorphism $\pi_1(M) \to H_1(M, \mathbb{Z})$. Its deck transformation group is $H_1(M, \mathbb{Z})$ and $H^1(\widetilde{M}, \mathbb{Z}) = \{0\}$. When $\pi_1(M)$ is abelian, $\widehat{M} = \widetilde{M}$. A closed curve in $\widehat{M}$ projects to a closed curve in M with trivial homology.

If $M_1 \xrightarrow{p} M$ is a covering, denote by $L_1 := L \circ dp : TM_1 \to \mathbb{R}$ the lifted lagrangian to TM_1.

2-7.1 Lemma. *If $M_1 \xrightarrow{p} M$ is a covering, then $c(L_1) \leq c(L)$.*

Proof: The lemma follows form the fact that closed curves on N project to closed curves on M. □

2-7.2 Proposition. *If M_1 is a finite covering of M_2 then $c(L_1) = c(L_2)$.*

Proof: We know that $c(L_1) \leq c(L_2)$. Suppose that the strict inequality holds and let k be such that $c(L_1) < k < c(L_2)$. Hence there exists a closed curve γ in M_2 with negative $(L_2 + k)$-action. Since M_1 is a finite covering of M_2 some iterate of γ lifts to a closed curve in M_1 with negative $(L_1 + k)$-action which contradicts $c(L_1) < k$. □

2-7.3 Proposition. [60]

$$c_0(L) = c_a(L) = \textit{critical value of the abelian cover.}$$

Then we have

$$c_u(L) \leq c_a(L) = c_0(L) \leq c(L - \omega) \qquad \forall [\omega] \in H^1(M, \mathbb{R}),$$

where c_u is the critical value of the lift of the lagrangian to the universal cover. When $c_u(L) < c_0(L)$, the method in equation (2.28) gives some minimizing measures which are not Mather minimizing. For symmetric lagrangians $c(L) = e_0 = c_0(L) = c_u(L)$. Mañé [39] gives an example in which $e_0 < c_a(L) = c_0(L) < c(L)$. G. Paternain and M. Paternain [60] give an example in which $c_u(L) < c_a(L)$.

Proof: Let ω be a closed form in M. Since $H_1(M, \mathbb{R}) = \{0\}$, the lift $\widehat{\omega}$ of ω to $\widehat{M}$ is exact, then

$$c_a(L) := c(\widehat{L}) = c(\widehat{L} - \widehat{\omega}) \le c(L - \omega).$$

Hence

$$c_a(L) \le \min_{\omega \in H^1(M,\mathbb{R})} c(L - \omega) = c_0(L).$$

Moreover,

$$\begin{aligned}
-c_a(L) &= \inf \left\{ A_{\widehat{L}}(\mu) \,|\, \mu \in \overline{\mathcal{C}(\widehat{M})} \right\} \\
&= \inf\{ A_{\widehat{L}}(\mu_\gamma) \,|\, \mu_\gamma \in \mathcal{C}(\widehat{M}) \} && \text{by proposition 2-4.2,} \\
&= \inf\{ A_L(\mu_\gamma) \,|\, \mu_\gamma \in \mathcal{C}(M),\ \rho(\mu) = 0 \},
\end{aligned}$$

because a closed curve γ on M has homology $[\gamma] = 0$ if and only if it has a closed lift to $\widehat{M}$. Then

$$\begin{aligned}
-c_a(L) &= \inf\{ A_L(\mu) \,|\, \mu \in \overline{\mathcal{C}(M)},\ \rho(\mu) = 0 \}, && \text{by 2-4.2 and 2-6.1} \\
&\le \min\{ A_L(\mu) \,|\, \mu \in \mathcal{M}(L),\ \rho(\mu) = 0 \} && \text{because } \mathcal{M}(L) \subset \overline{\mathcal{C}(M)} \\
&= -\beta(0) = c_0(L).
\end{aligned}$$

□

The *real abelian cover* is the covering $\check{M}$ of M with $h : \pi(M) \to H_1(M, \mathbb{R})$ is the Hurewicz homomorphism. It is an intermediate covering $\widetilde{M} \to \check{M} \to M$ and the deck transformations of $\widehat{M} \to \check{M}$ are given by the torsion[4] of $H_1(M, \mathbb{Z})$. Hence $\widehat{M} \to \check{M}$ is a finite cover so that they have the same critical value $c_a(L) = c_0(L)$.

[4] i.e. the elements of finite order $\mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k} \subseteq H_1(M, \mathbb{Z})$.

Chapter 3

Globally minimizing orbits.

3-1 Tonelli's theorem.

Given x, $y \in M$ and $T > 0$, let

$$\mathcal{C}_T(x,y) := \big\{\, \gamma \in \mathcal{C}^{ac}([0,T],M) \mid \gamma(0)=x,\ \gamma(T)=y \,\big\}.$$

We say that $\gamma \in \mathcal{C}_T(x,y)$ is a *Tonelli minimizer* if

$$A_L(\gamma) = \min_{\eta \in \mathcal{C}_T(x,y)} A_L(\eta)\,.$$

In this section we shall prove

3-1.1 Tonelli's Theorem.
For all x, $y \in M$ and $T > 0$ there exists a Tonelli minimizer on $\mathcal{C}_T(x,y)$.

The only difference in the proof of this theorem when M is noncompact is corollary 3-1.8. An independent proof of this corollary is given in remark 3-1.9.

The idea of Tonelli's theorem is to prove that the sets

$$\mathcal{A}(c) := \big\{\, \gamma \in \mathcal{C}_T(x,y) \mid A_L(\gamma) \le c \,\big\} \tag{3.1}$$

are compact in the C^0-topology. Then a Tonelli minimizer will be a curve in

$$\bigcap_{c\geq\alpha}\mathcal{A}(c)\neq\varnothing\,,$$

where $\alpha=\inf_{\eta\in\mathcal{C}_T(x,y)}A_L(\eta)\geq\inf L>-\infty$.

An addendum to Tonelli's theorem due to Mather [46] states that these sets are compact in the topology of absolutely continuous curves. Given γ_1, $\gamma_2\in\mathcal{C}^{ac}([0,T],M)$ define their *absolutely continuous distance* by

$$d_1(\gamma_1,\gamma_2):=\sup_{t\in[0,T]}d_M\big(\gamma_1(t),\gamma_2(t)\big)+\int_0^T d_{TM}\big([\gamma_1(t),\dot\gamma_1(t)],[\gamma_2(t),\dot\gamma_2(t)]\big)\,dt.$$

3-1.2 Theorem (Mather [46]). *For any x, $y\in M$, $T>0$, $b\in\mathbb{R}$, the set*

$$\mathcal{A}(b):=\big\{\,\gamma\in\mathcal{C}_T(x,y)\,|\,A_L(\gamma)\leq b\,\big\}$$

is compact in the C^0-topology.

This theorem is proved in 3-1.12. We quote here the following proposition (addendum on page 175 of Mather [46]).

3-1.3 Proposition. (Mather [46])

If $N\subseteq M$ is a compact subset and γ_1, γ_2, ... is a sequence in $\mathcal{C}^{ac}([a,b],N)$ which converges C^0 to γ and $A_L(\gamma_i)$ converges to $A_L(\gamma)$, then γ_1, $\gamma_2,\ldots$ converges in the d_1-topology to γ.

In fact, the set $\mathcal{A}(b)$ in proposition 3-1.2 is not compact in the d_1-topology as the following example shows. Since the action functional is only lower semicontinuous on $\mathcal{C}_T(x,y)$ a priori it is not possible to ensure the convergence of the action in a C^0-convergent sequence obtained from theorem 3-1.2. Unless, for example, if the action is converging to the minimal action (e.g. proposition 2-4.4).

3-1.4 Example. *$\mathcal{A}(b)$ is not compact in the d_1-topology.*

Let $L = \frac{1}{2}|v|^2$ be the riemannian lagrangian on $\mathbb{R}^2$ with the flat metric. Let $\eta(t) = (t, 0)$ and $\gamma_n(t) = (t, \frac{1}{n}\sin(2\pi n t))$, $t \in [0,1]$. The action $A_L(\gamma_n) = 1 + 2\pi^2$ is bounded. $\gamma_n \to \eta$ in the C^0-topology.

The length of γ_n is bounded below by a polygonal curve joining the maxima and minimums of its second component. Hence $\ell(\gamma_n) \geq 2n\sqrt{4/n^2 + 1/(4n^2)} > 4$. Therefore $\ell(\gamma_n) \not\to \ell(\eta)$, and thus $\gamma_n \not\to \eta$ in the d_1-topology. Moreover, since a reparametrization preserves length, there is no reparametrization of the γ_n's which converges in the d_1 topology to η.

We shall split the proof of Tonelli's theorem in several parts:

3-1.5 Definition.

A family $\mathcal{F} \subseteq C^0([a,b], M)$ is *absolutely equicontinuous* if $\forall\, \varepsilon > 0$ $\exists\, \delta > 0$ such that

$$\sum_{i=1}^{N} |t_i - s_i| < \delta \quad \Longrightarrow \quad \sum_{i=1}^{N} d(x_{s_i}, x_{t_i}) < \varepsilon,$$

whenever $]s_1, t_1[, \ldots,]s_N, t_N[$ are disjoint intervals in $[a,b]$.

3-1.6 Remark.

(i) An absolutely equicontinuous family is equicontinuous.

(ii) A uniform limit of absolutely equicontinuous functions is absolutely continuous.

3-1.7 Lemma. *For all $c \in \mathbb{R}$ and $T > 0$, the family*

$$\mathcal{F}(c) := \left\{\, \gamma \in \mathcal{C}^{ac}([0,T], M) \mid A_L(\gamma) \leq c \,\right\}$$

is absolutely equicontinuous.

Proof: Since by the superlinearity, the lagrangian L is bounded below; by adding a constant we may assume that $L \geq 0$. For $a > 0$ let

$$K(a) := \inf \Big\{ \frac{L(x,v)}{|v|} \;\Big|\; (x,v) \in TM, \, |v| \geq a \Big\}. \tag{3.2}$$

The superlinearity implies that $\lim_{a\to+\infty} K(a) = +\infty$. Given $\varepsilon > 0$ let $a > 0$ be such that

$$\frac{c}{K(a)} < \frac{\varepsilon}{2}.$$

Let $0 \leq s_1 < t_1 \leq \cdots \leq s_N < t_N \leq T$, $J := \cup_{i=1}^N [s_i, t_i]$ and $E := J \cap \big[|\dot{x}| > a\big]$, then $L(x_s, \dot{x}_s) \geq K(a)\,|\dot{x}_s|$ for $s \in E$. We have that

$$\begin{aligned}
K(a) \sum_{i=1}^N d(x_s, x_t) &\leq K(a) \int_E |\dot{x}| + K(a) \int_{J\setminus E} |\dot{x}| \\
&\leq \int_E L(x, \dot{x}) + a \cdot K(a)\, m(J) \\
&\leq c + a \cdot K(a)\, m(J), \qquad \text{(because } L \geq 0\text{)},
\end{aligned}$$

where m is the Lebesgue measure on $[0,T]$

$$\sum_{i=1}^N d(x_{s_i}, x_{t_i}) \leq \textstyle\int_J |\dot{x}_s| \leq \frac{c}{K(a)} + a\, m(J) \leq \frac{\varepsilon}{2} + a\, m(J). \tag{3.3}$$

This implies the absolute equicontinuity of $\mathcal{F}(c)$.

□

In order to apply the Arzela-Ascoli theorem we need a compact range, for this we have:

3-1.8 Corollary. *For all $c \in \mathbb{R}$ and $T > 0$ there is $R > 0$ such that for all x, $y \in M$,*

$$\mathcal{A}(c) \subseteq \mathcal{C}^{ac}\big([0,T], \overline{B}(x,R)\big),$$

where $\overline{B}(x,R) := \{\, z \in M \mid d_M(x,z) \leq R \,\}$.

Proof: Inequality (3.3) for $N = 1$ and $J = [s,t]$ is $d(x_s, x_t) \le \frac{\varepsilon}{2} + a\,|t-s|$. It is enough to take $R = \frac{\varepsilon}{2} + a\,T$. $\square$

3-1.9 Remark.

Corollary 3-1.8 is the only difference for the proof of Tonelli's theorem when M is non-compact. Another proof for corollary 3-1.8 is the following:

Adding a constant we may assume that $L \ge 0$. There is $B > 0$ such that $L(x,v) \ge |v| - B$ for all $(x,v) \in TM$. Then for $0 \le s \le t \le T$, we have that

$$d(x_s, x_t) \le \int_s^t |\dot{x}| \le BT + \int_s^t L(x,\dot{x}) \le BT + c.$$

$\square$

Recall that

$$\mathcal{F}(c) := \left\{ \gamma \in \mathcal{C}^{ac}([0,T], M) \mid A_L(\gamma) \le c \right\}$$

3-1.10 Theorem.

If $\gamma_n \in \mathcal{F}(c)$ and $\gamma_n \to \gamma$ in the uniform topology, then $\gamma \in \mathcal{F}(c)$.

We shall need the following lemma. We may assume that $M = \mathbb{R}^n$,

3-1.11 Lemma. *Given K compact, $a > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in K$, $|x - y| \le \delta$, $|v| \le a$ and $w \in \mathbb{R}^n$, then*

$$L(y,w) \ge L(x,v) + L_v(x,v)\,(w - v) - \varepsilon. \tag{3.4}$$

Proof of theorem 3-1.10:

It is immediate from the definition 3-1.5 that a uniform limit of absolutely equicontinuous curves is absolutely continuous. We may assume that $\gamma_n([0,T])$ is contained in a compact neighbourhood K of $\gamma([0,T])$. By the superlinearity we may assume that $L \ge 0$. Let $\varepsilon > 0$ and $E = \big[|\dot{\gamma}| \le a\big]$, then by lemma 3-1.11, for n large,

$$\int_E \big[L(\dot{\gamma}) + L_v(\dot{\gamma})(\dot{\gamma}_n - \dot{\gamma}) - \varepsilon\big] \le \int_E L(\dot{\gamma}_n) \le c \qquad \text{(since } L \ge 0\text{)}. \tag{3.5}$$

Claim:

$$\lim_n \int_E L_v(\dot\gamma)\,(\dot\gamma_n - \dot\gamma) = 0.$$

Letting $n \to +\infty$ on (3.5), we have that

$$\int_E L(\dot\gamma) - \varepsilon\, T \le c.$$

Since $E \uparrow [0,T]$ when $a \to +\infty$ and $L \ge 0$, then

$$\int_0^T L(\dot\gamma) = \lim_{a\to+\infty} \int_E L(\dot\gamma) \le c + \varepsilon\, T.$$

Now let $\varepsilon \to 0$.

We now prove the claim. We have to prove that

$$\lim_n \int \psi\, \dot z_n = 0, \tag{3.6}$$

where $\psi = L_v(\dot\gamma)\cdot 1_E$ is bounded, $z_n = \gamma_n - \gamma$ and 1_E is the characteristic function of E. Since $\|z_n\|_\infty \to 0$, if $A = [a,b]$ is an interval then

$$\lim_n \int_A \dot z_n = \lim_n z_n\Big|_a^b = 0. \tag{3.7}$$

Since γ is absolutely continuous, $\dot\gamma \in \mathcal{L}^1$. From $\dot\gamma \in \mathcal{L}^1$ and inequality (3.3), given $\varepsilon > 0$ there is $\delta > 0$ such that

$$m(D) < \delta \quad \Longrightarrow \quad \int_D |\dot z_n| < \varepsilon, \qquad \forall n \in \mathbb{N}. \tag{3.8}$$

Equations (3.7) and (3.8) imply that (3.7) holds for any Borel set A, approximating A by a finite union of intervals. Hence (3.6) holds if ψ is a simple function $\sum_k a_k\, 1_{A_k}$.

Let f be a simple function such that $\|f\|_\infty \le 2\,\|\psi\|_\infty$ and $\int |\psi - f| < \varepsilon^2$. Let $B = [|\psi - f| > \varepsilon]$, then $m(B) < \varepsilon$.

$$\left|\int (\psi - f)\, \dot z_n\right| \le \varepsilon \int_{B^c} |\dot z_n| + 3\,\|\psi\|_\infty \int_B |\dot z_n| \le h(\varepsilon).$$

By (3.8) one can take $h(\varepsilon)$ with $\lim_\varepsilon h(\varepsilon)=0$. Then

$$\left|\lim_n \int \psi\, \dot z_n\right| \le \lim_n \left|\int f\, \dot z_n\right| + \limsup_n \left|\int (\psi - f)\, \dot z_n\right| \le 0 + h(\varepsilon) \xrightarrow{\varepsilon} 0.$$

□

3-1.12. Proof of Tonelli's theorem:

By lemma 3-1.7, the family $\mathcal{A}(c)$ in (3.1) is equicontinuous, and by corollary 3-1.8, the curves in $\mathcal{A}(c)$ have a uniform compact range. By Arzelá-Ascoli's theorem and theorem 3-1.10, $\mathcal{A}(c)$ is compact. Then

$$\gamma \in \bigcap_{c \ge \inf_{\mathcal{C}_T(x,y)} A_L} \mathcal{A}(c)$$

is a Tonelli's minimizer on $\mathcal{C}_T(x,y)$. □

Proof of lemma 3-1.11:

Let

$$C_1 := \sup\{\, |L_v(x,v)| \mid x \in K,\ |v| \le a \,\}$$
$$C_2 := \sup\{\, L(x,v) - L_v(x,v)\cdot v \mid x \in K,\ |v| \le a \,\}.$$

Let $b > 0$ be such that

$$K(b)\cdot r \ge C_2 + C_1\, r \qquad \text{for all } r \ge b,$$

where $K(b)$ is from (3.2). Then if $y \in M$ and $|w| > b$,

$$\begin{aligned} L(y,w) &\ge K(b)\, |w| \\ &\ge C_2 + C_1\, |w| \\ &\ge C_2 + L_v(x,v)\cdot w \\ &\ge L(x,v) + L_v(x,v)\cdot(w-v) \qquad \text{for } |w| \ge b. \end{aligned}$$

This gives (3.4) when $|w| \ge b$. Since L is convex,

$$L(x,w) \ge L(x,v) + L_v(x,v)\cdot(w-v) \qquad \forall\, w \in \mathbb{R}^n.$$

Then there is $\delta > 0$ such that for $|x-y| \le \delta$, $|v| \le a$ and $|w| \le b$ inequality (3.4) holds.

□

3-2 A priori compactness.

The following lemma, due to Mather [46] for Tonelli minimizers in the non-autonomous case, will be very useful. In the autonomous case its proof is very simple.

3-2.1 Lemma.
For $C > 0$ there exists $A = A(C) > 0$ such that if x, $y \in M$ and $\gamma \in \mathcal{C}_T(x, y)$ is a solution of the Euler-Lagrange equation with $A_L(\gamma) \leq C\,T$, then $|\dot\gamma(t)| < A$ for all $t \in [0, T]$.

Proof: By the superlinearity there is $D > 0$ such that $L(x, v) \geq |v| - D$ for all $(x, v) \in TM$. Since $A_L(\gamma) \leq C\,T$, the mean value theorem implies that there is $t_0 \in]0, T[$ such that

$$|\dot\gamma(t_0)| \leq D + C.$$

The conservation of the energy and the uniform bounds (1.7) and (1.6) imply that there is $A = A(C) > 0$ such that $|\dot\gamma| \leq A$.

□

For $k \geq c(L)$ and x, $y \in M$, define

$$\Phi_k(x, y; T) := \inf_{\gamma \in \mathcal{C}_T(x,y)} A_{L+k}(\gamma).$$

3-2.2 Corollary. *Given $\varepsilon > 0$, there are constants $A(\varepsilon)$, $B(\varepsilon)$, $C(\varepsilon) > 0$ such that if $T \geq \varepsilon$, $d(x, y) < R$ and $k \in \mathbb{R}$, then*

(i) $\Phi_k(x, y; T) \leq [C(\varepsilon, R) + k]\,T$.

(ii) If $\gamma \in \mathcal{C}_T(x, y)$ is a solution of the Euler-Lagrange equation such that $A_{L+k}(\gamma) \leq [C(\varepsilon, R) + k]\,T + 1$, then $|\dot\gamma| \leq A(\varepsilon, R)$ and $E(\gamma, \dot\gamma) \leq B(\varepsilon, R)$.

Proof: Comparing with the action of a geodesic on $\mathcal{C}_T(x,y)$, we get (i) with

$$C(\varepsilon,R)=\sup\big\{\,L(v)\;\big|\;|v|\le\tfrac{d(x,y)}{\varepsilon}\le\tfrac{R}{\varepsilon}\,\big\}.$$

Using (i) and lemma 3-2.1,we obtain $A(\varepsilon,R)$. Using $A(\varepsilon,R)$ and inequality (1.7) we obtain $B(\varepsilon,R)$. □

3-2.3 Lemma.

There exists $A>0$ such that if $x,\,y\in M$ and $\gamma\in\mathcal{C}_T(x,y)$ is a solution of the Euler-Lagrange equation with

$$A_{L+c}(\gamma)\le\Phi_c(x,y)+d_M(x,y),$$

then (a) $T>\frac{1}{A}\,d_M(x,y)$.

(b) $|\dot\gamma(t)|<A$ for all $t\in[0,T]$.

Proof: Let $\eta:[0,d(x,y)]\to M$ be a minimal geodesic with $|\dot\eta|\equiv 1$. Let $\ell(r)$ be from (1.1) and $D=\ell(1)+c+2$. From the superlinearity condition there is $B>0$ such that

$$L(x,v)+c>D\,|v|-B,\qquad\forall (x,v)\in TM.$$

Then

$$\begin{aligned}
[\ell(1)+c]\;d(x,y)&\ge A_{L+c}(\eta)\ge\Phi_c(x,y) &&(3.9)\\
&\ge A_{L+c}(\gamma)-d(x,y) &&(3.10)\\
&\ge\int_0^T\big(D\,|\dot\gamma|-B\,\big)\,dt-d(x,y)\\
&\ge D\;d(x,y)-B\,T-d(x,y).
\end{aligned}$$

Hence

$$T\ge\tfrac{D-\ell-c-1}{B}\;d(x,y)\ge\tfrac{1}{B}\;d(x,y).$$

From (3.9) and (3.10), we get that

$$\begin{aligned} A_L(\gamma) &\leq \big[\, \ell(1) + c + 1 \,\big]\, d(x,y) - c\,T, \\ &\leq \Big\{\, B\,[\, \ell(1) + c + 1\,] - c \,\Big\}\, T. \end{aligned}$$

Then lemma 3-2.1 completes the proof. □

3-3 Energy of time-free minimizers.

A curve $\gamma \in \mathcal{C}(x,y)$ is a *global minimizer* or *time free* minimizer for $L+k$ if $k \geq c(L)$ and $A_{L+k}(\gamma) = \Phi_k(x,y)$.

3-3.1 Proposition. *A time-free minimizer for $L+k$ has energy $E \equiv k$.*

We need the following

3-3.2 Lemma. *Let : $x : [0,T] \to M$ be an absolutely continuous curve and $k \in \mathbb{R}$. For $\lambda > 0$, let $x_\lambda(t) := x(\lambda t)$ and $\mathcal{A}(\lambda) := A_{L+k}(x_\lambda)$. Then*

$$\mathcal{A}'(1) = \int_0^T \big[\, E(x,\dot{x}) - k \,\big]\, dt.$$

Proof: Since $\dot{x}_\lambda(t) = \lambda\, \dot{x}(\lambda t)$, then

$$\mathcal{A}(\lambda) = \int_0^{\frac{T}{\lambda}} \big[\, L(x(\lambda t),\, \lambda\, \dot{x}(\lambda t)) + k \,\big]\, dt.$$

Differentiating $\mathcal{A}(\lambda)$ and evaluating at $\lambda = 1$, we have that

$$\mathcal{A}'(1) = -T\, \big[\, L(x(T), \dot{x}(T)) + k \,\big] + \int_0^T \big[\, L_x\, t\, \dot{x} + L_v\, (\dot{x} + t\, \ddot{x}) \,\big]\; dt.$$

Integrating by parts the term $(L_x\, \dot{x} + L_v \ddot{x})\, t = \big(\frac{d}{dt} L\big)\, t$, we have that

$$\begin{aligned} \mathcal{A}'(1) &= -T\, \big[\, L(x(T), \dot{x}(T)) + k \,\big] + L\, t\, \big|_0^T + \int_0^T \big(\, L_v\, \dot{x} - L \big)\; dt \\ &= -T\, k + \int_0^T E(x,\dot{x})\; dt = \int_0^T \big[\, E(x,\dot{x}) - k \,\big]\; dt. \end{aligned}$$

□

Proof of proposition 3-3.1.

Since γ is a solution of the Euler-Lagrange equation its energy $E(\gamma,\dot{\gamma})$ is constant. Since it minimizes with free time, the derivative in lemma 3-3.2 must be zero. So that $E(\gamma,\dot{\gamma}) \equiv k$. □

3-3.3 Corollary.

Let $x \in \mathcal{C}^{ac}([0,1], M)$ *and* $k > 0$*. For* $T > 0$*, write* $y_T(t) = x(\frac{t}{T}) : [0,T] \to M$ *and* $\mathcal{B}(T) = A_{L+k}(y_T)$*. Then*

$$\mathcal{B}'(T) = -\frac{1}{T}\int_0^T \big[\, E(y_T, \dot{y}_T) - k \,\big]\, dt.$$

Proof: Using $\lambda = \frac{T}{S}$ on lemma 3-3.2, we have that $\frac{d}{dS} = -\frac{T}{S^2}\,\frac{d}{d\lambda}$. Thus

$$\tfrac{d}{dS}\big|_{S=T}\mathcal{B} = -\frac{1}{T}\,\tfrac{d}{d\lambda}\big|_{\lambda=1}\mathcal{A} = -\frac{1}{T}\int_0^T \big[\, E(y_T, \dot{y}_T) - k \,\big]\, dt.$$

□

3-4 The finite-time potential.

Recall that if $k \geq c(L)$, x, $y \in M$ and $T > 0$,

$$\Phi_k(x, y; T) := \inf_{\gamma \in \mathcal{C}_T(x,y)} A_{L+k}(\gamma).$$

Here we shall prove the following proposition. See also corollary 4-11.8.

3-4.1 Proposition.

1. *For $k \in \mathbb{R}$, a compact subset $R \subseteq M$ and $\varepsilon > 0$, the function $(x, y, t) \mapsto \Phi_k(x, y; t)$ is Lipschitz on $R \times R \times [\varepsilon, +\infty[$ for any $\varepsilon > 0$.*

2. $\Phi_k(x, y; T) = \Phi_c(x, y; T) + (k - c)\, T$, *for $k \geq c(L)$, x, $y \in M$.*

3. $\lim_{\varepsilon \to 0^+} \Phi_k(x, y; \varepsilon) = +\infty$, *for $k \geq c(L)$, $x \neq y$.*

4. $\lim_{T \to +\infty} \Phi_k(x, y; T) = +\infty$, *for $k > c(L)$, $x, y \in M$.*

5. $\lim_{T \to +\infty} \Phi_k(x, y; T) = -\infty$, *for $k < c(L)$, $x, y \in M$.*

6. *When M is compact, the limits in items 4 and 5 are uniform in (x, y).*

Proof: Item 2 follows from the fact that both action potentials satify an equivalent variational principle. We now compute the limits.

3. Given $A > 0$, let $B > 0$ be such that $L(x, v) > A\,|v| - B$. Then

$$\begin{aligned}\Phi_k(x, y; \varepsilon) = \inf_{\gamma \in \mathcal{C}_\varepsilon(x,y)} A_{L+k}(\gamma) &\geq \inf_{\gamma} \int_0^\varepsilon A\,|\dot\gamma| - B + k \\ &\geq A\, d(x, y) + (k - B)\, \varepsilon.\end{aligned}$$

Thus, $\liminf_{\varepsilon \to 0^+} \Phi_k(x, y; \varepsilon) \geq A\, d(x, y)$. Now let $A \to +\infty$.

4. Observe that if $k > c(L)$,

$$\lim_{T\to+\infty} \Phi_k(x,y;T) \geq \lim_{T\to+\infty} \big[\, \Phi_c(x,y) + (k-c)\,T\,\big] = +\infty.$$

When M is compact, this limit is uniform because

$$\Phi_k(x,y;T) \geq \inf_{x,y\in M} \Phi_c(x,y) + (k-c)\,T.$$

5. For $k < c(L)$ there exists a closed curve $\gamma \in \mathcal{C}_S(z,z)$, $S > 0$, $z \in M$ such that $A_{L+c}(\gamma) < 0$. Then for $T = nS + \tau$, with $S \leq \tau < 2S$, we have that

$$\Phi_k(x,y;T) \leq \Phi_k(x,z;\tfrac{\tau}{2}) + n\,A_{L+c}(\gamma) + \Phi_k(z,y;\tfrac{\tau}{2}).$$

This implies that $\lim_{T\to+\infty} \Phi_k(x,y;T) = -\infty$. The uniformity of this limit for (x,y) on compact subsets follows from the Lipschitz condition of item 1 that we now prove.

Fix $\varepsilon > 0$. We prove now that the function $]\varepsilon,+\infty[\ni t \mapsto \Phi_k(x,y;t)$ is uniformly Lipschitz. If $T > \varepsilon$ and $\gamma \in \mathcal{C}_T(x,y)$ is a Tonelli minimizer, from corollary 3-2.2 and inequality (1.6) there exists $D = D(\varepsilon,R) > 0$ such that $|E(\gamma,\dot\gamma) - k| \leq D(\varepsilon,R) + |k|$. Denote $h(s) := \Phi_k(x,y;s)$. If $\gamma_s(t) := \gamma(\frac{T}{s}t)$, $t \in [0,s]$, then $h(s) \leq A_{L+k}(\gamma_s) =: \mathcal{B}(s)$. Using corollary 3-3.3 we have that

$$\begin{aligned} f(T) :&= \limsup_{\delta\to 0} \frac{h(T+\delta)-h(T)}{\delta} \\ &\leq \mathcal{B}'(T) = \frac{1}{T}\int_0^T \big[\, k - E(\gamma,\dot\gamma)\,\big]\, dt \\ &\leq\; D(\varepsilon,R) + |k|\,. \end{aligned}$$

If S, $T > \varepsilon$ we have that

$$\begin{aligned} \Phi_k(x,y;S) &\leq \Phi_k(x,y;T) + \int_T^S f(t)\, dt \\ &\leq \Phi_k(x,y;T) + \big[\, D(\varepsilon,R) + |k|\,\big]\; |T-S|. \end{aligned}$$

Since we can reverse the roles of S and T, this implies the uniform Lipschitz condition for $T \mapsto \Phi_k(x, y; T)$.

Now we prove the Lipschitz condition on a neighbourhood of y_0 for $y \mapsto \Phi_k(x_0, y; T_0)$. The proof for a neighbourhood of x_0 is similar. Since R is compact, this is enough to show that the function is Lipschitz on $R \times R \times]\varepsilon, +\infty[$.

Fix $0 < \delta < \frac{\varepsilon}{2}$ small. Let $\lambda \in \mathcal{C}_T(x_0, y_0)$ be a Tonelli minimizer. Observe that λ realizes $\Phi_k(x_0, y_0, T)$. Let $\gamma : [0, 1] \to M$ be a length minimizing geodesic joining y_0 to y, with $|\dot{\gamma}| \equiv 1$. Let $\eta(s, t)$, $s \in [0, 1]$, $t \in [T - \delta, T]$ be a variation by solutions of (E-L) such that $\eta(s, T - \varepsilon) = \lambda(T - \delta)$, $\eta(s, T) = \gamma(s)$ and $\eta(0, t) = \lambda(t)$. Let

$$\mathcal{E}(s) := A_{L+k}\big(\lambda|_{[0,T-\delta]} * \eta(s, \cdot)|_{[T-\delta,T]}\big).$$

Then

$$\begin{aligned}
\mathcal{E}'(s) &= \frac{d}{ds} \int_{T-\delta}^{T} L\big(\tfrac{\partial \eta}{\partial t}(s, t)\big)\, dt = \int_{T-\delta}^{T} L_x\, \tfrac{\partial \eta}{\partial s} + L_v\, \tfrac{\partial^2 \eta}{\partial s\, \partial t} \\
&= L_v\, \tfrac{\partial \eta}{\partial s}\Big|_{T-\delta}^{T} + \int_{T-\delta}^{T} \big(L_x - \tfrac{d}{dt} L_v\big)\, \tfrac{\partial \eta}{\partial s} \\
&= L_v\big(\gamma(s), \tfrac{\partial \eta}{\partial t}(s, T)\big) \cdot \dot{\gamma}(s).
\end{aligned}$$

By Weierstrass theorem **??**, if δ and $d(y, y_0)$ are small enough, the curves $t \mapsto \eta(s, t)$ are Tonelli minimizers — and thus they realize $\Phi_k(y_0, y, \delta)$. Then by corollary 3-2.2, there exists $A(\delta, R) > 0$ such that $|\frac{\partial \eta}{\partial t}(s, T)| < A(\delta, R)$. By lemma 1-4.4, $\|L_v(x, \frac{\partial \eta}{\partial t}(s, T))\| \le f(A(\delta, R))$. Thus

$$\begin{aligned}
\Phi_k(x_0, y; T) &\le \Phi_k(x_0, y_0; T) + \int_0^1 \mathcal{E}'(s)\, ds \\
&\le \Phi_k(x_0, y_0; T) + \int_0^1 f(A(\delta, R))\ |\dot{\gamma}(s)|\ ds \\
&= \Phi_k(x_0, y_0; T) + f(A(\delta, R))\, d(y, y_0).
\end{aligned}$$

The value od δ can be taken locally constant on a neighbourhood of y_0. Changing the roles of y and y_0 we obtain that $f(A(\delta, R))$ is a local Lipschitz constant for $y \mapsto \Phi_k(x, y; T)$, $x, y \in R$. □

3-5 Global Minimizers.

Here we construct curves that realize the action potential.
For $k < c(L)$, $\Phi_k \equiv -\infty$, so there are no minimizers.

3-5.1 Proposition.
If $k > c(L)$ and $x,\ y \in M$, $x \neq y$, then there is $\gamma \in \mathcal{C}(x,y)$ such that

$$A_{L+k}(\gamma) = \Phi_k(x,y).$$

Moreover, the energy of γ is $E(\gamma,\dot\gamma) \equiv k$.

Proof: Let $f(t) := \Phi_k(x,y;t)$. By proposition 3-4.1, $f(t)$ is continuous and $f(t) \to +\infty$ when $t \to 0^+$ or $t \to +\infty$. Hence it attains its minimum at some $T > 0$. Moreover, $\Phi_k(x,y) = \inf_{t>0} \Phi_k(x,y;t) = \Phi_k(x,y;T)$. Now take a Tonelli minimizer γ on $\mathcal{C}_T(x,y)$. From lemma 3-3.2, the energy of γ is k. $\square$

We now study minimizers at $k = c(L)$. Observe that for $c = c(L)$ and any absolutely continuous curve $\gamma \in \mathcal{C}(x,y)$, we have that

$$A_{L+c}(\gamma) \geq \Phi_c(x,y) \geq -\Phi_c(y,x). \tag{3.11}$$

3-5.2 Definition. Set $c = c(L)$.

An absolutely continuous curve $\gamma \in \mathcal{C}(x,y)$ is said *semistatic* if

$$A_{L+c}(\gamma) = \Phi_c(x,y).$$

An absolutely continuous curve $\gamma \in \mathcal{C}(x,y)$ is said *static* if

$$A_{L+c}(\gamma) = -\Phi_c(y,x).$$

These names are justified by the following remark: For mechanic lagrangians $L = \frac{1}{2}|v|_x^2 - U(x)$, static orbits are the fixed points $(x_0, 0)$ of the lagrangian flow where $U(x)$ is maximal; and semistatic orbits lie in the stable or unstable manifolds of those fixed points.

By the triangle inequality for Φ_c the definition of semistatic curve $x:[a,b]\to M$ is equivalent to

$$A_{L+c}\big(x|_{[s,t]}\big)=\Phi_c\big(x(s),x(t)\big),\qquad \forall\, a\le s\le t\le b. \tag{3.12}$$

Inequality (3.11) implies that static curves are semistatic.

Moreover, a curve $\gamma\in\mathcal{C}(x,y)$ is *static* if

(a) γ is semistatic, and

(b) $d_c(x,y)=\Phi_c(x,y)+\Phi_c(y,x)=0$.

From proposition 3-3.1 we get

3-5.3 Corollary. *Semistatic curves have energy $E\equiv c(L)$.*

3-5.4 Definition.

$$\begin{aligned}
\mathcal{M}&=\cup\big\{\operatorname{supp}(\mu)\;\big|\;\mu\in\mathcal{M}(L),\;\; A_L(\mu)=-c(L)\,\big\}\\
\widetilde{\mathcal{N}}=\Sigma(L)&:=\big\{\,w\in TM\;\big|\;x_w:\mathbb{R}\to M\text{ is semistatic }\big\}\\
\mathcal{A}=\widehat{\Sigma}(L)&:=\big\{\,w\in TM\;\big|\;x_w:\mathbb{R}\to M\text{ is static }\big\}\\
\Sigma^-(L)&:=\big\{\,w\in TM\;\big|\;x_w:]-\infty,0]\to M\text{ is semistatic }\big\}\\
\Sigma^+(L)&:=\big\{\,w\in TM\;\big|\;x_w:[0,+\infty[\to M\text{ is semistatic }\big\}
\end{aligned}$$

We call $\mathcal{M}$ the *Mather set*, $\widetilde{\mathcal{N}}$ the *Mañé set*, $\mathcal{P}=\pi(\widehat{\Sigma}(L))$ the *Peierls set*[1] and $\mathcal{A}=\widehat{\Sigma}(L)$ the *Aubry set*.

Using the characterization of minimizing measures 3-6.1 and corollary 3-5.3 we have that[2]

$$\mathcal{M}\subseteq\mathcal{A}\subseteq\widetilde{\mathcal{N}}\subseteq\acute{\mathcal{E}}\;, \tag{3.13}$$

where $\mathcal{M}$ is the Mather set, $\mathcal{A}$ is the Aubry set, $\widetilde{\mathcal{N}}$ is the Mañé set and $\acute{\mathcal{E}}$ is the energy level $\acute{\mathcal{E}}=[E\equiv c(L)]$. All these inclusions can be made

[1]The name is justified by proposition 3-7.1.*5*.
[2]The typographical relationship was observed by Albert Fathi.

proper constructing examples of embedded flows as in equation (1.18) and adding a properly chosen potential $\phi(x)$.

Denote by $\alpha(v)$ and $\omega(v)$ the α and ω-limits of v under the Euler-Lagrange flow.

3-5.5 Proposition.
A local static is a global static, i.e. if $x_v|_{[a,b]}$ is static then $v \in \widehat{\Sigma}(L)$ (i.e. the whole orbit is static).

Proof: Let $\eta(t) = \pi\,\varphi_t(v)$ and let $\gamma_n \in \mathcal{C}_{T_n}(\eta(b), \eta(a))$ be solutions of (E-L) with

$$A_{L+c}(\gamma_n) < \Phi_c(\eta(b), \eta(a)) + \tfrac{1}{n}.$$

By the a priori bounds 3-2.3 $|\dot\gamma_n| < A$. We can assume that $\dot\gamma_n(0) \to w$. Let $\xi(s) = \pi\,\varphi_s(w)$. If $w \neq \dot\eta(b)$ then the curve $\eta|_{[b-\varepsilon,b]} * \xi|_{[0,\varepsilon]}$ is not C^1, and hence by remark 1-2.2, it can not be a Tonelli minimizer. Thus

$$\Phi_c(\eta(b-\varepsilon), \xi(\varepsilon)) < A_{L+c}(\eta|_{[b-\varepsilon,b]}) + A_{L+c}(\xi|_{[0,\varepsilon]}).$$

$$\begin{aligned}
\Phi_c(\eta(a), \eta(a)) &\leq \Phi_c(\eta(a), \eta(b-\varepsilon)) + \Phi_c(\eta(b-\varepsilon), \xi(\varepsilon)) + \Phi_c(\xi(\varepsilon), \eta(a)) \\
&< A_{L+c}(\eta_{[a,b-\varepsilon]}) + A_{L+c}(\eta|_{[b-\varepsilon,b]}) + A_{L+c}(\xi|_{[0,\varepsilon]}) + \liminf_n A_{L+c}(\gamma_n|_{[\varepsilon,T_n]}) \\
&\leq A_{L+c}(\eta|_{[a,b]}) + \lim_n \big(\gamma_n|_{[0,\varepsilon]} * \gamma_n|_{[\varepsilon,T_n]} \big) \\
&\leq -\Phi_c(\eta(b), \eta(a)) + \Phi_c(\eta(b), \eta(a)) = 0,
\end{aligned}$$

which contradicts proposition 2-1.1(3). Thus $w = \dot\eta(b)$ and similarly $\lim_n \dot\gamma_n(T_n) = \dot\eta(a)$.

If $\limsup T_n < +\infty$, we can assume that $\tau = \lim_n T_n > 0$ exists. In this case η is a (semistatic) periodic orbit of period $\tau + b - a$ and then it is static.

Now suppose that $\lim_n T_n = +\infty$. If $s > 0$, we have that

$$\begin{aligned}
&A_{L+c}(\eta|_{[a-s,b+s]}) + \Phi_c(\eta(b+s), \eta(a-s)) \leq \\
&\quad \leq \lim_n \left\{ A_{L+c}(\gamma_n|_{[T_n-s,T_n]}) + A_{L+c}(\eta) + A_{L+c}(\gamma_n|_{[0,s]}) \right\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \Phi_c(\eta(b+s), \eta(a-s)) \\
&\quad \leq \Phi_c(\eta(a), \eta(b)) \\
&\qquad + \lim_n \left\{ A_{L+c}(\gamma_n|_{[0,s]}) + A_{L+c}(\gamma_n|_{[s,T_n-s]}) + A_{L+c}(\gamma_n|_{[T_n-s,T_n]}) \right\} \\
&\quad \leq \Phi_c(\eta(a), \eta(b)) + \Phi_c(\eta(b), \eta(a)) = 0.
\end{aligned}$$

Thus $\eta_{[a-s,b+s]}$ is static. $\square$

3-5.6 Definition. Set $c = c(L)$.
An absolutely continuous curve $\gamma : [0, +\infty[\to M$ (resp. $\eta :]-\infty, 0] \to M$) is a *ray* if $\gamma|_{[0,t]}$ (resp. $\eta|_{[-t,0]}$) is a Tonelli minimizer for all $t > 0$; i.e.

$$A_{L+c}(\gamma|_{[0,t]}) = \Phi_c(\gamma(0), \gamma(t); t) \quad \text{ for all } t > 0.$$

Clearly a semi-infinite semistatic curve is a ray. We shall see in corollary 4-11.9 that rays are semistatic.

3-5.7 Proposition.
If $v \in \Sigma$ is semistatic, then $\alpha(v) \subset \widehat{\Sigma}(L)$ and $\omega(v) \subset \widehat{\Sigma}(L)$. Moreover $\alpha(v)$ and $\omega(v)$ are each contained in a static class.

Proof: We prove only that $\omega(v) \subset \widehat{\Sigma}$. Let $\gamma(t) = \pi\, \varphi_t(v)$. Suppose that $t_n \to +\infty$ and $\dot{\gamma}(t_n) \to w \in TM$. Let $\eta(t) = \pi\, \varphi_t(w)$. Since γ and η are solutions the Euler-Lagrange equation, then $\gamma|_{[t_n-s,t_n+s]} \xrightarrow[C^1]{} \eta|_{[-s,s]}$.

Then

$$\begin{aligned}
A_{L+c}\big(\eta|_{[-s,s]}\big) + \Phi_c(\eta(s),\eta(-s)) = &\\
&= \lim_n \big\{ \, A_{L+c}(\gamma|_{[t_n-s,t_n+s]}) + \lim_m A_{L+c}(\gamma|_{[t_n+s,t_m-s]}) \, \big\} \\
&= \lim_n \ \lim_m \ A_{L+c}(\gamma|_{[t_n-s,t_m-s]}) \\
&= \lim_n \ \lim_m \ \Phi_c(\gamma(t_n-s),\gamma(t_m-s)) \\
&= \Phi_c(\eta(-s),\eta(-s)) = 0.
\end{aligned}$$

Thus $w \in \widehat{\Sigma}(L)$. Let $u \in \omega(v)$. We may assume that $\dot\gamma(s_n) \to u$ with $t_n < s_n < t_{n+1}$. Then

$$\begin{aligned}
d_c(\pi w,\pi u) &= \Phi_c(\pi w,\pi u) + \Phi_c(\pi u,\pi w) \\
&= \lim_n \ A_{L+c}(\gamma|_{[t_n,s_n]}) + A_{L+c}(\gamma|_{[s_n,t_{n+1}]}) \\
&= \lim_n \ A_{L+c}(\gamma_n|_{[t_n,t_{n+1}]}) = \Phi_c(\pi w,\pi w) = 0.
\end{aligned}$$

Thus w and u are in the same static class. $\square$

3-6 Characterization of minimizing measures.

Recall that minimizing measures μ satisfy $A_L(\mu) = -c(L)$ and that by theorem 2-4.1 they are invariant.

3-6.1 Theorem (Mañé [39]).
$\mu \in \mathcal{M}(L)$ is a minimizing measure if and only if $\operatorname{supp}(\mu) \subseteq \widehat{\Sigma}(L)$.

Proof: Since $\widehat{\Sigma}$ is closed, it is enough to prove the theorem for ergodic measures. Suppose that $\mu \in \mathcal{M}(L)$ is ergodic and $\operatorname{supp}(\mu) \subseteq \Sigma(L)$. Since μ is finite, by Birkhoff's theorem there is a set of total μ-measure A such that if $\theta \in A$ then $\liminf_{T\to+\infty} d_{TM}(\theta, \varphi_T\theta) = 0$ and

$$\begin{aligned}\int_M L + c\, d\mu &= \lim_{T\to+\infty} \frac{1}{T}\int_0^T \big[\, L(\varphi_t\,\theta) + c\,\big]\, dt \\ &\le \liminf_{T\to+\infty} \tfrac{1}{T}\, \Phi_c\big(\pi(\varphi_T\,\theta), \pi(\theta)\big) = 0.\end{aligned}$$

Now suppose that $\mu \in \mathcal{M}(L)$ is minimizing. Then $\int L + c\ d\mu = 0$. Applying corollary 3-6.5 to $F = L + c$ and $X = TM$, we get that there is a set $A \subset TM$ of total μ-measure such that if $\theta \in A$ then there is a sequence $T_n \to +\infty$ such that $d(\theta, \varphi_{T_n}\,\theta) \to 0$ and

$$\lim_n \int_0^{T_n} \big[\, L(\varphi_t\,\theta) + c\,\big]\, dt = 0.$$

Then

$$\begin{aligned}0 \le d_c(\pi\,\theta,\, \pi\varphi_1\,\theta) &= \lim_n \big[\, \Phi_c(\pi\,\theta, \pi\,\varphi_1\,\theta) + \Phi_c(\pi\,\varphi_1\,\theta, \pi\,\varphi_{T_n}\,\theta)\,\big] \\ &\le \lim_n \int_0^{T_n} \big[\, L(\varphi_t\,\theta) + c\,\big]\, dt = 0.\end{aligned}$$

This implies that θ is static. Since $\widehat{\Sigma}(L)$ is closed and A is dense in $\operatorname{supp}(\mu)$ then $\operatorname{supp}(\mu) \subseteq \widehat{\Sigma}(L)$. □

It follows from theorem 2-5.1 that

3-6.2 Corollary.
If M is compact then $\widehat{\Sigma}(L) \neq \varnothing$.

Combining theorem 3-6.1 with corollary 3-5.3, we get

3-6.3 Corollary (Dias Carneiro [9])**.** *If μ is a minimizing measure then it is supported in the energy level $E(\operatorname{supp}(\mu)) = c(L)$.*

3-6.4 Lemma. *Let $(X, \mathfrak{B}, \nu)$ be a probability space, f an ergodic measure preserving map and $F : X \to \mathbb{R}$ an integrable function. Given $A \in \mathfrak{B}$ with $\nu(A) > 0$ denote by $\hat{A}$ the set of point $p \in A$ such that for all $\varepsilon > 0$ there exists an integer $N > 0$ such that $f^N(p) \in A$ and*

$$\Big| \sum_{j=0}^{N-1} F\big(f^j(p)\big) - N \int F\, d\nu \Big| < \varepsilon\,.$$

Then $\nu(\hat{A}) = \nu(A)$.

Proof: Without loss of generality we can assume that $\int F\, d\nu = 0$. For $p \in X$ denote

$$S_N F(p) := \sum_{n=0}^{N-1} F\big(f^n(p)\big)\,.$$

Let

$$A(\varepsilon) := \big\{\, p \in A \,|\, \exists N > 0 \text{ such that } f^N(p) \in A \text{ and } |S_N F(p)| < \varepsilon \,\big\}\,.$$

It is enough to prove that $\nu(A(\varepsilon)) = \nu(A)$, because $\hat{A} = \bigcap_n A(\frac{1}{n})$. Let $\hat{X}$ be the set of points for which the Birkhoff's theorem holds for F and the characteristic functions of A and of $A(\varepsilon)$. Take $x \in A \cap \hat{X}$ and let $N_1 < N_2 < \cdots$ be the integers for which $f^{N_i}(x) \in A$. Define $\delta(k)$ by

$$N_k\, \delta(k) = |S_{N_k} F(x)|\,.$$

Since $x \in \hat{X}$ we have that $\lim_{k\to+\infty} \delta(k) = 0$. Set

$$\begin{aligned} c_k &:= S_{N_k} F(x)\,, \\ \mathcal{S} &:= \big\{\, k \in \mathbb{N} \;\big|\; \forall \ell > k,\ |c_\ell - c_k| \geq \varepsilon \,\big\}, \\ S(k) &:= \big\{\, 1 \leq j \leq k-1 \;\big|\; j \in \mathcal{S} \,\big\}\,. \end{aligned}$$

Observe that if $j \notin S(k)$, then there is an $\ell > j$ such that $|c_\ell - c_j| \leq \varepsilon$, so that

$$\big| S_{N_\ell - N_j} F\left(f^{N_j}(x)\right)\big| = |c_\ell - c_j| \leq \varepsilon\,.$$

Hence

$$j \notin S(k) \Longrightarrow f^{N_j}(x) \in A(\varepsilon)\,.$$

This implies that

$$\begin{aligned} \nu\big(A - A(\varepsilon)\big) &= \lim_{k\to+\infty} \frac{1}{N_k}\ \#\big\{\, 0 \leq j < N_k \,|\, f^j(x) \in A - A(\varepsilon) \,\big\} \\ &\leq \frac{1}{N_k}\ \#S(k)\,. \end{aligned} \tag{3.14}$$

If $\mathcal{S}$ is finite, from inequality (3.14) we get that $\nu\big(A - A(\varepsilon)\big) = 0$, concluding the proof of the lemma.

Assume that $\mathcal{S}$ is infinite. This implies that the set $\{\, c_k \,|\, k \in \mathcal{S} \,\}$ is unbounded. Choose an infinite sequence $\mathcal{K}$ in $\mathcal{S}$ such that for all $k \in \mathcal{K}$ we have,

$$|c_k| = \max_{j \in S(k)} |c_j|.$$

Then, for $k \in \mathcal{K}$,

$$\tfrac{1}{2}\,(2\varepsilon)\ \#S(k) \leq |c_k| = \delta(k)\, N_k.$$

From (3.14), we get that

$$\nu\big(A - A(\varepsilon)\big) \leq \lim_{k \in \mathcal{K}} \frac{\delta(k)}{\varepsilon} = 0.$$

□

3-6.5 Corollary. *If besides the hypothesis of lemma 3-6.4, X is a complete separable metric space, and $\mathfrak{B}$ is its Borel σ-algebra, then for a.e. $x \in X$ the following property holds: for all $\varepsilon > 0$ there exists $N > 0$ such that $d(f^N(x), x) < \varepsilon$ and*

$$\left| \sum_{j=0}^{N-1} F\big(f^j(x)\big) - N \int F \, d\nu \right| < \varepsilon$$

Proof: Given $\varepsilon > 0$ let $\{V_n(\varepsilon)\}$ be a countable basis of neighbourhoods with diameter $< \varepsilon$ and let $\hat{V}_n$ be associated to V_n as in lemma 3-6.4. Then the full measure subset $\bigcap\limits_m \bigcup\limits_n \hat{V}_n(\frac{1}{m})$ satisfies the required property. $\square$

3-7 The Peierls barrier.

For $T > 0$ and $x,\ y \in M$ define

$$h_T(x,y) = \Phi_c(x,y;T) := \inf_{\gamma\in\mathcal{C}_T(x,y)} A_{L+c}(\gamma).$$

So that the curves which realize $h_T(x,y)$ are the Tonelli minimizers on $\mathcal{C}_T(x,y)$. Define the *Peierls barrier* as

$$h(x,y) := \liminf_{T\to+\infty} h_T(x,y).$$

The difference between the action potential and the Peierls barrier is that in the Peierls barrier the curves must be defined on large time intervals. Clearly

$$h(x,y) \ge \Phi_c(x,y).$$

3-7.1 Proposition.
If $h : M \times M \to \mathbb{R}$ is finite, then

1. *h is Lipschitz.*
2. *$\forall\ x,\ y \in M$, $h(x,x) \ge \Phi_c(x,y)$, in particular $h(x,x) \ge 0$, $\forall\ x \in M$.*
3. *$h(x,z) \le h(x,y) + h(y,z), \qquad \forall\ x,\ y,\ z \in M$.*
4. *$h(x,y) \le \Phi_c(x,p) + h(p,q) + \Phi_c(q,y), \qquad \forall\ x,\ y,\ p,\ q \in M$.*
5. *$h(x,x) = 0 \iff x \in \pi(\widehat{\Sigma}) = \mathcal{P}$.*
6. *If $\widehat{\Sigma} \neq \varnothing$, $h(x,y) \le \inf_{p\in\pi(\widehat{\Sigma})} \Phi_c(x,p) + \Phi_c(p,y)$.*

Proof: Item *2* is trivial. Observe that for all S, $T > 0$ and $y \in M$,

$$h_{T+S}(x,z) \le h_T(x,y) + h_S(y,z).$$

Taking $\liminf_{T\to+\infty}$ we get that

$$h(x,z) \le h(x,y) + h_S(y,z), \qquad \text{for all } S > 0.$$

Taking $\liminf_{S\to+\infty}$, we obtain item *3*.

1. Taking the infimum on $S>0$, we get that

$$\begin{aligned} h(x,z) &\le h(x,y)+\Phi_c(y,z) \qquad \forall\, x,\, y,\, z\in M.\\ &\le h(x,y)+A\, d_M(y,z), \end{aligned}$$

where A is a Lipschitz constant for Φ_c. Changing the roles of x, y, z, we obtain that h is Lipschitz.

4. Observe that

$$\inf_{S>T} h_S(x,y)\le \Phi_c(x,p)+h_T(p,q)+\Phi_c(q,x).$$

Taking $\liminf_{T\to+\infty}$ we get item *4*.

5. We first prove that if $p\in\mathcal{P}=\pi(\widehat{\Sigma})$, then $h(p,p)=0$. Take $v\in\widehat{\Sigma}$ such that $\pi(v)=p$ and $y\in\pi(\omega\text{-limit}(v))$. Let $\gamma(t):=\pi\,\varphi_t(v)$ and choose $t_n\uparrow+\infty$ such that $\gamma(t_n)\to y$. Then

$$\begin{aligned} 0\le h(p,p) &\le h(p,y)+\Phi_c(y,p)\\ &\le \lim_n A_{L+c}(\gamma|_{[0,t_n]})+\Phi_c(y,p)\\ &\le \lim_n -\Phi_c(\gamma(t_n),p)+\Phi_c(y,p)=0. \end{aligned}$$

Conversely, if $h(x,x)=0$, then there exists a sequence of Tonelli minimizers $\gamma_n\in\mathcal{C}(x,x;T_n)$ with $T_n\to+\infty$ and $A_{L+c}(\gamma_n)\xrightarrow{n}0$. By lemma 3-2.3, $|\dot\gamma|$ is uniformly bounded. Let v be an accumulation point of $\dot\gamma_n(0)$ and $\eta(t):=\pi\,\varphi_t(v)$. Then if $\dot\gamma_{n_k}(0)\xrightarrow{k}v$, for any $s>0$ we have that

$$\begin{aligned} 0 &\le \Phi_c(x,\pi\,\varphi_s\,v)+\Phi_c(\pi\,\varphi_s\,v,x)\\ &\le A_{L+c}\big(\eta|_{[0,s]}\big)+\Phi_c(\pi\,\varphi_s\,v,x)\\ &\le \lim_k A_{L+c}\big(\gamma_{n_k}|_{[0,s]}\big)+A_{L+c}\big(\gamma_{n_k}|_{[s,T_n]}\big)\\ &=0. \end{aligned}$$

Thus $v\in\widehat{\Sigma}$.

6. Using items *4* and *5*, we get that

$$h(x,y) \le \inf_{p\in\pi(\widehat{\Sigma})} \big[\, \Phi_c(x,p) + 0 + \Phi_c(p,y)\,\big].$$

$\square$

3-7.2 Proposition. *If M is compact, then*

$$h(x,y) = \inf_{p\in\pi(\widehat{\Sigma})} \big[\Phi_c(x,p) + \Phi_c(p,y)\big].$$

Proof:

From proposition 3-7.1.*6* we have that

$$h(x,y) \le \inf_{p\in\pi(\widehat{\Sigma})} \big[\, \Phi_c(x,p) + \Phi_c(p,y)\,\big].$$

In particular $h(x,y) < +\infty$ for all x, $y \in M$. Now let $\gamma_n \in \mathcal{C}_{T_n}(x,y)$ with $T_n \to +\infty$ and $A_{L+c}(\gamma_n) \to h(x,y) < +\infty$. Then $\frac{1}{T} A_{L+c}(\gamma_n) \to 0$. Let μ be a weak limit of a subsequence of the measures μ_{γ_n}. Then μ is minimizing. Let $q \in \pi\big(\mathrm{supp}(\mu)\big)$ and $q_n \in \gamma_n([0,T_n])$ be such that $\lim_n q_n = q$. Then,

$$\begin{aligned}\Phi_c(x,q) + \Phi_c(q,y) &\le \Phi_c(x,q_n) + \Phi_c(q_n,y) + 2A\, d(q_n,q)\\ &\le A_{L+c}(\gamma_n) + 2A\, d(q_n,q).\end{aligned}$$

Letting $n \to \infty$, we get that

$$\Phi_c(x,q) + \Phi_c(q,y) \le h(x,y).$$

$\square$

3-8 Graph Properties.

In this section we shall see that the projection $\pi : \widehat{\Sigma} \to M$ is injective. We shall call $\mathcal{P} := \pi(\widehat{\Sigma})$ the *Peierls set.* [3] Thus the projection $\pi|_{\widehat{\Sigma}}$ gives an identification $\mathcal{P} \approx \widehat{\Sigma}$.

For $v \in TM$, write $x_v(t) = \pi\, \varphi_t(v)$. Given $\varepsilon > 0$, let

$$\Sigma^\varepsilon := \{w \in TM \mid x_w : [0, \varepsilon) \to M \text{ or } x_w : (-\varepsilon, 0] \to M \text{ is semistatic } \}.$$

3-8.1 Theorem. (Mañé) [39]

For all $p \in \pi(\widehat{\Sigma})$ there exists a unique $\xi(p) \in T_pM$ such that $(p, \xi(p)) \in \Sigma^\varepsilon$, in particular $(p, \xi(p)) \in \widehat{\Sigma}$ and $\widehat{\Sigma} =$ graph(ξ).

Moreover, the map $\xi : \pi(\widehat{\Sigma}) \to \Sigma$ is Lipschitz.

The proofs of the injectivity of π in this book only need that the solutions of the Euler-Lagrange equation are differentiable[4]. The reader may provide those proofs as exercises. The proof of the Lipschitz condition need the following lemma, due to Mather. For the proof see [46] or Mañé [36].

3-8.2 Mather's Crossing lemma. *[46]*

Given $A > 0$ there exists $K > 0$ $\varepsilon_1 > 0$ and $\delta > 0$ with the following property: if $|v_i| < A$, $(p_i, v_i) \in TM$, $i = 1, 2$ satisfy $d(p_1, p_2) < \delta$ and $d((p_1, v_1), (p_2, v_2)) \geq K^{-1} d(p_1, p_2)$ then, if $a \in \mathbb{R}$ and $x_i : \mathbb{R} \to M$, $i = 1, 2$, are the solutions of L with $x_i(a) = p_i$, $\dot{x}_i(a) = v_i$, there exist solutions $\gamma_i : [a - \varepsilon, a + \varepsilon] \to M$ of L with $0 < \varepsilon < \varepsilon_1$, satisfying

$$\begin{aligned} \gamma_1(a-\varepsilon) &= x_1(a-\varepsilon) \quad , \quad \gamma_1(a+\varepsilon) = x_2(a+\varepsilon)\,, \\ \gamma_2(a-\varepsilon) &= x_2(a-\varepsilon) \quad , \quad \gamma_2(a+\varepsilon) = x_1(a+\varepsilon)\,, \\ A_L(x_1|_{[a-\varepsilon,a+\varepsilon]}) &+ A_L(x_2|_{[a-\varepsilon,a+\varepsilon]}) > A_L(\gamma_1) + A_L(\gamma_2) \end{aligned}$$

[3] This name is justified by proposition 3-7.1(*5*).

[4] and hence a non-differentiable curve can not be a Tonelli minimizer.

Proof of theorem 3-8.1:

We prove that if $(p,v) \in \widehat{\Sigma}$, $(q,w) \in \Sigma^\varepsilon$, and $d(p,q) < \delta$, then

$$d_{TM}\big((p,v),(q,w)\big) < K\, d_M(p,q)\,.$$

Observe that this implies the theorem. For simplicity, we only prove the case in which $x_v|_{[-\varepsilon,0]}$ is semistatic. Suppose it is false. Then by lemma 3-8.2 there exist α, $\beta : [-\varepsilon,\varepsilon] \to M$ such that

$$\begin{aligned} \alpha(-\varepsilon) &= x_w(-\varepsilon) =: q_{-\varepsilon}\ , & \alpha(0) &= p\,,\\ \beta(-\varepsilon) &= x_v(-\varepsilon)\ =: p_\varepsilon\ , & \beta(0) &= q\,, \end{aligned}$$

and

$$A_L(\alpha) + A_L(\beta) < A_L(x_w|_{[-\varepsilon,0]}) + A_L(x_v|_{[-\varepsilon,0]}).$$

So

$$\begin{aligned} \Phi_c(q_{-\varepsilon},p) + \Phi_c(p_{-\varepsilon},q) &< \Phi_c(q_{-\varepsilon},q) + \Phi_c(p_{-\varepsilon},p)\\ &= \Phi_c(q_{-\varepsilon},q) - \Phi_c(p,p_{-\varepsilon}) \end{aligned}$$

Thus

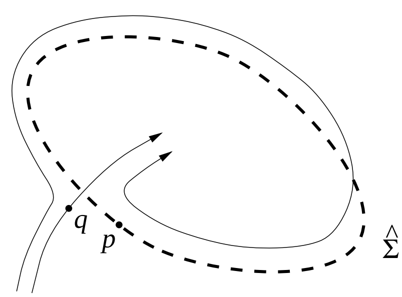

FIG. 1: GRAPH PROPERTY.

$$\Phi_c(q_{-\varepsilon},q) \le \Phi_c(q_{-\varepsilon},p) + \Phi_c(p,p_{-\varepsilon}) + \Phi_c(p_{-\varepsilon},q) < \Phi_c(q_{-\varepsilon},q)$$

which is a contradiction. $\square$

Using the graph property 3-8.1 we can define an equivalence relation on $\widehat{\Sigma}$ by

$$u, v \in \widehat{\Sigma}, \quad u \equiv v \qquad \Longleftrightarrow \qquad d_c(\pi(u), \pi(v)) = 0.$$

The equivalence classes are called *static classes*. The continuity of the pseudo-metric d_c implies that a static class is closed, and it is invariant by proposition 3-5.5.

For $v \in TM$ denote by $\omega(v)$ its ω-limit. Let Γ be a static class, the set

$$\Gamma^+ = \left\{ v \in \Sigma^+(L) \mid \omega(v) \subseteq \Gamma \right\}$$

is called the (forward) *basin* of Γ. Clearly Γ^+ is forward invariant. Let

$$\begin{aligned} \Gamma_0^+ &= \textstyle\bigcup_{t>0} \varphi_t(\Gamma^+) \\ &= \textstyle\bigcup_{\varepsilon>0} \left\{ v \in TM \mid x_v|_{]-\varepsilon,+\infty[} \text{ is semistatic and } \omega(v) \subseteq \Gamma \right\}. \end{aligned}$$

The set $\pi(\Gamma^+ \setminus \Gamma_0^+)$ is called the *cut locus* of Γ^+.

3-8.3 Theorem. (Mañé)[39]

For every static class Γ, the projection $\pi : \Gamma_0^+ \to M$ is injective with Lipschitz inverse.

The projection $\pi : \Gamma^+ \to M$ may not be surjective. But when M is compact for generic lagrangians $\pi(\Gamma^+) = \pi(\Gamma^-) = M$ because there is only one static class (cf. theorem 7-0.1.(B)). But π may not be injective on $\Gamma^+ \setminus \Gamma_0^+$ even for generic lagrangians.

Proof: We prove that for K as in lemma 3-8.2, if $v\,, w \in \Gamma_0^+$ then

$$d_{TM}(v, w) \leq K\, d_M(\pi(v), \pi(w)). \tag{3.15}$$

Suppose it is false. Then there are v, $w \in \Gamma_0^+$ such that inequality (3.15) does not hold. Let $\varepsilon > 0$ be such that $x_v|_{[-\varepsilon,+\infty[}$ and $x_w|_{[-\varepsilon,+\infty[}$ are semistatic. By lemma 3-8.2, there exist $\alpha \in \mathcal{C}_{2\varepsilon}(x_v(-\varepsilon), x_w(\varepsilon))$ and $\beta \in \mathcal{C}_{2\varepsilon}(x_w(-\varepsilon), x_v(\varepsilon))$ such that

$$A_{L+c}(\alpha) + A_{L+c}(\beta) + \delta < A_{L+c}(x_v|_{[-\varepsilon,\varepsilon]}) + A_{L+c}(x_w|_{[-\varepsilon,\varepsilon]}),$$

for some $\delta > 0$. Let p, $q \in \pi(\Gamma)$ and $s_n\, t_n \to +\infty$ be such that $x_v(s_n) \to +\infty$ and $x_v(s_n) \to p$, $x_w(t_n) \to q$. Then

$$\begin{aligned}\Phi_c(x_v(-\varepsilon), x_w(t_n)) + \Phi_c(x_w(-\varepsilon), x_v(s_n)) + \delta \qquad\qquad&\\ \leq A_{L+c}(\alpha * x_w|_{[\varepsilon,t_n]}) + A_{L+c}(\beta * x_v|_{[\varepsilon,s_n]}) + \delta&\\ < A_{L+c}(x_v|_{[-\varepsilon,s_n]}) + A_{L+c}(x_w|_{[-\varepsilon,t_n]})&\\ = \Phi_c(x_v(-\varepsilon), x_v(s_n)) + \Phi_c(x_w(-\varepsilon), x_w(t_n)).&\end{aligned}$$

Letting $n \to \infty$ and adding $d_c(p, q) = 0$, we have that

$$\begin{aligned}\Phi_c(x_v(-\varepsilon), p) + \Phi_c(x_w(-\varepsilon), q) \qquad\qquad&\\ \leq \Phi_c(x_v(-\varepsilon), q) + \Phi_c(x_w(-\varepsilon), p) + \Phi_c(q, p) + \Phi_c(p, q)&\\ < \Phi_c(x_v(-\varepsilon), p) + \Phi_c(x_w(-\varepsilon), q).&\end{aligned}$$

$\square$

3-9 Coboundary Property.

The coboundary property was first presented by R. Mañé in [38] and further developed in [39] and by A. Fathi.

3-9.1 Theorem. (Mañé) [39]

If $c = c(L)$, then $(L+c)\big|_{\widehat{\Sigma}}$ is a Lipschitz coboundary. More precisely, taking any $p \in M$ and defining $G : \widehat{\Sigma} \to \mathbb{R}$ by

$$G(w) = \Phi_c\big(p, \pi(w)\big),$$

then

$$(L+c)\big|_{\widehat{\Sigma}} = \frac{dG}{df}\,,$$

where

$$\frac{dG}{d\varphi}(w) := \lim_{h\to 0}\ \frac{1}{h}\big[\, G(\varphi_h(w) - G(w)\,\big]\,.$$

Proof: Let $w \in \widehat{\Sigma}$ and define $F_w(v) := \Phi_c(\pi(w), \pi(v))$. We have that

$$\begin{aligned}
\frac{dF_w}{d\varphi}\bigg|_w &= \lim_{h\to 0}\ \tfrac{1}{h}\left[\, F_w(\varphi_h\, w) - F_w(w)\,\right] \\
&= \lim_{h\to 0}\ \tfrac{1}{h}\left[\, \Phi_c(\pi w, \pi\varphi_h\, w) - \Phi_c(\pi w, \pi w)\,\right] \\
&= \lim_{h\to 0}\ \tfrac{1}{h}\, S_{L+c}\left(x_w|_{[0,h]}\right) \\
&= \lim_{h\to 0}\ \frac{1}{h}\int_0^h \big[L\big(x_w(s), \dot{x}_w(s)\big)\ + c\big]\, ds \\
&= L(w) + c.
\end{aligned}$$

We claim that for any $p \in M$ and any $w \in \widehat{\Sigma}$, $h \in \mathbb{R}$,

$$\begin{aligned}
G(\varphi_h\, w) &= \Phi_c\big(p, \pi(\varphi_h\, w)\big) = \Phi_c\big(p, \pi(w)\big) + \Phi_c\big(\pi(w), \pi(\varphi_h\, w)\big) \\
G(\varphi_h\, w) &= \Phi_c\big(p, \pi(w)\big) + F_w\big(\varphi_h(w)\big). \qquad (3.16)
\end{aligned}$$

This is enough to prove the theorem because then

$$\frac{dG}{d\varphi}\bigg|_w = \frac{d}{dh} F_h(\varphi_h\, w)\bigg|_{h=0} = \frac{F_w}{d\varphi}\bigg|_w = L(w)\, + c,$$

and G is Lipschitz by proposition 2-1.1.

We now prove (3.16). Let $q := \pi(w)$, $x := \pi(\varphi_h\, w)$. We have to prove that

$$\Phi_c(p,x) = \Phi_c(p,q) + \Phi_c(q,x). \tag{3.17}$$

Since the points q and x can be joined by the static curve $x_w|_{[0,h]}$, then

$$\Phi_c(x,q) = -\Phi_c(q,x).$$

Using twice the triangle inequality for Φ_c we get that

$$\Phi_c(p,q) \le \Phi_c(p,x) + \Phi_c(x,q) = \Phi_c(p,x) - \Phi_c(q,x) \le \Phi_c(p,q).$$

This implies (3.17). $\square$

3-10 Covering Properties.

3-10.1 Theorem. $\pi(\Sigma^+(L)) = M$.

But in general $\pi : \Sigma^+(L) \to M$ is not injective.

Proof: First suppose that $\widehat{\Sigma}(L) \neq \varnothing$. Take $p \in \pi(\widehat{\Sigma})$. Given $x \in M \setminus \pi(\widehat{\Sigma})$, take a Tonelli minimizer $\gamma_n \in \mathcal{C}_{T_n}(x,p)$ such that

$$A_{L+c}(\gamma_n) < \Phi_c(x,p) + \tfrac{1}{n}.$$

By the a priori bounds 3-2.3, $|\dot\gamma_n| < A$ and $T_n > \frac{1}{A}\, d(x,p)$. Let $v = \lim_k \dot\gamma_{n_k}(0)$ be an accumulation point of $\langle\dot\gamma_n(0)\rangle$. Let $\eta(t) := \pi\,\varphi_t(v)$. Then, if $0 < s < \liminf_k T_{n_k}$, we have that

$$\begin{aligned} A_{L+c}\big(\eta|_{[0,s]}\big) &= \lim_k A_{L+c}\big(\gamma_{n_k}|_{[0,s]}\big) \\ &\le \lim_k \big[\, \Phi_c\big(\gamma_{n_k}(0), \gamma_{n_k}(s)\big) + \tfrac{1}{n_k} \,\big] \\ &= \Phi_c\big(\eta(0), \eta(s)\big). \end{aligned}$$

Then η is semistatic on $[0,S]$, where $S = \liminf_k T_{n_k}$. If $S < +\infty$ then $\eta(S) = \lim_k \gamma_{n_k}(T_k) = p$. Since $x \notin \pi(\widehat{\Sigma})$, this contradicts the graph property 3-8.1; hence $S = +\infty$. Thus $\eta|_{[0,+\infty[}$ is semistatic and $v \in \Sigma^+$.

If $\widehat{\Sigma} = \varnothing$, then by corollary 3-6.2, M is non-compact. Let $x \in M$ and $\langle y_n\rangle \subseteq M$ such that $d_M(x,y_n) \to +\infty$. Let $\gamma_n \in \mathcal{C}_{T_n}(x,y_n)$ be a Tonelli minimizer such that

$$A_{L+c}(\gamma_n) < \Phi_c(x,y_n) + \tfrac{1}{n}.$$

Then by lemma 3-2.3, $|\dot\gamma_n| < A$, and hence $T_n \to +\infty$. The rest of the proof is similar to the case above. □

By corollary 3-5.3 $E(\Sigma^+) = c(L)$, using (1.5) we get that $c(L) \ge e_0$ and then

$$e_0 \le c_u \le c_a \le c_0 \le c(L). \tag{3.18}$$

3-11 Recurrence Properties.

Let $\mathbf{\Lambda}$ be the set of static classes. Define a reflexive partial order $\preccurlyeq$ in $\mathbf{\Lambda}$ by

(a) $\preccurlyeq$ is reflexive.

(b) $\preccurlyeq$ is transitive.

(c) If there is $v \in \Sigma$ with the α-limit set $\alpha(v) \subseteq \Lambda_i$ and ω-limit set $\omega(v) \subseteq \Lambda_j$, then $\Lambda_i \preccurlyeq \Lambda_j$.

3-11.1 Theorem.

Suppose that M is compact and the number of static classes is finite. Then given Λ_i and Λ_j in $\mathbf{\Lambda}$, we have that $\Lambda_i \preccurlyeq \Lambda_j$.

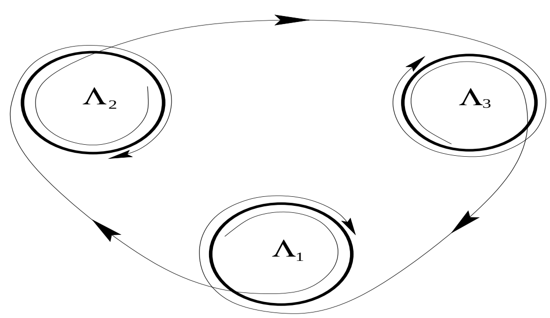

FIG. 2: CONNECTING ORBITS BETWEEN STATIC CLASSES.
The three closed curves represent the static classes and the other curves represent semistatic orbits connecting them.

Theorem 3-11.1 could be restated by saying that if the cardinality of $\mathbf{\Lambda}$ is finite, then given two static classes Λ_i and Λ_j there exist classes $\Lambda_i = \Lambda_1, \ldots, \Lambda_n = \Lambda_j$ and semistatic vectors $v_1, \ldots, v_{n-1} \in \Sigma$ such that for all $1 \leq k \leq n-1$ we have that $\alpha(v_k) \subseteq \Lambda_k$ and $\omega(v_k) \subseteq \Lambda_{k+1}$. In other words, between two static classes there exists a chain of static classes connected by heteroclinic semistatic orbits (cf. figure 2).

A proof of the following theorem can be found in [12]

3-11.2 Theorem. *If M is compact, then*

1. *$\Sigma(L)$ is chain transitive.*
2. *$\widehat{\Sigma}(L)$ is chain recurrent.*

Now we proceed to prove theorem 3-11.1. Assume for the rest of this section that M is compact.

3-11.3 Proposition.
If $v \in \Sigma$ is semistatic, then $\alpha(v) \subset \widehat{\Sigma}(L)$ and $\omega(v) \subset \widehat{\Sigma}(L)$. Moreover $\alpha(v)$ and $\omega(v)$ are each contained in a static class.

Proof: We prove only that $\omega(v) \subset \widehat{\Sigma}$. Let $\gamma(t) = \pi\,\varphi_t(v)$. Suppose that $t_n \to +\infty$ and $\dot\gamma(t_n) \to w \in TM$. Let $\eta(t) = \pi\,\varphi_t(w)$. Since γ and η are solutions the Euler-Lagrange equation, then $\gamma|_{[t_n-s,t_n+s]} \xrightarrow[C^1]{} \eta|_{[-s,s]}$. Then

$$
\begin{aligned}
A_{L+c}\big(\eta|_{[-s,s]}\big) + \Phi_c(\eta(s),\eta(-s)) = \\
&= \lim_n \big\{ A_{L+c}(\gamma|_{[t_n-s,t_n+s]}) + \lim_m A_{L+c}(\gamma|_{[t_n+s,t_m-s]}) \big\} \\
&= \lim_n \lim_m A_{L+c}(\gamma|_{[t_n-s,t_m-s]}) \\
&= \lim_n \lim_m \Phi_c(\gamma(t_n-s),\gamma(t_m-s)) \\
&= \Phi_c(\eta(-s),\eta(-s)) = 0.
\end{aligned}
$$

Thus $w \in \widehat{\Sigma}(L)$. Let $u \in \omega(v)$. We may assume that $\dot\gamma(s_n) \to u$ with $t_n < s_n < t_{n+1}$. Then

$$
\begin{aligned}
d_c(\pi w,\pi u) &= \Phi_c(\pi w,\pi u) + \Phi_c(\pi u,\pi w) \\
&= \lim_n A_{L+c}(\gamma|_{[t_n,s_n]}) + A_{L+c}(\gamma|_{[s_n,t_{n+1}]}) \\
&= \lim_n A_{L+c}(\gamma_n|_{[t_n,t_{n+1}]}) = \Phi_c(\pi w,\pi w) = 0.
\end{aligned}
$$

Thus w and u are in the same static class. ☐

3-11.4 Proposition. *Every static class is connected.*

Proof: Let Λ be a static class and suppose that it is not connected. Let U_1, U_2 be disjoint open sets such that $\Lambda \subseteq U_1 \cup U_2$ and $\Lambda \cap U_i \neq \varnothing$, $i = 1, 2$. Let $p_i \in \pi(U_i \cap \Lambda)$, $i = 1, 2$. Since U_1 and U_2 are disjoint sets we can take a solution $x_{v_n} : [a_n, b_n] \to M$, $a_n < 0 < b_n$ of (E-L) such that $x_{v_n}(0) \notin \pi(U_1 \cup U_2)$, $x_{v_n}(a_n) = p_1$, $x_{v_n}(b_n) = p_2$ and

$$A_{L+c}(x_{v_n}) \leq \Phi_c(p_1, p_2) + \tfrac{1}{n}. \tag{3.19}$$

Let u be a limit point of v_n, then $x_u : \mathbb{R} \to M$ is semistatic (see the proof of claim 2 item (a)). Then, for $a_n \leq s \leq t \leq b_n$,

$$d_c(p_1, p_2) \leq \Phi_c(p_1, x_{v_n}(s)) + \Phi_c(x_{v_n}(s), x_{v_n}(t)) + \Phi_c(x_{v_n}(t), p_2) + \Phi_c(p_2, p_1),$$

therefore

$$\begin{aligned} d_c(p_1, p_2) &\leq \Phi_c(p_2, p_1) \\ &\qquad + \liminf_n \left[\Phi_c(p_1, x_{v_n}(s)) + \Phi_c(x_{v_n}(s), x_{v_n}(t)) + \Phi_c(x_{v_n}(t), p_2)\right] \\ &\leq \Phi_c(p_2, p_1) + \liminf_n A_{L+c}(x_{v_n}) \\ &\leq d_c(p_1, p_2) = 0, \end{aligned}$$

where in the last inequality we used (3.19). Hence

$$\Phi_c(p_1, x_u(s)) + \Phi_c(x_u(s), x_u(t)) + \Phi_c(x_u(t), p_2) + \Phi_c(p_2, p_1) = 0.$$

Combining the last equation with the triangle inequality we obtain

$$\begin{aligned} &d_c(x_u(s), x_u(t)) \leq \\ &\quad \leq \Phi_c(x_u(s), x_u(t)) + \left[\Phi_c(x_u(t), p_2) + \Phi_c(p_2, p_1) + \Phi_c(p_1, x_u(s))\right] = 0. \end{aligned}$$

So that $u \in \widehat{\Sigma}$. Moreover, for $s = 0, t = 1$:

$$\begin{aligned} &d_c(x_u(0), p_1) \leq \\ &\quad \leq \Phi_c(p_1, x_u(0)) + \left[\Phi_c(x_u(0), x_u(1)) + \Phi_c(x_u(1), p_2) + \Phi_c(p_2, p_1)\right] = 0. \end{aligned}$$

Hence $x_u(0) \in \pi(\Lambda)$. On the other hand $x_u(0) \notin \pi(U_1 \cup U_2)$. This contradicts the fact that $\Lambda \subseteq U_1 \cup U_2$. $\square$

Proof of theorem 3-11.1.

Given $v \in TM$ denote by $\alpha(v)$ and $\omega(v)$ its α and ω-limits respectively. By proposition 3-11.4 the static classes are connected. Hence if we assume that there are only finitely many of them, the connected components of $\widehat{\Sigma}$ are finite and must coincide with the static classes. For $\varepsilon > 0$, let $\widehat{\Sigma}(\varepsilon)$ be the ε-neighborhood of $\widehat{\Sigma}$, i.e.

$$\widehat{\Sigma}(\varepsilon) := \{\, v \in TM \mid d_{TM}(v, \widehat{\Sigma}) < \varepsilon \,\}.$$

Fix $\varepsilon > 0$ small enough such that the connected components of $\widehat{\Sigma}(\varepsilon)$ are the ε-neighborhoods of the static classes. So that for $0 < \delta < \varepsilon$, $\widehat{\Sigma}(\delta) = \sum_{i=1}^{N(\varepsilon)} \Lambda_i(\delta)$, where $\Lambda_i(\delta)$ are disjoint open sets containing exactly one static class and the number of components $N(\varepsilon)$ is fixed for all $0 < \delta < \varepsilon$.

Now suppose that the theorem is false. Let Λ_i, $\Lambda_k \in \mathbf{\Lambda}$ be such that $\Lambda_i \not\preccurlyeq \Lambda_k$. Let

$$\mathbb{A} := \bigcup_{\{\, \Lambda_j \in \mathbf{\Lambda} \,|\, \Lambda_i \preccurlyeq \Lambda_j \,\}} \Lambda_j \quad , \quad \mathbb{B} := \bigcup_{\{\, \Lambda_j \in \mathbf{\Lambda} \,|\, \Lambda_i \not\preccurlyeq \Lambda_j \,\}} \Lambda_j .$$

Given $v \in \Sigma$ with $\alpha(v) \subseteq \mathbb{A}$ and $0 < \delta < \varepsilon$, define inductively $s_k(v)$, $t_k(v)$, $T_k(v)$ as follows. Let

$$s_1(v) := \inf\{\, s \in \mathbb{R} \mid f_s(v) \notin \mathbb{A}(\varepsilon) \,\} \in \mathbb{R} \cup \{+\infty\}.$$

If $s_k(v) < +\infty$, $k \geq 1$, define

$$\begin{aligned} t_k(v) &:= \sup\{\, t < s_k(v) \mid f_t(v) \in \mathbb{A}(\delta) \,\}, \\ T_k(v) &:= \inf\{\, t > s_k(v) \mid f_t(v) \in \mathbb{A}(\delta) \,\}. \end{aligned}$$

Observe that $s_k(v) < +\infty$ implies that $T_k(v) < +\infty$ because by the definition of $\mathbb{B}$ and the transitivity of $\preccurlyeq$ we have that $\omega(v) \subseteq \mathbb{A}$. Define

$$A_k = A_k(\delta) := \sup\{\, |\, T_k(v) - t_k(v) \,| \ : \ v \in \Sigma,\ \alpha(v) \subseteq \mathbb{A},\ s_k(v) < +\infty \,\},$$

if $s_k(v) = +\infty$ for all $v \in \Sigma$ with $\alpha(v) \subseteq \mathbb{A}$, write $A_\ell(\delta) \equiv 0$ for all $\ell \geq k$. Now set:

$$s_{k+1}(v) := \inf\{\, s > T_k(v) \,|\, f_t(v) \notin \mathbb{A}(\varepsilon) \,\}.$$

Observe that $s_k(v)$, $t_k(v)$ and $T_k(v)$ are invariant under f_t.

We split the rest of the proof of theorem 3-11.1 into the following claims:

Claim 1. $A_k(\delta) < +\infty$ *for all* $k = 1, 2, \dots$ *and all* $0 < \delta < \varepsilon$.

Define

$$\mathbb{M} := \{\, v \,|\; v \in \Sigma,\ \alpha(v) \subseteq \mathbb{A} \,\}.$$

Claim 2.

(a) $\overline{\mathbb{M}} \cap \mathbb{B} \neq \varnothing$.

(b) $\limsup_k A_k(\delta) = \sup_k A_k(\delta) = +\infty$.

Claim 3. *There exist sequences* $v_n \in \Sigma$, $0 < s_n < t_n$ *such that* $v_n \to u_1 \in \mathbb{A}$, $f_{s_n}(v_n) \to u_2 \notin \mathbb{A}(\varepsilon)$, $f_{t_n}(v_n) \to u_3 \in \mathbb{A}$ *and* $d_c(\pi u_1, \pi u_3) = 0$.

We now use claim 3 to complete the proof of theorem 3-11.1. If $u_1 \in \Lambda_j \subseteq \mathbb{A}$, we shall prove that $u_2 \in \Lambda_j \setminus \mathbb{A}(\varepsilon)$, obtaining a contradiction and thus proving theorem 3-11.1. It is enough to show that $d_c(\pi u_1, \pi u_2) = 0$. Indeed

$$\begin{aligned}
d_c(\pi u_1, \pi u_2) &= \Phi_c(\pi u_1, \pi u_2) + \Phi_c(\pi u_2, \pi u_1) \\
&\leq \Phi_c(\pi u_1, \pi u_2) + \Phi_c(\pi u_2, \pi u_3) + \Phi_c(\pi u_3, \pi u_1) \\
&\leq \lim_n \big[\Phi_c(\pi v_n, \pi f_{s_n}(v_n)) + \Phi_c(\pi f_{s_n}(v_n), \pi f_{t_n}(v_n))\big] \\
&\qquad\qquad\qquad\qquad + \Phi_c(\pi u_3, \pi u_1) \\
&= \lim_n \Phi_c(\pi v_n, \pi f_{t_n}(v_n)) + \Phi_c(\pi u_3, \pi u_1) \\
&= d_c(\pi u_1, \pi u_3) = 0,
\end{aligned}$$

where the fourth equation holds because v_n is a semistatic vector.

$\square$

Proof of claim 1:

Suppose that $A_i < +\infty$ for $i = 1, \dots, k-1$ and $A_k = +\infty$. The case $k = 1$ is similar. Then there exists $v_n \in \Sigma$, with $\alpha(v_n) \subset \mathbb{A}$ and $T_k(v_n) - t_k(v_n) \to +\infty$. We can assume that $t_k(v_n) = 0$ and that v_n converges (Σ is compact). Let $u = \lim_n v_n \in \partial\mathbb{A}(\delta)$. Then for all n, we have that

$$m\{\, t < 0 \,|\, f_t(v_n) \notin \mathbb{A}(\varepsilon) \,\} \le \sum_{i=1}^{k-1} A_i\,,$$

where m is the Lebesgue measure on $\mathbb{R}$. This implies that

$$m\{\, t < 0 \,|\, f_t(u) \notin \mathbb{A}(\varepsilon) \,\} \le \sum_{i=1}^{k-1} A_i$$

and hence $\alpha(u) \cap \overline{\mathbb{A}(\varepsilon)} \neq \varnothing$. By proposition 3-11.3, $\alpha(u) \subset \mathbb{A}$. Since $f_t(v_n) \notin \mathbb{A}(\varepsilon)$ for $0 < t < T_k(v_n)$ and $T_k(v_n) \to +\infty$, then $f_t(u) \notin \mathbb{A}(\varepsilon)$ for all $t > 0$ and hence $\omega(u) \subseteq \mathbb{B}$. But then the orbit of u contradicts the definition of $\mathbb{B}$.

$\square$

Proof of claim 2:

(a) Let $p \in \pi\mathbb{A}$, $q \in \pi\mathbb{B}$. For $n > 0$, let $x_{v_n} : [a_n, b_n] \to M$ be a solution of (E-L) such that $x_{v_n}(a_n) = p$, $x_{v_n}(b_n) = q$ and

$$A_{L+c}(x_{v_n}) \le \Phi_c(p, q) + \tfrac{1}{n}.$$

This implies that

$$A_{L+c}\left(x_{v_n}|_{[s,t]}\right) \le \Phi_c\big(x_{v_n}(s), x_{v_n}(t)\big) + \tfrac{1}{n} \tag{3.20}$$

for all $a_n \le s \le t \le b_n$. We can assume that

$$\inf\{\, s > a_n \,|\, x_{v_n}(s) \in \mathbb{B}(\delta) \,\} = 0,$$

and that the sequence v_n converges (cf. lemma 3-2.3). Let $u = \lim_n v_n \in \pi^{-1}(\partial\, \pi\, \mathbb{B}(\delta))$. Taking limits in (3.20) we obtain that $x_u|_{[s,t]}$ is semistatic for all $\liminf_n a_n \le s \le t \le \limsup_n b_n$.

Any limit point w of $\dot{x}_{v_n}(a_n) = f_{a_n}(v_n)$ satisfies $\pi(w) = p \in \pi\mathbb{A}$, and by the graph property (theorem 3-8.1), $w \in \mathbb{A}$. Similarly, any limit point of $f_{b_n}(v_n)$ is in $\mathbb{B}$. Since $\mathbb{A} \cup \mathbb{B}$ is invariant and $u \notin \mathbb{A} \cup \mathbb{B}$, then $\lim_n a_n = -\infty$, $\lim_n b_n = +\infty$. Hence $u \in \Sigma$. Since $f_t(v_n) \notin \mathbb{B}(\delta)$ for all $a_n \le t < 0$ and $a_n \to -\infty$, then $f_t(u) \notin \mathbb{B}(\delta)$ for all $t < 0$. Hence $\alpha(u) \subseteq \mathbb{A}$ and thus $u \in \mathbb{M}$. Since $u \in \pi^{-1}(\partial\, \pi\, \mathbb{B}(\delta))$ there exists $z \in \mathbb{B}$ such that $d_M(\pi(u), \pi(z)) \le \delta$. Since $z \in \widehat{\Sigma}$ and $u \in \Sigma$, by theorem 3-8.1 we have that

$$d_{TM}\big((\pi(u), u), (\pi(z), z)\big) \le K\,\delta.$$

Thus $u \in \mathbb{M} \cap \mathbb{B}(K\delta)$. Letting $\delta \to 0$, we obtain that $\overline{\mathbb{M}} \cap \mathbb{B} \neq \varnothing$.

(b) By claim 1 it is enough to show that $\sup_k A_k(\delta) = +\infty$. If $\sup_k A_k(\delta) < T$, then $\mathbb{M} \subseteq \mathbb{M}(\delta, T)$, where

$$\mathbb{M}(\delta, T) = \{\, v \in \Sigma \mid f_{[-T,T]}(v) \cap \mathbb{A}(\delta) \neq \varnothing \,\}.$$

Then $\mathbb{M} \cap \mathbb{B} \subseteq \mathbb{M}(\delta, T) \cap \mathbb{B} = \varnothing$, because $\mathbb{B}$ is invariant and $\mathbb{B} \cap \mathbb{A}(\delta) = \varnothing$. This contradicts item (a).

□

Proof of claim 3:

Given $0 < \delta < \varepsilon$, by claim 2(b) there exists $k > N(\varepsilon)$ such that $A_k(\delta) > 0$. Hence there is $v = v_\delta \in \Sigma$ with $\alpha(v) \subset \mathbb{A}$, such that the orbit of v leaves $\mathbb{A}(\varepsilon)$ and returns to $\mathbb{A}(\delta)$ at least k times. Since $k > N(\varepsilon)$ there is one component $\Lambda_j(\delta) \subseteq \mathbb{A}(\delta)$ with two of these returns, i.e. there exist $\tau_1(\delta) < s(\delta) < \tau_2(\delta)$ with $f_{\tau_1}(v) \in \Lambda_j(\delta)$, $f_s(v) \notin \mathbb{A}(\varepsilon)$ and $f_{\tau_2}(v) \in \Lambda_j(\delta)$. We can choose v_δ so that $\tau_1(\delta) = 0$. Now, there exists a sequence such that the repeated component $\Lambda_j \subset \Lambda_j(\delta_n)$ is always the same. Let $s_n := s(\delta_n)$, $t_n := \tau_2(\delta_n)$ and choose a subsequence such that v_n, $f_{s_n}(v_n)$ and $f_{t_n}(v_n)$ converge. Let $u_1 = \lim_n v_n \in \cap_n \Lambda_j(\delta_n) = \Lambda_j$, $u_3 = \lim_n f_{t_n}(v_n) \in \Lambda_j$ and $u_2 = \lim_n f_{s_n}(v_n) \notin \mathbb{A}(\varepsilon)$. Since $u_1, u_3 \in \Lambda_j$, then $d_c(\pi u_1, \pi u_3) = 0$.

□

Chapter 4

The Hamiltonian viewpoint.

4-1 The Hamilton-Jacobi equation.

Let ω be the canonical symplectic form on T^*M. A subspace λ of T_pT^*M is called *isotropic* if $\omega(X,Y)=0$ for all X,Y on λ. Since ω is nondegenerate, the isotropic subspaces have dimension $\leq n$, half of the dimension of T^*M. Isotropic spaces of dimension n are called *lagrangian* subspaces We say that a submanifold $W \subset T^*M$ is *lagrangian* if at each point $\theta \in W$, its tangent space $T_\theta W$ is a lagrangian subspace of $T_\theta T^*M$. In particular, $\dim W = \dim M = n$.

4-1.1 Theorem (Hamilton-Jacobi)**.**
If the hamiltonian H is constant on a lagrangian submanifold N, then N is invariant under the hamiltonian flow.

Proof: We only have to show that the hamiltonian vector field X is tangent to N. Since H is constant on N, then $dH|_{TN} \equiv 0$. Since $\omega(X,\cdot) = dH$, then $\omega(X,\xi)=0$ for all $\xi \in TN$. Since the tangent spaces to N are lagrangian, they are maximal isotropic subspaces, therefore $X \in TN$. $\square$

Some distinguished n-dimensional manifolds on T^*M are the graph submanifolds, which are of the form

$$G_\eta = \{\ (x, \eta_x) \,|\, x \in M\ \} \subset T^*M, \tag{4.1}$$

where η_x is a 1-form on M. A *lagrangian graph* is a lagrangian graph submanifold. In fact,

4-1.2 Lemma. *G_η is a lagrangian graph if and only if the form η is closed:*

$$G_\eta \textit{ is lagrangian } \iff d\eta \equiv 0$$

Proof: Choose local coordinates $q_1, \ldots, q_n$ of M. Then $\eta(q) = \sum_k p_k(q)\, dq_k$. A basis of the tangent space to the graph G_η is given by $E_i = \left(\frac{\partial}{\partial q_i}, \sum_k \frac{\partial p_k}{\partial q_i} \frac{\partial}{\partial p_k}\right)$. Applying $\omega = dp \wedge dq$,

$$\omega(E_i, E_j) = \tfrac{\partial p_i}{\partial q_j} - \tfrac{\partial p_j}{\partial q_i}.$$

Since

$$d\eta = \sum_{i<j} \left(\tfrac{\partial p_i}{\partial q_j} - \tfrac{\partial p_j}{\partial q_i}\right)\, dq_j \wedge dq_i\,,$$

then $\omega|_{TG_\eta} \equiv 0 \iff d\eta \equiv 0$.

□

Thus, we can associate a cohomology class $[\eta] \in H^1(M, \mathbb{R})$ to each lagrangian graph G_η. Lagrangian graphs with zero cohomology class are the graphs of the exact 1-forms: G_{df}, with $\eta = df$ and $f : M \to \mathbb{R}$ a smooth function. These are called *exact lagrangian graphs.*

The Hamilton-Jacobi equation for autonomous hamiltonians is

$$H(x, d_x u) = k, \qquad u : M \to \mathbb{R}. \tag{H-J}$$

Thus a smooth solution of the Hamilton-Jacobi equation corresponds to an exact invariant lagrangian graph.

4-2 Dominated functions.

We say that a function u is *dominated* by $L+k$, and write $u \prec L+k$ if

$$u(y) - u(x) \leq \Phi_k(x,y) \quad \text{for all } x, y \in M.$$

The triangle inequality implies that the functions $u(x) = \Phi_k(y,x)$ and $v(x) = -\Phi_k(x,y)$ are dominated, for any $y \in M$.

4-2.1 Lemma.

1. *If $u \prec L+k$, then u is Lipschitz with the same Lipschitz constant as Φ_c. In particular, a family of dominated functions is equicontinuous.*
2. *If $u \prec L+k$ then $H(x, d_x u) \leq k$ at any differentiability point x of u.*

Proof:

1. We have that $u(y) - u(x) \leq \Phi_c(x,y) \leq A\, d_M(x,y)$, where A is a Lipschitz constant for Φ_c. Changing the roles of x and y, we get that u is Lipschitz.

2. We have that

$$u(y) - u(x) \leq \int_\gamma L(\gamma, \dot\gamma) + k$$

for all curves $\gamma \in \mathcal{C}(x,y)$. This implies that

$$d_x u \cdot v \leq L(x,v) + k$$

for all $v \in T_x M$ when u is differentiable at $x \in M$. Since

$$H(x, d_x u) = \sup\{\, d_x u \cdot v - L(x,v) \,|\, v \in T_x M \,\},$$

then $H(x, d_x u) \leq k$. □

4-2.2 EXERCISES:

1. If $u : M \to \mathbb{R}$ is differentiable, then
$$H(x, d_x u) \leq k \Longleftrightarrow u \prec L + k.$$

2. Fix $x_0 \in M$. For $k > c(L)$ choose $f_k \in C^\infty(M, \mathbb{R})$ such that $f(x_0) = 0$ and $H(df_k) < k$ (cf. 4-4.4). Let $u(x) := \limsup_{k \to c(L)} f_k(x)$.
Then $u < +\infty$, u is Lipschitz and $H(x, d_x u) \leq c(L)$ for a.e. $x \in M$.

4-2.3 Definition. Given a dominated function $u \prec L + k$, we say that an absolutely continuous curve $\gamma : [a, b] \to M$ *realizes* u, if

$$u(\gamma(t)) - u(\gamma(s)) = A_{L+k}(\gamma|_{[s,t]}), \qquad \text{for all } a \leq s \leq t \leq b. \tag{4.2}$$

Observe that such realizing curves must be global minimizers. In particular for $k = c(L)$, they are semistatic.

The following proposition shows that we actually get a solution of (H-J) if there are (semistatic) curves which realize a dominated function u.

4-2.4 Proposition. *Suppose that $u \prec L + k$.*

1. *If $\gamma :]-\varepsilon, \varepsilon[\to M$ realizes u, then u is differentiable at $\gamma(0)$.*

2. *If $\gamma :]-\varepsilon, 0] \to M$ or $\gamma : [0, \varepsilon[\to M$ realizes u and u is differentiable at $x = \gamma(0)$, then $d_x u = L_v(x, \dot\gamma(0))$ and $H(x, d_x u) = k$.*

4-2.5 Remarks.

1. Equation $d_x u = L_v(x, \dot\gamma(0))$ means that the tangent vector $(x, \dot\gamma(0))$ of any a.c. curve γ realizing u is sent by the Legendre transform to $d_x u$.

2. In particular, since the functions $u(x) = \Phi_k(p, x)$ (resp. $v(x) = -\Phi_c(x, p)$) are dominated, then they are differentiable at any point which is not at the (backward) (resp. forward) $(L + k)$-cut locus of p.

3. Observe that the energy $E(x,\dot\gamma(0)) = H(x,d_xu)$. In proposition 4-9.7, we show that any semistatic orbit realizes some dominated function. Thus we obtain another proof for $\Sigma\subset E^{-1}\{c\}$, i.e. that the semistatic orbits have energy $c(L)$.

Proof: *1.* Let $w\in T_xM$ and let $\eta(s,t)$ be a variation of γ fixing the endpoints $\gamma(-\varepsilon)$, $\gamma(\varepsilon)$ such that $\eta(0,t)=\gamma(t)$ and $\frac{\partial}{\partial s}\eta(0,0)=w$. Define

$$\mathcal{A}(s) := \int_{-\varepsilon}^0 L(\tfrac{\partial}{\partial t}\eta(s,t)) + k \;\; dt.$$

Then, integrating by parts and using the Euler-Lagrange equation (E-L),

$$\mathcal{A}'(0) = L_v\;\xi|_{-\varepsilon}^0 + \int_{-\varepsilon}^0 \left[L_x - \tfrac{d}{dt}L_v\right]\,\xi\;dt = L_v(x,\dot\gamma(0))\cdot w\,,$$

where $\xi(t):=\frac{\partial}{\partial s}\eta(0,t)$. Also

$$\begin{aligned}\tfrac{1}{s}\big[\,u(\eta(s,0))-u(x)\,\big] &= \tfrac{1}{s}\big[u(\eta(s,0))-u(\gamma(-\varepsilon))+u(\gamma(-\varepsilon))-u(\gamma(0))\,\big]\\ &\le \tfrac{1}{s}\,[\,\mathcal{A}(s)-\mathcal{A}(0)\,]\;,\end{aligned}$$

where we used that $u\prec L+k$ and (4.2). Hence

$$\limsup_{s\to 0}\tfrac{1}{s}\left[\,u(\eta(s,0))-u(x)\,\right]\le\mathcal{A}'(0). \tag{4.3}$$

Similarly, if $\mathcal{B}(s):=A_{L+k}\big(\eta(s,\cdot)|_{[0,\varepsilon]}\big)$, then

$$u(\gamma(\varepsilon))-u(\eta(s,0))-u(\gamma(\varepsilon))+u(x)\le\mathcal{B}(s)-\mathcal{B}(0),$$

$$\limsup_{s\to 0}\tfrac{1}{s}\left[\,u(x)-u(\eta(s,0))\,\right]\le\mathcal{B}'(0)=-L_v(x,\dot\gamma(0))\cdot w.$$

Hence

$$\liminf_{s\to 0}\tfrac{1}{s}\left[\,u(\eta(s,0))-u(x)\,\right]\ge L_v(x,\dot\gamma(0))\cdot w. \tag{4.4}$$

From (4.3) and (4.4) we get that u is differentiable at $x=\gamma(0)$. and $d_xu=L_v(x,\dot\gamma(0))$.

2. Now assume that $\gamma :]-\varepsilon, 0] \to M$ realizes u and that u is differentiable at $x = \gamma(0)$. The same argument as in 4.3 shows that

$$d_x u \cdot w \leq L_v(x, \dot\gamma(0)) \cdot w \qquad \text{for all } w \in T_x M.$$

Applying this inequality to $-w$ and combining both inequalities we get that $d_x u = L_v(x, \dot\gamma(0))$.

Now, since $u \prec L + k$, by lemma 4-2.1, $H(x, d_x u) \leq k$. Since

$$u(\gamma(0)) - u(\gamma(t)) = A_{L+k}(\gamma|_{[t,0]}) = \int_t^0 \big[\, L(\gamma(s), \dot\gamma(s)) + k \,\big]\, ds,$$

then

$$d_x u \cdot \dot\gamma(0) = L(\gamma(0), \dot\gamma(0)) + k.$$

Hence

$$H(x, d_x u) = \sup_{v \in T_x M} \{\, d_x u \cdot v - L(x, v) \,\} \geq k.$$

$\square$

4-3 Weak solutions of the Hamilton-Jacobi equation.

We shall see in corollary 4-4.7 that there are no weakly differentiable subsolutions of (H-J) for $k < c(L)$.

In the next proposition we show that when $k \geq c(L)$ there are always Lipschitz solutions of (H-J). On the other hand, in theorem 4-8.4 we show that when M is compact the only energy level that supports a differentiable solution is $k = c(L)$. **When M is non-compact there may be differentiable solutions on $k > c(L)$, as example ?? shows**.

4-3.1 Proposition. *If $k \geq c(L)$, then for any $y \in M$, the function $u(x) = \Phi_k(y, x)$ satisfies $H(x, d_x u) = k$ for a.e. $x \in M$.*

Proof: Since u is Lipschitz, by by Rademacher's theorem [19], it is differentiable at Lebesgue-almost every point. Since u is dominated by proposition 4-2.4.*2*, it is enough to see that u is one-sided realized at every point.

If $k > c(L)$, by proposition 3-5.1, for all $x \in M$, $x \neq y$, there exists a finite-time global minimizer $\gamma \in \mathcal{C}_T(y, x)$ with $A_{L+k}(\gamma) = \Phi_k(y, x)$. By the triangle inequality, the function $\delta(t) = A_{L+k}(\gamma|_{[0,t]}) - \Phi_k(y, \gamma(t))$ is increasing. Also $\delta(t) \geq 0$ and $\delta(T) = 0$. So that $\delta(t) \equiv 0$ and hence γ backward-realizes u at x.

If $k = c(L)$ then u may be realized by an infinite semistatic orbit as follows. Let $\gamma_n \in \mathcal{C}(y, x)$, $\gamma_n : [-T_n, 0] \to M$ be a Tonelli minimizer such that

$$\Phi_c(y, x) \leq A_{L+c}(\gamma_n) \leq \Phi_c(y, x) + \tfrac{1}{n}.$$

This implies that

$$\Phi_c(y, \gamma(s)) \leq A_{L+c}(\gamma_n|_{[-T_n,s]}) \leq \Phi_c(y, \gamma(s)) + \tfrac{1}{n},$$

for all $-T_n \leq s \leq 0$. Thus

$$\left|u(x) - u(\gamma_n(s)) - A_{L+c}(\gamma_n|_{[s,0]})\right| \leq \tfrac{1}{n}. \tag{4.5}$$

By proposition 3-2.3, $|\gamma_n| < A$ for all n. Hence, if $x \neq y$, $\liminf_n T_n > 0$. We can assume that $\dot\gamma_n(0) \to v \in T_xM$ and $T_n > \varepsilon > 0$. Let $\lambda(t) = \pi\,\varphi_t(v)$. Then $\gamma_n|_{[-\varepsilon,0]} \to \lambda|_{[-\varepsilon,0]}$ in the C^1-topology. Letting $n \to \infty$ on (4.5), we get that λ realizes u.

$\square$

4-4 Lagrangian graphs.

We say that a function $u : M \to \mathbb{R}$ is a *subsolution* of the Hamilton-Jacobi equation if

$$H(x, d_x u) \le k,$$

We shall prove that for $k > c(L)$ there is always a C^∞ subsolution of the Hamilton-Jacobi equation and for $k < c(L)$ there are no (weakly) differentiable subsolutions. Hence

4-4.1 Theorem. *If M is any covering of a closed manifold, then*

$$\begin{aligned} c(L) &= \inf_{f \in C^\infty(M,\mathbb{R})} \sup_{x \in M} H(x, d_x f) \\ &= \inf\{k \in \mathbb{R} : \text{ there exists } f \in C^\infty(M,\mathbb{R}) \text{ such that } H(df) < k\}. \end{aligned}$$

where H is the hamiltonian associated with L.

Theorem 4-4.1 could be restated by saying that $c(L)$ is the infimum of the values of $k \in \mathbb{R}$ for which $H^{-1}(-\infty, k)$ contains an exact lagrangian graph. This is a very geometric way of describing the critical value.

In exercise 4-2.2. there is an elementary construction of a weak subsolution for $k = c(L)$. Theorem 4-4.1 is an immediate consequence of lemma 4-4.2 and proposition 4-4.4 below.

4-4.2 Lemma. *If there exists a C^1 function $f : M \to \mathbb{R}$ such that $H(df) \le k$, then $k \ge c(L)$.*

Proof: Recall that

$$H(x, p) = \max_{v \in T_x M} \{p(v) - L(x, v)\}.$$

Since $H(df) \le k$ it follows that for all $(x, v) \in TM$,

$$d_x f(v) - L(x, v) \le k.$$

Therefore, if $\gamma : [0,T] \to M$ is any absolutely continuous closed curve with $T > 0$, we have

$$\int_0^T (L(\gamma,\dot\gamma)+k)\,dt = \int_0^T \big[\,L(\gamma,\dot\gamma)+k-d_\gamma f(\dot\gamma)\,\big]\,dt \ge 0,$$

and thus $k \ge c(L)$. $\square$

4-4.3 Remark. The utility of a differentiable subsolution of the Hamilton-Jacobi equation can be seen in lemma 4-4.2. If $H(du) \le c(L)$, then the lagrangian L can be replaced by the lagrangian

$$\mathbb{L}(x,v) := L(x,v) - d_x u(v) + c(L) \ge 0$$

The new lagrangian $\mathbb{L}$ is positive, has the same minimizing measures as L, its α and β functions are translates of those for L. The static set $\widehat{\Sigma}(\mathbb{L})$ is contained in the level set $\mathbb{L} = 0$.

4-4.4 Proposition.
For any $k > c(L)$ there exists $f \in C^\infty(M,\mathbb{R})$ such that $H(df) < k$.

Proof: Set $c = c(L)$. Fix $q \in M$ and let $u(x) := \Phi_c(q,x)$. By the triangle inequality, $u \prec L + c$. By lemma 4-2.1.*2*, $H(d_x u) \le c$ at any point $x \in M$ where $u(x)$ is differentiable.

We proceed to regularize u. Since u is Lipschitz, by Rademacher's theorem (cf. [19]) it is differentiable at Lebesgue almost every point. Moreover it is weakly differentiable (cf. [19, Section 4.2.3]), that is, for any C^∞ function $\varphi : U \to \mathbb{R}$ with compact support, equation (4.6) holds. The next lemma completes the proof. $\square$

4-4.5 Lemma. *Let M be a riemannian covering of a compact manifold and suppose that $\sup_{|v|\le k} \left\| \frac{\partial L}{\partial x}(x,v) \right\| < +\infty$. If $u : M \to \mathbb{R}$ is weakly differentiable and*

$$H(x, d_x u) \le k \text{ for a.e. } x \in M,$$

then for all $\delta > 0$ there exists $f \in C^\infty(M, \mathbb{R})$ such that $H(x, d_x f) < k+\delta$ for all $x \in M$.

4-4.6 Remark. If M is a covering $M \xrightarrow{\rho} N$ of a compact manifold N and the lagrangian L on M is lifted from a lagrangian ℓ on N: $L = \ell \circ d\rho$, then the condition on $\frac{\partial L}{\partial x}$ follows. In general it may not be true as the lagrangian $L(x, v) = \frac{1}{2}|v|^2 + \sin(x^2)$ on $\mathbb{R}$ shows.

The condition on M can be replaced by some bounds in the riemannian metric of M, see appendix **??**.

Proof: We shall explain first how to prove the proposition in the case in which M is compact and then we will lift the construction to an arbitrary covering M.

We can assume that $M \subset \mathbb{R}^N$. Let U be a tubular neighbourhood of M in $\mathbb{R}^N$, and $\rho : U \to M$ a C^∞ projection along the normal bundle. Extend $u(x)$ to U by $\overline{u}(z) = u\big(\rho(z)\big)$. Then $\overline{u}(z)$ is also weakly differentiable.

Extend the lagrangian to U by

$$\overline{L}(z, v) = L\big(\rho(z), d_z\rho(v)\big) + \tfrac{1}{2}\, |v - d_z\rho(v)|^2\,.$$

Then the corresponding hamiltonian satisfies $\overline{H}(z, p \circ d_z\rho) = H\big(\rho(z), p\big)$ for $p \in T^*_{\rho(z)}M$. At any point of differentiability of $\overline{u}$, we have that $d_z\overline{u} = d_{\rho(z)}u \circ d_z\rho$, and $\overline{H}(d_z\overline{u}) = H(d_{\rho(z)}u) \le k$.

Let $\varepsilon > 0$ be such that

(a) The 3ε-neighbourhood of M in $\mathbb{R}^N$ is contained in U.

(b) If $x \in M$, $(y, p) \in T^*\mathbb{R}^N = \mathbb{R}^{2N}$, $\overline{H}(y, p) \le k$ and $d_{\mathbb{R}^N}(x, y) < \varepsilon$, then $\overline{H}(x, p) < k + \delta$.

Let $\psi : \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that $\psi(x) \ge 0$, support$(\psi) \subset (-\varepsilon, \varepsilon)$ and $\int_{\mathbb{R}^N} \psi(|x|)\, dx = 1$. Let $K : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be $K(x, y) = \psi(|x - y|)$. Let N_ε be the ε-neighbourhood of M in $\mathbb{R}^N$. Define f :

$N_\varepsilon \to \mathbb{R}$ by

$$f(x) = \int_{\mathbb{R}^N} \overline{u}(y)\; K(x,y)\; dy\,.$$

Then f is C^∞ on N_ε.

Observe that $\partial_x K(x,y) = -\partial_y K(x,y)$. Since $\overline{u}(y)$ is weakly differentiable, for any C^∞ function $\varphi : U \to \mathbb{R}$ with compact support

$$\int_{\mathbb{R}^N} (\varphi\; d\overline{u} + \overline{u}\; d\varphi)\; dx = 0\,. \tag{4.6}$$

Hence

$$-\int_{\mathbb{R}^N} \overline{u}(y)\,\partial_y K(x,y)\; dy = \int_{\mathbb{R}^N} K(x,y)\; d_y\overline{u}\; dy.$$

Now, since

$$d_x f = \int_{\mathbb{R}^N} \overline{u}(y)\,\partial_x K(x,y)\; dy,$$

we obtain

$$d_x f = \int_{\mathbb{R}^N} K(x,y)\; d_y\overline{u}\; dy.$$

From the choice of $\varepsilon > 0$ we have that $\overline{H}(x, d_y\overline{u}) < k + \delta$ for almost every $y \in \operatorname{supp} K(x,\cdot)$ and $x \in M$. Since $K(x,y)\,dy$ is a probability measure, by Jensen's inequality

$$H(d_x f) \le \overline{H}(d_x f) \le \int_{\mathbb{R}^N} \overline{H}(x, d_y\overline{u})\, K(x,y)\; dy < k + \delta\,.$$

for all $x \in M$.

Now, suppose that M is a covering of a compact manifold N with covering projection p. Assume that $N \subseteq \mathbb{R}^N$. We can regularize our function u similarly as we shall now explain. For $\widetilde{x} \in M$ let x be the projection of $\widetilde{x}$ to N and let μ_x be the Borel probability measure on N defined by

$$\int_N \varphi\, d\mu_x = \int_{\mathbb{R}^N} (\varphi \circ \rho)(y)\; K(x,y)\; dy,$$

for any continuous function $\varphi : N \to \mathbb{R}$. Then the support of μ_x satisfies

$$\operatorname{supp}(\mu_x) \subset \{y \in N \,:\, d_N(x,y) < \varepsilon\}.$$

Let $\widehat{\mu}_{\widetilde{x}}$ be the Borel probability measure on M uniquely defined by the conditions: $\operatorname{supp}(\widehat{\mu}_{\widetilde{x}}) \subset \{\widehat{y} \in M \,:\, d_M(\widetilde{x},\widehat{y}) < \varepsilon\}$ and $p_*\widehat{\mu}_{\widetilde{x}} = \mu_x$. Then we have

$$\frac{d}{d\widetilde{x}} \int_M \varphi \, d\widehat{\mu}_{\widetilde{x}} = \int_M d_{\widehat{y}}\varphi \, d\widehat{\mu}_{\widetilde{x}}(\widehat{y}),$$

for any weakly differentiable function $\varphi : M \to \mathbb{R}$. The condition (b) above is now granted by the bound on $\frac{\partial L}{\partial x}$. Then the same arguments as above show that

$$f(\widetilde{x}) = \int_M u(\widehat{y}) \, d\widehat{\mu}_{\widetilde{x}}(\widehat{y}),$$

satisfies $H(d_{\widetilde{x}} f) < k$. □

Combining lemma 4-4.5 with lemma 4-4.2, we obtain:

4-4.7 Corollary.
There are no weakly differentiable subsolutions of (H-J) *for* $k < c(L)$.

If we consider lagrangian graphs with other cohomology classes, we obtain Mather's alpha function:

4-4.8 Corollary. *If M is compact,*

$$\alpha(\kappa) = \inf_{[\omega]=\kappa} \sup_{x \in M} H(x, \omega(x)).$$

In particular the critical value of the abelian cover $c_0 = \min_\kappa \alpha(\kappa)$ is the infimum of the energy levels which contain a lagrangian graph of any cohomology class in is interior.

Proof: Let us fix a closed one form ω_0 such that $[\omega_0] = \kappa$. By equality (2.30) we have that $\alpha(\kappa) = c(L - \omega_0)$. Hence, it suffices to show that

$$c(L - \omega_0) = \inf_{[\omega]=\kappa} \sup_{x \in M} H(x, \omega(x)). \tag{4.7}$$

It is straightforward to check that the hamiltonian associated with $L-\omega_0$ is $H(x, p+\omega_0(x))$. Since all the closed one forms in the class κ are given by $\omega_0 + df$ where f ranges among all smooth functions, equality (4.7) is now an immediate consequence of theorem 4-4.1.

□

4-5 Contact flows.

Let N be a $2n{+}1$ smooth manifold and let α be a non-degenerate 1-form, i.e. $\alpha \wedge (d\alpha)^n$ is a volume form for N.

$\dim \ker d\alpha \equiv 1$**, where**

$$\ker d_x\alpha := \{\, v \in T_xN \mid d_x\alpha(v,w) = 0,\ \forall w \in T_xN \,\}, \qquad x \in N.$$

Define a vector field Y on N by $Y(x) \in \ker d_x\alpha$,

---

etc

4-5.1 Proposition. *The hamiltonian flow on the energy level $[H \equiv k]$ is a reparametrization of a Θ-preserving flow if and only if $k > c(L)$.*

¿También se tiene desigualdad estricta cuando M no es compacta?

Proof: Let $\Sigma = [H \equiv k]$. Observe that

$$\tfrac{d}{dt}\psi_t^*\Theta = \mathcal{L}_X\Theta|_\Sigma = i_X\, d\Theta + d\, i_X\Theta = dH|_\Sigma + d\Theta(X) = d\Theta(X)|_\Sigma.$$

Thus a reparametrization of ψ_t preserves Θ if and only if its vector field is a constant multiple of $Y = \frac{1}{\Theta(X)}\, X$. Such reparametrization exists if and only if $\Theta(X) \neq 0$ on Σ.

Since H is convex, the sets $\Sigma_x := T_x^*M \cap \Sigma$ have compact convex interiors. The outward normal vector to Σ_x at (x,p) is $H_p(x,p)$. Observe that $\Theta(X) = p \cdot H_p$. Then if the point $(x,0)$ lies on the exterior of Σ_x and (x,p) is the tangency point of a tangent line to Σ_x passing through $(x,0)$, we have that $\Theta(X(x,p)) = 0$. Thus $\Theta(X) \neq 0$ on Σ implies that the zero section of T^*M lies in $H < k$. Since the zero section is the derivative of a constant function, by theorem 4-4.1, $k \geq c(L)$.

Moreover, if the zero section lies inside $H < k$, then $\Theta(X) > 0$.

Now suppose that $k > c(L)$. By theorem 4-4.1, there exists $f \in C^\infty(M, \mathbb{R})$ with $H(df) < k$. Define the new convex hamiltonian $\mathbb{H}(x,p) := H(x, p + d_x f)$. Then the energy level $[\mathbb{H} \equiv k]$ contains the zero section. Thus, if $\mathbb{X}$ is its hamiltonian vector field, then $\Theta(\mathbb{X}) > 0$. Let $\mathbb{L} := L - df$. Let $\mathbb{H}$ be the hamiltonian of $\mathbb{L}$ and $\mathbb{X}$ its hamiltonian flow. Then

$\square$

4-6 Finsler metrics.

In this section we prove that if $k > c(L)$ then the Euler-Lagrange flow on the energy level $E \equiv k$ is a reparametrization of the geodesic flow on the unit tangent bundle of a Finsler metric. This allows to borrow theorems from Finsler geometry.

We give first a lagrangian proof and afterwards a hamiltonian proof, with a flavor of symplectic geometry.

When $k > c(L)$, the subsolution $H(df) < k$ obtained in proposition 4-4.4 can be used to replace the lagrangian L by the lagrangian $\mathbb{L} = L - df$, then $\mathbb{L} + k > 0$. These lagrangians have the same energy function and equivalent variational principles. Hence they have the same lagrangian flow, minimizing orbits, and the same action functional on closed curves and invariant measures. Their action potentials are related by

$$\mathbb{P}_k(x, y) = \Phi_k(x, y) + f(y) - f(x).$$

Given a Finsler metric $\sqrt{F}$ and an absolutely continuous curve γ, define its Finsler length as

$$l_F(\gamma) = \int \sqrt{F(\dot{\gamma})}.$$

Observe that since the Finsler metric is homogeneous of degree one, the definition does not depend on the parametrization of the curve. Define the Finsler distance as

$$D_F(x, y) = \inf_{\gamma \in \mathcal{C}(x,y)} l_F(\gamma).$$

4-6.1 Theorem. *[14, 30] If $k > c(L)$ then the lagrangian flow on the energy level $E \equiv k$ is a reparametrization of the geodesic flow of a Finsler metric on its unit tangent bundle.*

Moreover, if $f \in C^\infty(M, \mathbb{R})$ is such that $H(df) < k$, the Finsler lagrangian F can be taken to be

$$\sqrt{F(x, v)} = L(x, v) + k - d_x f(v)$$

on $E(x,v)=k$, and then

$$\Phi_k(x,y)=D_F(x,y)+f(y)-f(x), \qquad \textit{for all } x,y\in M. \tag{4.8}$$

If $k>-\inf L$ then f can chosen $f\equiv 0$.

Proof: By theorem 4-4.1, if $k>c(L)$ then there exists $f\in C^\infty(M,\mathbb{R})$ such that $H(df)<k$. Observe that

$$H(x,0)=\max_{v\in T_xM}\big[\,0\cdot v-L(x,v)\,\big]=-\inf_{v\in T_xM}L(x,v).$$

So that if $k>-\inf L$, we can choose $f\equiv 0$.

Observe that if $H(df)<k$ then $k>c(L)>e_0$. So that the zero section $M\times 0\subset E^{-1}\{k\}\subset TM$. Also, $L+k-df>0$. Then we can define a Finsler metric on TM by $\sqrt{F}=L+k-df$ on $E^{-1}\{k\}$ and extend it by homogeneity. Since $k>c(L)=c(L-df)$, by proposition 3-5.1, for any $x\neq y$ there exists a global minimizer on $\mathcal{C}(x,y)$ for $L+k-df$ which has energy k. By the homogeneity of $\sqrt{F}$, we can restrict the curves in the definition of D_F to those with energy k. Thus $\Phi_k-\Delta f=D_F$.

To show that the lagrangian flow on $E\equiv k$ is a reparametrization of the geodesic flow on the unit tangent bundle of $\sqrt{F}$, we only need to prove that sufficiently small Euler-Lagrange solutions with energy k are geodesics of $\sqrt{F}$. Let $\mathbb{L}=L-df$. The equality (4.8) implies that any $(\mathbb{L}+k)$-global minimizer is a geodesic for $\sqrt{F}$. So it is enough to prove that sufficiently small orbits with energy k are global minimizers.

Fix $x\in M$ and a small neighbourhood $\mathcal{N}(x)$ of x such that for all $y\in\mathcal{N}(x)$ there exists a unique Euler-Lagrange solution contained in $\mathcal{N}(x)$, with energy k and joining x to y. Let $\mathbb{P}_k$ be the action potential for $\mathbb{L}$ and let

$$\begin{aligned}\varepsilon&=\inf\{\,\mathbb{P}_k(x,z)\,|\,z\notin\mathcal{N}(x)\,\}>0,\\ \mathcal{M}(x)&=\{\,y\in M\,|\,\mathbb{P}_k(x,y)<\tfrac{\varepsilon}{2}\,\}.\end{aligned}$$

Then $\mathcal{M}(x)$ is a neighborhood of $\mathcal{N}(x)$ with $\mathcal{M}(x)\subset\mathcal{N}(x)$. By the triangle inequality, any $(\mathbb{L}+k)$-global minimizer joining x to a point $y\in$

$\mathcal{M}(x)$ must be contained in $\mathcal{N}(x)$. By proposition 3-5.1 such minimizer exists. Hence all the small solutions contained in $\mathcal{N}(x)$ joining x to points $y \in \mathcal{M}(x)$ are global minimizers. $\square$

Now we shall give a hamiltonian proof of theorem 4-6.1.

First we need a hamiltonian characterization of Finsler lagrangians. We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is a *Finsler energy* if $f(x) > 0$ when $x \neq 0$ and f is positively homogeneous of order 2. Observe that L is a Finsler lagrangian if and only if it is a Finsler energy on each tangent space $T_x M$

4-6.2 Lemma. *Let $f \in C^2(M, \mathbb{R})$ be strictly convex and superlinear. Then f is a Finsler energy if and only if its convex dual f^* is a Finsler energy.*

Proof: If f is homogeneous, then, writing $v = \lambda\, w$, $\lambda > 0$, we have

$$f^*(\lambda\, p) = \max_v \big[\, \lambda p\, v - f(v) \,\big] = \max_w \big[\, \lambda^2 p\, w - \lambda^2 f(w) \,\big] = \lambda^2 f^*(p).$$

Let $\mathcal{L}_f$ be the Legendre transform of f. Observe that

$$f^* \circ \mathcal{L}_f(p) = f'(p) \cdot p - f(p) = \tfrac{d}{dt} f(tp)\big|_{t=1} - f(p) = 2\, f(p) - f(p) = f(p).$$

So that $f^* > 0$.

Since f is strictly convex and superlinear, then $f^{**} = f$. Thus the argument above shows that if f^* is homogeneous then so is f. $\square$

4-6.3 Lemma. *If two convex hamiltonians have a common regular level set Σ, then their hamiltonian vector fields are parallel on Σ.*

Proof: Suppose that $H^{-1}\{k\} = G^{-1}\{\ell\} = \Sigma$. If $p \in \Sigma$, then $d_p H = \ker d_p G = T_p \Sigma$. Since Σ is a regular energy level, these derivatives are nonzero. Moreover, since H and G are convex, they are positive on vectors pointing outwards Σ. Thus $d_p H = \lambda(p)\, d_p G$ for some $\lambda(p) > 0$. Also,

$$\omega(X_H, \cdot) = d_p H = \lambda(p)\, d_p G = \lambda(p)\, \omega(X_G, \cdot)$$

So that $X_H = \lambda(p)\, X_G$. $\square$

4-6.4 Lemma. *If F is a Finsler lagrangian, then the orbits of its lagrangian flow are reparametrizations of the unit speed geodesics of F.*

Proof: We prove it for a Finsler hamiltonian G. The result follows because the Euler-Lagrange flow of F is the hamiltonian flow of its energy function on TM with respect to the symplectic form $\mathcal{L}_F^*(\omega)$, where ω is the canonical symplectic form on T^*M.

Let G be a Finsler hamiltonian. Since $G(x, \lambda p) = \lambda G(x, p)$, then

$$d_{\lambda p} G(w) = \tfrac{d}{ds} G(x, \lambda p + sw)\big|_{s=1} = \tfrac{d}{ds} \lambda^2\, G(x, p + s\tfrac{w}{\lambda})\big|_{s=1} = \lambda\, d_p G(w).$$

If X is the hamiltonian flow of G, then $\omega(X(p), \cdot) = d_p G$. Therefore $X(\lambda\, p) = \lambda X(p)$. □

Hamiltonian proof of theorem 4-6.1:

If $k > c(L)$, by theorem 4-4.1, there exists $f \in C^\infty(M, \mathbb{R})$ with $H(df) < k$. Let $\mathbb{H}(x, p) \overset{\text{def}}{=} H(x, p + d_x f)$. Then $\mathbb{H}^{-1}(]-\infty, k[)$ contains the zero section of T^*M. Define a new hamiltonian $G : T^*M - M \times 0 \to \mathbb{R}$ by $G \equiv \frac{1}{4}$ on $\mathbb{H}^{-1}\{k\}$ and $G(x, \lambda p) = \lambda^2\, G(x, p)$ for all $\lambda \geq 0$. By lemma 4-6.2, the convex dual G^* of G is a Finsler metric on TM.

Since by definition $G^{-1}\{\frac{1}{4}\} = \mathbb{H}^{-1}\{k\}$, it follows from lemma 4-6.3 that the hamiltonian flows of G and $\mathbb{H}^{-1}\{k\}$ coincide up to reparametrization on the energy level $G^{-1}\{\frac{1}{4}\} = \mathbb{H}^{-1}\{k\}$.

The Legendre transforms $\mathcal{L}_G(x, p) = (x, G_p)$ and $\mathcal{L}_{\mathbb{H}}(x, p) = (x, \mathbb{H}_p)$ on $\mathbb{H}^{-1}\{k\}$ satisfy

$$\begin{aligned} &G_p \cdot w = 0 = \mathbb{H}_p \cdot p, && \text{for all } w \in T_p(T_X^* M \cap \mathbb{H}^{-1}\{k\}); \\ &G_p \cdot p = 2\, G(p) = \tfrac{1}{2} > 0, && \mathbb{H}_p \cdot p = \Theta(\mathbb{X}) > 0, \quad \text{(by 4-5.1)}. \end{aligned} \tag{4.9}$$

So that $\mathcal{L}_G(x, p) = \lambda\, \mathcal{L}_{\mathbb{H}}(x, p)$ for some $\lambda(x, p) > 0$. Also $\mathcal{L}_{\mathbb{H}}$ conjugates the hamiltonian flow on $\mathbb{H}^{-1}\{k\}$ to the lagrangian flow of L on $E^{-1}\{k\}$. By lemma 4-6.4 the orbits of the Euler-Lagrange flow of L on $E^{-1}\{k\}$ are reparametrization of unit speed geodesic of G^*.

We now compute G^*. From (4.9), $\mathbb{H}_p = 2\,\Theta(\mathbb{X})\,G_p$, thus

$$\begin{aligned} G^*(x, H_p) &= G^*(x, 2\,\Theta(\mathbb{X})\,G_p) = 4\,\Theta(\mathbb{X})^2\,G^*(x, G_p) \\ &= 4\,\Theta(\mathbb{X})^2\,G(p) = \Theta(\mathbb{X})^2 = (p \cdot \mathbb{H}_p)^2 \\ &= (\mathbb{L}_v \cdot v)^2 = (\mathbb{L} + k)^2. \end{aligned}$$

□

Let $h : T^*M \to T^*M$ be the map $h(x, p) = (x, p + d_x f)$. Then the hamiltonian flows ϕ_t of H and ψ_t of $H_{df}(x, p) = H(x, p+d_x f)$ are related by $h \circ \phi_t = \psi_t \circ h$. Thus, the hamiltonian flow of H is *conjugate* to a Finsler hamiltonian flow.

A hamiltonian level set can be made a Finsler level set if and only if it contains the zero section. On the other hand, a lagrangian energy level $[E = k]$ with $k > e_0$ always contains the zero section.

4-7 Anosov energy levels.

An *Anosov energy level* is a regular energy level on which the flow ϕ_t is an Anosov flow.

4-7.1 Theorem. *If the energy level* $\mathbb{E}^{-1}(k)$ *is Anosov, then*

$$k > c_u(\lambda).$$

Proof: Suppose that the energy level k is Anosov and set $\Sigma \stackrel{\text{def}}{=} \mathbb{H}^{-1}(k)$. Let $\pi : T^*N \to N$ denote the canonical projection. G.P. Paternain and M. Paternain proved in [57] that Σ must project onto the whole manifold N and that the weak stable foliation $\mathcal{W}^s$ of ϕ_t^* is *transverse* to the fibers of the fibration by $(n-1)$-spheres given by

$$\pi|_\Sigma : \Sigma \to N.$$

Let $\widetilde{N}$ be the universal covering of N. Let $\widetilde{\Sigma}$ denote the energy level k of the lifted hamiltonian H. We also have a fibration by $(n-1)$-spheres

$$\widetilde{\pi}|_{\widetilde{\Sigma}} : \widetilde{\Sigma} \to \widetilde{N}.$$

Let $\widetilde{\mathcal{W}^s}$ be the lifted foliation which is in turn a weak stable foliation for the hamiltonian flow of H restricted to $\widetilde{\Sigma}$. The foliation $\widetilde{\mathcal{W}^s}$ is also transverse to the fibration $\widetilde{\pi}|_{\widetilde{\Sigma}} : \widetilde{\Sigma} \to \widetilde{N}$. Since the fibers are compact a result of Ehresmann (cf. [7]) implies that for every $(x,p) \in \widetilde{\Sigma}$ the map

$$\widetilde{\pi}|_{\widetilde{\mathcal{W}^s}(x,p)} : \widetilde{\mathcal{W}^s}(x,p) \to \widetilde{N},$$

is a covering map. Since $\widetilde{N}$ is simply connected, $\widetilde{\pi}|_{\widetilde{\mathcal{W}^s}(x,p)}$ is in fact a diffeomorphism and $\widetilde{\mathcal{W}^s}(x,p)$ is simply connected. Consequently, $\widetilde{\mathcal{W}^s}(x,p)$ intersects each fiber of the fibration $\widetilde{\pi}|_{\widetilde{\Sigma}} : \widetilde{\Sigma} \to \widetilde{N}$ at just one point. In other words, each leaf $\widetilde{\mathcal{W}^s}(x,p)$ is the graph of a one form. On the other hand it is well known that the weak stable leaves of an Anosov energy level are lagrangian submanifolds. Since any closed one form in the universal covering must be exact, it follows that each leaf $\widetilde{\mathcal{W}}^s(x,p)$ is an exact lagrangian graph. The theorem now follows from lemma 4-4.2 and the fact that by structural stability there exists $\varepsilon > 0$ such that for all $k' \in (k-\varepsilon, k+\varepsilon)$ the energy level k' is Anosov. $\square$

For $e \in \mathbb{R}$, let $\mathcal{A}_e$ be the set of $\phi \in C^\infty(M)$ such that the flow of $H+\phi$ is Anosov in $(H+\phi)^{-1}(e)$ and let $\mathcal{B}_e$ be the set of $\phi \in C^\infty(M)$ such that $(H+\phi)^{-1}(e)$ contains no conjugate points. As is well known $\mathcal{A}_e$ is open in C^k topology and $\mathcal{B}_e$ is closed. On the other hand G. and M. Paternain [54] have shown that $\mathcal{A}_e$ is contained in $\mathcal{B}_e$. It is proved in [15] the following

4-7.2 Theorem. *The interior of $\mathcal{B}_e$ in the C^2 topology is $\mathcal{A}_e$.*

This theorem is an extension to the Hamiltonian setting of a result of R. O. Ruggiero for the geodesic flow [64]. Theorems 4-7.2 and 4-7.1 have as corollary:

4-7.3 Corollary. *Given a convex superlinear lagrangian L, $k < c_u(L)$ and $\varepsilon > 0$ there exists a smooth function $\psi : N \to \mathbb{R}$ with $|\psi|_{C^2} < \varepsilon$ and such that the energy level k of $L + \psi$ possesses conjugate points.*

Proof: Suppose now that there exists $\epsilon > 0$ such that for every ψ with $|\psi|_{C^2} < \epsilon$, the energy level k of $\lambda + \psi$ has no conjugate points. The main result in [15] says that in this case the energy level k of λ must be Anosov thus contradicting theorem 4-7.1. $\square$

4-7.4 Proposition. *If k is a regular value of the energy such that $k < e$, then the energy level k has conjugate points.*

Proof: If an orbit does not have conjugate points then there exist along it two subbundles called the *Green subbundles*. They have the following properties: they are invariant, lagrangian and they have dimension $n = \dim N$. Moreover, they are contained in the same energy level as the orbit and they do not intersect the vertical subbundle (cf. [13]). If k is a regular value of the energy with $k < e$, then $\pi(\mathbb{E}^{-1}(k))$ is a manifold with boundary and at the boundary the vertical subspace is completely contained in the energy level. Therefore the orbits that begin at the boundary must have conjugate points, because at the boundary two n-dimensional subspaces contained in the energy level (which is $(2n-1)$-dimensional) must intersect. $\square$

4-8 The weak KAM Theory.

In the rest of this chapter we develop the theory of global weak KAM solutions, discovered by Albert Fathi. Recall that for an autonomous hamiltonian $H : T^*M \to \mathbb{R}$, the Hamilton-Jacobi equation is

$$H(x, d_x u) = k, \tag{H-J}$$

where $u : U \subseteq M \to \mathbb{R}$. Here we are interested on *global* solutions of (H-J), i.e. $u : M \to \mathbb{R}$ satisfying (H-J).

It may not be possible to obtain a smooth global solution of (H-J). Instead, for certain values of k, we shall find *weak* solutions of (H-J), which are Lipschitz. By Rademacher's theorem [19], a Lipschitz function is Lebesgue almost everywhere differentiable so that (H-J) makes sense in a.e. point.

In fact, we have seen in corollary 4-4.7 that there are no weakly differentiable[1] global solutions for $k \le c(L)$. In theorem 4-8.4 we shall see that when M is compact, there are no $C^{1+\text{Lip}}$ solutions unless $k = c(L)$. In proposition 4-9.7 we show that there are always Lipschitz solutions for $k = c(L)$, and in proposition 4-10.2 we show that when M is non-compact there are Lipschitz solutions for $k > c(L)$.

Given a dominated function $u \prec L + c$ define the sets

$$\Gamma_0^+(u) := \{\, v \in \Sigma^+ \mid u(x_v(t)) - u(x_v(0)) = \Phi_c(x_v(0), x_v(t)),\ \forall t > 0 \,\},$$

$$\Gamma_0^-(u) := \{\, v \in \Sigma^- \mid u(x_v(0)) - u(x_v(t)) = \Phi_c(x_v(t), x_v(0)),\ \forall t < 0 \,\},$$

$$\Gamma^+(u) := \bigcup_{t>0} \varphi_t\big(\Gamma^+(u)\big) \quad , \qquad \Gamma^-(u) := \bigcup_{t<0} \varphi_t\big(\Gamma^+(u)\big),$$

where $x_v(t) = \pi\, \varphi_t(v)$. We call $\Gamma^+(u)$ (resp. $\Gamma^+(u)$) the *basin* of u and $\pi\big(\Gamma_0^+(u) \setminus \Gamma^+(u)\big)$ (resp. $\pi\big(\Gamma_0^+(u) \setminus \Gamma^+(u)\big)$) the *cut locus* of u.

4-8.1 Definition.
A function $u_- : M \to \mathbb{R}$ is a *backward weak KAM* solution of (H-J) if

[1] In particular, by Rademacher's theorem, there are no Lipschitz global solutions.

1. $u_- \prec L + c$.
2. $\pi\big(\Gamma_0^-(u_-)\big) = M$.

A function $u_+ : M \to \mathbb{R}$ is a *forward weak KAM* solution of (H-J) if

1. $u_+ \prec L + c$.
2. $\pi\big(\Gamma_0^+(u_+)\big) = M$.

4-8.2 Remark.
From the domination condition it follows that u is Lipschitz and that the curve γ is semistatic. From proposition 4-2.4, at an interior point x of such a curve γ, the function u is differentiable and $H(x, d_x u) = c$. Moreover, item *2* in proposition 4-2.4 shows that if u is differentiable at an endpoint of a curve γ, then $H(x, d_x u) = c$. By Rademacher's theorem [19], u is differentiable at (Lebesgue) almost every point in M. So that u is indeed a weak solution of the Hamilton-Jacobi equation for $k = c(L)$.

4-8.3 Theorem.
If $u \in \mathfrak{S}^+$ (resp. $u \in \mathfrak{S}^-$) is a weak KAM solution, then

1. *u is Lipschitz and hence differentiable (Lebesgue)-almost everywhere.*
2. *$u \prec L + c$.*
3. *$H(x, d_x u) = c(L)$ at any differentiability point x of u.*
4. **Covering Property:** *$\pi(\Gamma_0^+(u)) = M$.*
5. **Graph Property:** *$\pi : \Gamma^+(u) \to M$ is injective and its inverse is Lipschitz, with Lipschitz constant depending only on L.*
6. **Smoothness Property:** *u is differentiable on $\Gamma^+(u)$ and its derivative $d_x u$ is the image of $(\pi|_{\Gamma^+(u)})^{-1}(x)$ under the Legendre transform $\mathcal{L}$ of L. In particular, the energy of $\Gamma_0^+(u)$ is $c(L)$.*

Proof: Items *2* and *4* are the definition of $u \in \mathfrak{S}^+$. Item *1* follows from proposition 4-2.1.*1* and Rademacher's theorem [19]. Item *3* follows from proposition 4-2.4.*2* and the fact that by item *4*, u is one-sided realized at every point. Item *6* follows from proposition 4-2.4 and remark 4-2.5.

We prove item *5*. Let (z_1, v_1), $(z_2, v_2) \in \Gamma^+(u)$ and suppose that $d_{TM}(v_1, v_2) > K\, d_M(z_1, z_2)$, where K is from lemma 3-8.2 and the A that we input on lemma 3-8.2 is from lemma 3-2.3. Let $0 < \varepsilon < \varepsilon_1$, (with ε_1 from lemma 3-8.2) be such that $\varphi_{-\varepsilon}(z_i, v_i) \in \Gamma^+(u)$. Let $y_i = x_{v_i}(\varepsilon)$, $i = 1, 2$, then $u(y_i) = u(x_i) + \Phi_c(x_i, y_i)$, $i = 1, 2$. Then lemma 3-8.2 implies that

$$\Phi_c(x_1, y_2) + \Phi_c(x_2, y_1) < \Phi_c(x_1, y_1) + \Phi_c(x_2, y_2).$$

Adding $u(y_1) + u(y_2)$ and using that $u \prec L + c$, we get that

$$\begin{aligned} u(x_1) + u(x_2) &\le \Phi_c(x_1, y_2) + u(y_2) + \Phi_c(x_2, y_1) + u(y_1) \\ &< \Phi_c(x_1, y_1) + u(y_1) + \Phi_c(x_2, y_2) + u(y_2) \\ &= u(x_1) + u(x_2), \end{aligned}$$

which is a contradiction. This proves item *5*.

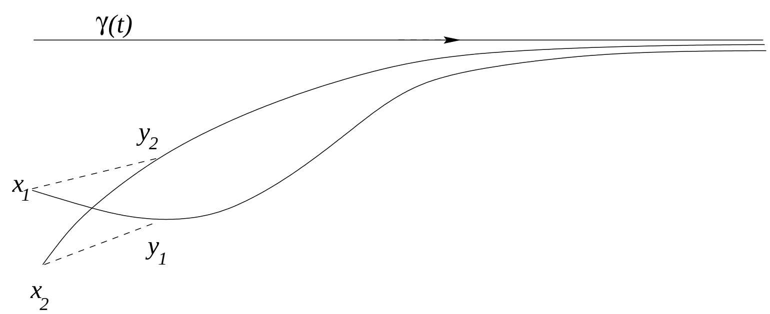

FIG. 1: GRAPH PROPERTY.

□

4-8.4 Theorem (Fathi [23]).
If $u \in C^{1+Lip}(M, \mathbb{R})$, M is compact and

$$H(x, d_x u) = k, \qquad \forall x \in M. \tag{4.10}$$

Then $k = c(L)$ and u is a weak KAM solution $u \in \mathfrak{S}^- \cap \mathfrak{S}^+$.
Conversely, if $u \in \mathfrak{S}^+ \cap \mathfrak{S}^-$, then $u \in C^{1+Lip}$.

Proof: From (4.10) we get that

$$\max_{v \in T_x M} d_x u \cdot v - L(x, v) = k. \tag{4.11}$$

The strict convexity of L on $T_x M$ implies that the maximum is attained at a unique vector $\xi(x) \in T_x M$. The implicit function theorem implies that the vector field $x \mapsto \xi(x)$ is Lipschitz. Thus it can be integrated to obtain a flow ψ_t on M.

From (4.11), $u \prec L + k$ and the flow lines of ψ_t realize $L + k$, i.e.

$$u\big(\psi_t(x)\big) = u\big(\psi_s(x)\big) + \int_s^t \big[L(\psi_\tau(x), \tfrac{d}{d\tau}\psi_\tau(x)) + k\big]\, d\tau, \ \forall s < t, \ \forall x \in M. \tag{4.12}$$

From (4.11), $L + k - du \geq 0$. This implies that

$$c(L - du) = c(L) \geq k.$$

Let μ be and invariant measure for the flow ψ_t. Observe that the measure $\nu := \xi_*(\mu)$ is holonomic and $L + k = du$ on $\operatorname{supp}(\nu)$. Then

$$\int (L + k)\, d\nu = \int (du)\, d\nu = 0.$$

Hence $k \leq c(L)$ and thus $k = c(L)$. Then by (4.12), $u \in \mathfrak{S}^- \cap \mathfrak{S}^+$.

Conversely, if $u \in \mathfrak{S}^- \cap \mathfrak{S}^+$, by 4-8.3.*6*, u is differentiable and $\mathcal{L}^{-1}(d_x u) = \xi(x)$. Moreover, $\Gamma^{\mp}(u) = M$ and by 4-8.3.*5*, ξ is Lipschitz. $\square$

4-9 Construction of weak KAM solutions

In this section we present three ways to construct weak KAM solutions: when the Peierls set is non-empty (in remark 4-9.3.*4*), when the Peierls barrier is finite (in proposition 4-9.2), and the general case (in proposition 4-9.7). In the horocycle flow (example 5-8), the Peierls barrier is finite, but the Peierls set is empty. In example 5-7, $h_c = +\infty$ and also $\mathcal{P} = \varnothing$.

When M is compact we characterize all weak KAM solutions in terms of their values on each static class.

We begin by observing that

4-9.1 Lemma.

1. *If* $\mathcal{U} \subseteq \mathfrak{S}^-$ *is such that* $v(x) := \inf_{u\in\mathcal{U}} u(x) > -\infty$*, for all* $x \in M$*; then* $v \in \mathfrak{S}^-$.
2. *If* $\mathcal{U} \subseteq \mathfrak{S}^+$ *is such that* $v(x) := \sup_{u\in\mathcal{U}} u(x) < +\infty$*, for all* $x \in M$*; then* $v \in \mathfrak{S}^+$.

Proof: We only prove item *1*. Since $u \prec L + c$ for all $u \in \mathcal{U}$, then

$$v(y) = \inf_{u\in\mathcal{U}} u(y) \le \inf_{u\in\mathcal{U}} u(x) + \Phi_c(x,y) = v(x) + \Phi_c(x,y). \tag{4.13}$$

Thus $v \prec L + c$.

Let $x \in M$ and choose $u_n \in \mathcal{U}$ such that $u_n(x) \to v(x)$. Choose $w_n \in \Gamma^-(u_n) \cap T_xM$. Since by lemma 3-2.3 $|w_n| < A$, we can assume that $w_n \to w \in T_xM$. By lemma 4-2.1.*1*, all the functions $u \in \mathcal{U}$ have the same Lipschitz constant K as Φ_c. For $t < 0$, we have that

$$\begin{aligned} v(x_w(t)) &\le \liminf_n u_n(x_{w_n}(t)) + K\, d_M(x_w(t), x_{w_n}(t)) \\ &= \liminf_n u_n(x) - \Phi_c(x_w(t), x) + K\, d_M(x_w(t), x_{w_n}(t)) \\ &= v(x) - \Phi_c(x_w(t), x) \le v(x_w(t)), \qquad \text{because } v \prec L + c. \end{aligned}$$

Hence $w \in \Gamma^-(v)$. □

4-9.a Finite Peierls barrier.

4-9.2 Proposition. *If $h_c < +\infty$ and $f : M \to \mathbb{R}$ is a continuous function. Suppose that*

$$\begin{aligned} v_-(x) &:= \inf_{z\in M} f(z) + h_c(z,x) > -\infty, \\ v_+(x) &:= \sup_{z\in M} f(z) - h_c(x,z) > -\infty. \end{aligned}$$

Then $v_- \in \mathfrak{S}^-$ and $v_+ \in \mathfrak{S}^+$.

Proof: We only prove that $v_- \in \mathfrak{S}^-$. By lemma 4-9.1 it is enough to prove that the functions $u(x) \mapsto h_c(z,x)$ are in $\mathfrak{S}^-$ for all $z \in M$.

By proposition 3-7.1.*4*, $u \prec L + c$. Now fix $x \in M$. Choose Tonelli minimizers $\gamma_n : [T_n, 0] \to M$ such that $\gamma_n \in \mathcal{C}(z,x)$, $T_n < -n$ and

$$A_{L+c}\big(\gamma_n|_{[T_n,0]}\big) \le h_c(z,x) + \tfrac{1}{n}.$$

By lemma 3-2.3, $|\dot\gamma_n(0)| < A$ for all n. We can assume that $\dot\gamma_n(0) \xrightarrow{n} w \in T_xM$. If $-n \le s \le 0$, then $s > T_n$ and

$$\begin{aligned} A_{L+c}\big(\gamma_n|_{[T_n,s]}\big) + \Phi_c(\gamma_n(s),x) &\le \\ &\le A_{L+c}\big(\gamma_n|_{[T_n,s]}\big) + A_{L+c}\big(\gamma_n|_{[s,0]}\big) \\ &\le h_c(z,x) + \tfrac{1}{n} \\ &\le h_c(z,\gamma_n(s)) + \Phi_c(\gamma_n(s),x) + \tfrac{1}{n}, \qquad \text{for } -n \le s < 0. \end{aligned}$$

Taking $\liminf_{n\to\infty}$, we get that

$$h_c(z, x_w(s)) + A_{L+c}\big(x_w|_{[s,0]}\big) = h_c(z,x).$$

Hence $w \in \Gamma^-(u)$. □

4-9.3 Remarks.

1. Observe that, since $\Phi_c(x, x) = 0$,

$$u \prec L + c \iff u(x) = \inf_{z \in M} u(z) + \Phi_c(z, x).$$

2. Item 4-9.3.*1* implies that the function h_c in proposition 4-9.2 can not be replaced by Φ_c. In fact, the function $u_z(x) = \Phi_c(z, x)$ satisfies $u_z \prec L + c$, but in general $u \notin \mathfrak{S}^-$, if z is not properly chosen.

3. For any $z \in M$ the function $u_z(x) = h_c(z, x) \in \mathfrak{S}^-$ and $v_z(x) := -h_c(x, z) \in \mathfrak{S}^+$.

4. If $p \in \mathcal{P}$ then $u_p(x) := \Phi_c(p, x) \in \mathfrak{S}^-$, because

$$\Phi_c(p, x) \le h_c(p, x) \le h_c(p, p) + \Phi_c(p, x) \le \Phi_c(p, x).$$

Similarly, $v_p(x) := -\Phi_c(x, p) \in \mathfrak{S}^+$.

4-9.b The compact case.

In the next theorem we characterize all weak KAM solutions when M is compact. They are determined by their values on one point of each static class.

Let $\mathbf{\Gamma} = \mathcal{P}/d_c$ be the set of static classes of L. For each $\gamma \in \mathbf{\Gamma}$ choose $p_\Gamma \in \mathbf{\Gamma}$ and let $\mathbb{P} = \{\, p_\Gamma \,|\, \Gamma \in \mathbf{\Gamma} \,\}$. We say that a function $f : \mathbb{P} \to \mathbb{R}$ is *dominated* ($f \prec L + c$) if $\quad f(p) \le f(q) + \Phi_c(q, p), \quad$ for all p, $q \in \mathbb{P}$.

4-9.4 Theorem.
If M is compact, the maps $\{\, f : \mathbb{P} \to \mathbb{R} \,|\, f \prec L + c \,\} \to \mathfrak{S}^-$,

$$f \longmapsto u_f(x) := \inf_{p \in \mathbb{P}} f(p) + \Phi_c(p, x),$$

and $\{\, f : \mathbb{P} \to \mathbb{R} \,|\, f \prec L + c \,\} \to \mathfrak{S}^+$,

$$f \longmapsto v_f(x) := \sup_{p \in \mathbb{P}} f(p) - \Phi_c(x, p),$$

are bijective isometries in the sup *norm.*

Proof: We only prove it for $f \mapsto u_f$.

The domination condition $f \prec L + c$ implies that $u_f > -\infty$. Then remark 4-9.3.*4* and lemma 4-9.1.*1* imply that $u_f \in \mathfrak{S}^-$.

The injectivity follows from

$$u_f(p) = \min_{q \in \mathbb{P}} f(q) + \Phi_c(q,p) = f(p) \qquad \forall p \in \mathbb{P},$$

because f is dominated.

To prove the surjectivity, let $u \in \mathfrak{S}^-$ and let $f = u|_{\mathbb{P}}$. Given $x \in M$ choose $w \in \Gamma_0^-(u) \cap T_x M$ and let $\gamma(t) = \pi\, \varphi_t(w)$. So that

$$u(x) - u(\gamma(t)) = A_{L+c}(\gamma|_{[t,0]}) = \Phi_c(\gamma(t), x) \quad \text{ for } t < 0. \tag{4.14}$$

Choose $q \in \pi[\alpha\text{-lim}(v)] \subset \mathcal{P}$ (by proposition 3-11.3), and $t_n \to -\infty$ such that $\gamma(t_n) \overset{n}{\to} q$. Using $t = t_n$ on equation (4.14), in the limit we have that

$$u(x) = u(q) + \Phi_c(q,x). \tag{4.15}$$

Now take $p \in \mathbb{P}$ such that $d_c(p,q) = 0$. Since $u \prec L + c$, then

$$u(q) \leq u(p) + \Phi_c(p,q) \leq u(q) + \Phi_c(q,p) + \Phi_c(p,q) = u(q).$$

So that

$$u(q) = u(p) + \Phi_c(p,q). \tag{4.16}$$

By the triangle inequality

$$\begin{aligned} \Phi_c(p,x) &\leq \Phi_c(p,q) + \Phi_c(q,x) \\ &\leq \Phi_c(p,q) + \Phi_c(q,p) + \Phi_c(p,x) = \Phi_c(p,x). \end{aligned}$$

So that

$$\Phi_c(p,x) = \Phi_c(p,q) + \Phi_c(q,x). \tag{4.17}$$

Combining equalities (4.15), (4.17) and (4.16), we have that

$$u(x) = u(p) + \Phi_c(p,x),$$

with $p \in \mathbb{P}$. So that $u_f \leq u$. But since $u \prec L + c$ and $f = u|_{\mathbb{P}}$, using remark 4-9.3.*1*, we have that $u \leq u_f$.

Now we see that $f \mapsto u_f$ is an isometry in the supremum norm $\| \ \|_0$. Given $x \in M$, choose $p_n \in \mathbb{P}$ such that $u_g(x) = \lim_n f(p_n) + \Phi_c(p_n, x)$. Since $u_f \prec L + c$, we have that

$$\begin{aligned} u_f(x) - u_g(x) &\leq \liminf_n f(p_n) + \Phi_c(p_n, x) - g(p_n) - \Phi_c(p_n, x) \\ &\leq \|f - g\|_0 \,. \end{aligned}$$

Changing the roles of f and g we get that $\|u_f - u_g\|_0 \leq \|f - g\|_0$. Since $u_f|_{\mathbb{P}} = f$ and $u_g|_{\mathbb{P}} = g$, then $\|u_f - u_g\|_0 \geq \|f - g\|_0$.

□

4-9.5 Corollary. *There is only one static class if and only if* $\mathfrak{S}^-$ *(resp.* $\mathfrak{S}^+$*) is unitary modulo an additive constant.*

This characterization of weak KAM solutions allows us to recover the following theorem: We say that two weak KAM solutions $u_- \in \mathfrak{S}^-$ and $u_+ \in \mathfrak{S}^+$ are *conjugate* if $u_- = u_+$ on $\mathcal{P}$ and denote it by $u_- \sim u_+$.

4-9.6 Corollary. (Fathi [20]) *If M is compact, then*

$$h(x,y) = \sup_{\substack{u_\mp \in \mathfrak{S}^\mp \\ u_- \sim u_+}} \{\, u_-(y) - u_+(x) \,\}.$$

Proof: If $u_+ \sim u_-$ and $p \in \mathcal{P}$, from the domination we get

$$\begin{aligned} u_+(p) &\leq u_=(x) + \Phi_c(x,p), \\ u_-(y) &\leq u_-(p) + \Phi_c(p,y). \end{aligned}$$

Adding these equations and using that $u_=(p) = u_-(p)$, we get

$$u_-(y) - u_+(x) \leq \Phi_c(x,p) + \Phi_c(p,y).$$

Taking $\inf_{p\in\mathcal{P}}$ and then $\sup_{u_=\sim u_-}$ we obtain

$$\sup_{u_+\sim u_-}\big\{\, u_-(y)-u_+(x)\,\big\}\le h(x,y).$$

On the other hand, let $u_=(z):=-h(z,y)$ and

$$\begin{aligned} u_-(z) :&= \min_{q\in\mathcal{P}}\big\{\, u_=(q)+\Phi_c(q,z)\,\big\} \qquad (4.18)\\ &= \min_{q\in\mathcal{P}}\big\{\, -h(q,y)+\Phi_c(q,z)\,\big\}\\ &= \min_{q\in\mathcal{P}}\big\{\, -\phi_c(q,y)+\Phi_c(q,z)\,\big\} \qquad (4.19)\end{aligned}$$

From remark 4-9.3.*3* and corollary 4-9.4, $u_\pm\in\mathfrak{S}^\pm$. Since u_+ is dominated, from (4.18) we get that $u_+\sim u_-$. From (4.19), $u_-(y)=0$ and hence $u_-(y)-u_+(x)=h(x,y)$.

□

4-9.c Busemann weak KAM solutions.

When $h_c=+\infty$, we use another method to obtain weak KAM solutions, resembling the constructions of Busemann functions in riemannian geometry. By proposition 3-10.1, $\Sigma^+\neq\varnothing$ and $\Sigma^-\neq\varnothing$ even when M is non-compact. We call the functions of proposition 4-9.7 *weak KAM Busemann* functions.

4-9.7 Proposition.

1. *If $w\in\Sigma^-(L)$ and $\gamma(t)=x_w(t)$, then*

$$\begin{aligned} u_w(x) &= \inf_{t<0}\,[\Phi_c(\gamma(t),x)-\Phi_c(\gamma(t),\gamma(0))]\\ &= \lim_{t\to-\infty}[\Phi_c(\gamma(t),x)-\Phi_c(\gamma(t),\gamma(0))]\end{aligned}$$

is in $\mathfrak{S}^-$.

2. If $w \in \Sigma^+(L)$ and $\gamma(t) = x_w(t)$, then

$$\begin{aligned} u_w(x) &= \sup_{t>0} \left[\Phi_c(\gamma(0), \gamma(t)) - \Phi_c(x, \gamma(t))\right] \\ &= \lim_{t\to+\infty} \left[\Phi_c(\gamma(0), \gamma(t)) - \Phi_c(x, \gamma(t))\right] \end{aligned}$$

is in $\mathfrak{S}^+$.

Proof: We only prove item *1.* We start by showing that the function

$$\delta(t) = \Phi_c(\gamma(t), x) - \Phi_c(\gamma(t), 0)$$

is increasing. If $s < t$, then

$$\begin{aligned} \delta(t) - \delta(s) &= \Phi_c(\gamma(t), x) - \Phi_c(\gamma(s), x) + \big[\, \Phi_c(\gamma(s), \gamma(0)) - \Phi_c(\gamma(t), \gamma(0)) \,\big] \\ &= \Phi_c(\gamma(t), x) - \Phi_c(\gamma(s), x) + \Phi_c(\gamma(s), \gamma(t)) \\ &\geq 0, \end{aligned}$$

where the last inequality follows from the triangle inequality applied to the triple $(\gamma(s), \gamma(t), x)$. By the triangle inequality, $\delta(t) \leq \Phi_c(\gamma(0), x)$, hence $\lim_{t\downarrow-\infty} \delta(t) = \inf_{t<0} \delta(t)$ and this limit is finite.
Since

$$\begin{aligned} u(y) &= \inf_{t<0} \Phi_c(\gamma(t), y) - \Phi_c(\gamma(t), \gamma(0)) \\ &\leq \inf_{t<0} \Phi_c(\gamma(t), x) + \Phi_c(x, y) - \Phi_c(\gamma(t), \gamma(0)) \\ &= u(x) + \Phi_c(x, y), \end{aligned}$$

then $u \prec L + c$.

Suppose that $x \in \mathcal{P} \neq \varnothing$. Let $(x, v) \in \widehat{\Sigma}$ and $t < 0$. Let $p = x_v(t)$ and $y \in M$. Since $d_c(x, p) = 0$, then

$$\begin{aligned} \Phi_c(y, x) &= \Phi(y, x) + \Phi_c(x, p) + \Phi_c(p, x) \\ &\geq \Phi_c(y, p) + \Phi_c(p, x) \geq \Phi_c(y, x). \end{aligned}$$

Hence $\Phi_c(y,x)=\Phi_c(y,p)+\Phi_c(p,x)$. For $y=\gamma(s)$ (and $p=x_v(t)$), we have that

$$\begin{aligned} u(x)-u(x_v(t)) &= \lim_{s\to+\infty}\left[\Phi_c(\gamma(s),x)-\Phi_c(\gamma(s),x_v(t))\right]=\Phi_c(x_v(t),x)\\ &= A_{L+c}\left(x_v|_{[t,0]}\right). \end{aligned}$$

Now let $x\in M\setminus\mathcal{P}$ and choose $y_n:[T_n,0]\to M$ a Tonelli minimizer such that $y_n(T_n)=\gamma(\text{-}n)$, $y_n(0)=x$ and

$$A_{L+c}(y_n|_{[T_n,0]})\le\Phi_c(\gamma(\text{-}n),x)+\tfrac{1}{n}.$$

This implies that

$$A_{L+c}(y_n|_{[s,t]})\le\Phi_c(y_n(s),y_n(t))+\tfrac{1}{n},\qquad \text{for } T_n\le s<t\le 0. \tag{4.20}$$

By lemma 3-2.3, $|\dot y_n|<A$. We can assume that $\dot y_n(0)\to v\in T_xM$. Then

$$A_{L+c}(x_v|_{[t,0]})=\Phi_c(\gamma(t),x)\qquad \text{for } \liminf_n T_n\le t\le 0. \tag{4.21}$$

We prove below that $\lim_n T_n=-\infty$. Then $v\in\Sigma^-(L)$. Observe that for $T_n\le s\le 0$ we have that

$$\begin{aligned} \Phi_c(\gamma(\text{-}n),x) &\le \Phi_c(\gamma(\text{-}n),y_n(s))+\Phi_c(y_n(s),x)\\ &\le A_{L+c}(y_n|_{[T_n,0]})\le\Phi_c(\gamma(\text{-}n),x)+\tfrac{1}{n}. \end{aligned}$$

Since $y\mapsto\Phi_c(z,y)$ is uniformly Lipschitz, we obtain that

$$\begin{aligned} u(x) &= \lim_n \Phi_c(\gamma(\text{-}n),x)-\Phi_c(\gamma(\text{-}n),\gamma(0))\\ &= \lim_n \Phi_c(\gamma(\text{-}n),x_v(s))+\Phi_c(x_v(s),x)-\Phi_c(\gamma(\text{-}n),\gamma(0))\\ &= u(x_v(s))+\Phi_c(x_v(s),x) \qquad \text{for all } s<0.\\ &= u(x_v(s))+A_{L+c}(x_v|_{[s,0]}) \qquad \text{because } v\in\Sigma^-. \end{aligned}$$

Now we prove that $\lim_n T_n=-\infty$. Suppose, for simplicity, that $\lim_n T_n=T_0>-\infty$. Since $\dot y_n(0)\to v$, then $y_n|_{[T_n,0]}\xrightarrow{C^1}x_v|_{[T_0,0]}$ and hence $\gamma(\text{-}n) = y_n(T_n) \to x_v(T_0) =: p$. Since by lemma 3-2.3 $|\dot\gamma|$ is bounded, we can assume that $\lim_n \dot\gamma(\text{-}n) = (p, w_1)$. By lemma 3-11.3, $w_1 \in \alpha\text{-}\lim(\dot\gamma) \subseteq \widehat\Sigma$. From (4.21), $\dot x_v(T_0) \in \Sigma^\varepsilon$. Since $\pi(w_1) = x_v(T_0) = p$, then lemma 3-8.1 implies that $\dot x_v(T_0) \in \widehat\Sigma$. Since $\widehat\Sigma$ is invariant, then $v \in \widehat\Sigma$ and hence $x = \pi(v) \in \pi(\widehat\Sigma) = \mathcal{P}$. This contradicts the hypothesis $x \in M \setminus \mathcal{P}$. $\square$

4-9.8 Corollary. *$\mathfrak{S}^- \neq \varnothing$ and $\mathfrak{S}^+ \neq \varnothing$.*

4-10 Higher energy levels.

The method in proposition 4-9.7 allows us to obtain analogous weak KAM solutions on energy levels $k > c(L)$ when M is non-compact.

Let

$$\Sigma^-(k) := \big\{\, v \in TM \mid A_{L+k}\big(x_v|_{[t,0]}\big) = \Phi_k\big(x_v(t), x_v(0)\big), \quad \forall t \le 0 \,\big\}.$$

For $u \prec L + k$ define

$$\Gamma^-(u,k) = \big\{\, v \in TM \mid A_{L+k}\big(x_v|_{[t,0]}\big) = u\big(x_v(0)\big) - u\big(x_v(t)\big), \quad \forall t \le 0 \,\big\}. \tag{4.22}$$

Let

$$\mathfrak{S}^-(k) = \big\{\, u : M \to \mathbb{R} \mid u \prec L + k \text{ and } \pi\big(\Gamma^-(u,k)\big) = M \,\big\}.$$

Define similarly $\Sigma^+(k)$, $\Gamma^+(u,k)$ and $\mathfrak{S}^+(k)$. Then the functions $u \in \mathfrak{S}^\pm$ satisfy Lipschitz, covering, graph and smoothness properties analogous to theorem 4-8.3.

Observe that we are requiring that the global minimizers in (4.22) are defined in the whole ray $]-\infty, 0]$. In the following lemma we show that there are not such weak KAM solutions when M is compact and $k > c(L)$. If we look for weak solutions of (H-J) with realizing curves defined only on *finite* intervals then the action potential $u(x) = \Phi_k(p,x)$, $v(x) = -\Phi_k(p,x)$ give such examples.

4-10.1 Proposition.
If M is compact, then $\Sigma^\pm(k) = \varnothing$ and hence $\mathfrak{S}^\pm(k) = \varnothing$ for all $k > c(L)$.
If M is non-compact then $\Sigma^\pm(k) \ne \varnothing$ for all $k > c(L)$.
Moreover, if $v \in \Sigma^\pm(k)$, then ω-$\lim(v) = \varnothing$ (resp. α-$\lim(v) = \varnothing$).

Proof: Suppose that $v \in \Sigma^+(k) \ne \varnothing$ and that $w \in \omega$-$\lim(v) \ne \varnothing$. Observe that the orbit of w can not be a fixed point (i.e. $w \ne 0$) because by (3.18), $k > c(L) \ge e_0$. Let $p = \pi(w)$ and choose $s > 0$ such that

$q = \pi(\varphi_s(w)) \neq p$. Then the same argument as in proposition 3-11.3 shows that

$$d_k(p,q) = \Phi_k(p,q) + \Phi_l(q,p) = 0,$$

which contradicts proposition 2-1.1.4.

This proves that $\Sigma^{\pm}(k) = \varnothing$ when M is compact. Observe that if $u_+ \in \mathfrak{S}^+(k)$ then $\varnothing \neq \Gamma^+(u,k) \subseteq \Sigma^+(k)$. Hence $\mathfrak{S}^{\pm}(k) = \varnothing$ when M is compact.

Assume now that M is non-compact. We show that $\Sigma^+(k) \neq 0$. Let $x \in M$ and $\langle y_n \rangle \subseteq M$ such that $d_M(x, y_n) \to +\infty$. Let $\gamma_n \in \mathcal{C}_{T_n}(x, y_n)$ be a Tonelli minimizer such that

$$A_{L+c}(\gamma_n) < \Phi_k(x, y_n) + \tfrac{1}{n}. \tag{4.23}$$

Then by lemma 3-2.3, $|\dot{\gamma}_n| < A$, and hence $T_n \to +\infty$. Let v be a density point of $\{\dot{\gamma}_n(0)\}$. Since $\gamma_n|_{[0,t]} \xrightarrow{C^1} x_v|_{[0,t]}$ for all $t > 0$, from (4.23) we obtain that $v \in \Sigma^+(k)$.

$\square$

By corollary 4-4.7, there are no weak solutions of (H-J) for $k < c(L)$. We complete the picture with the following:

4-10.2 Proposition.

If M is non-compact, then $\mathfrak{S}^{\pm}(k) \neq \varnothing$ for all $k \geq c(L)$.

If M is compact, then $\mathfrak{S}^{\pm}(k) = \varnothing$ for all $k > c(L)$.

Proof: Proposition 4-10.1 proves the case M compact.

Now we reproduce the proof of proposition 4-9.7. Let $w \in \Sigma^+(k) \neq \varnothing$, write $\gamma(t) := x_w(t)$ and

$$u_w(x) := \lim_{t \to +\infty} [\Phi_k(\gamma(0), \gamma(t)) - \Phi_k(x, \gamma(t))]$$

The limit exists because by the triangle inequality for Φ_k, the function

$$\delta(t) = \Phi_k(\gamma(0), \gamma(t)) - \Phi_k(x, \gamma(t))$$

is non-decreasing and it is bounded above by $\Phi_k(\gamma(0),x)$. The triangle inequality implies that $u \prec L+k$.

Now let $x \in M$ and choose $y_n : [0,T_n] \to M$ a Tonelli minimizer such that $y_n(0)=x$, $y_n(T_n)=\gamma(n)$ and

$$A_{L+c}(y_n|_{[0,T_n]}) \le \Phi_k(x,\gamma(n)) + \tfrac{1}{n}.$$

This implies that for all $0<t<T_n$,

$$\Phi_k(x,\gamma(n)) \le A_{L+k}\big(y_n|_{[0,t]}\big) + \Phi_k(y_n(t),\gamma(n)) \le \Phi_k(x,\gamma(n)) + \tfrac{1}{n}. \tag{4.24}$$

By lemma 3-2.3, $|\dot{y}_n| < A$. Since $y_n(T_n)=\gamma(n)$ and by proposition4-10.1, $\gamma(n)\to\infty$, then $\liminf_n T_n = +\infty$. Let $v \in T_xM$ be a density point of $\dot{y}_n(0)$.

Since $y_n|_{[0,t]} \xrightarrow{C^1} x_v|_{[0,t]}$, using (4.24) we get that

$$\begin{aligned} u(x_v(t)) &= \lim_n \Phi_k(\gamma(0),\gamma(n)) - \Phi_k(x_v(t),\gamma(n)) \\ &= \lim_n \Phi_k(\gamma(0),\gamma(n)) - \Phi_k(x,\gamma(n)) + A_{L+k}(y_n|_{[0,t]}) \\ &= u(x) + A_{L+k}(x_v|_{[0,t]}), \end{aligned}$$

for all $0<t<\liminf_n T_n = +\infty$.

□

4-11 The Lax-Oleinik semigroup.

The Lax-Oleinik semigroup was used to obtain weak KAM solutions by **Lions, Papanicolaou and Varadhan** [32] on tori $\mathbb{T}^n$ and later by Fathi [21] for compact manifolds.

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Through this section we shall assume that M is compact. The Lax-Oleinik semigroup $\langle T^- \rangle_{t\geq 0}$ is the semigroup of operators $T_t^- : C^0(M,\mathbb{R}) \to C^0(M,\mathbb{R})$ defined by

$$\begin{aligned} T_t^- u(x) &= \inf_{\gamma} \{ \, u(\gamma(0)) + \int_0^t L(\gamma(s),\dot\gamma(s)) + c \, ds \, \} \\ &= \min_{y\in M} \{ \, u(y) + \Phi_c(y,x;t) \, \}, \end{aligned}$$

where the infimum is taken on all piecewise differentiable curves $\gamma : [0,T] \to M$ with $\gamma(T) = x$. Similarly, define

$$T_t^+ u(x) = \max_{z\in M} \{ \, u(z) - \Phi_c(x,z;t) \, \}.$$

4-11.1 Proposition. *If M is compact.*

1. *The unique $c \in \mathbb{R}$ for which the semigroup $\langle T_t^- \rangle_{t\geq 0}$ has a fixed point is the critical value $c = c(L)$.*
2. *$u \in \mathfrak{S}^- \iff u$ is a fixed point of the semigroup $\langle T_t^- \rangle_{t\geq 0}$.*
3. *For all $\varepsilon > 0$ there exists a Lipschitz constant $K = K(\varepsilon) > 0$ such that $T_t^-(C^0(M,\mathbb{R})) \subseteq \mathrm{Lip}_K(M,\mathbb{R})$ for all $t > \varepsilon$.*
4. *T_t^- is a weak contraction: For all $t \geq 0$ and all $u, v \in C^0(M,\mathbb{R})$, $\left\| T_t^- u - T_t^- v \right\|_0 \leq \|u - v\|_0$.*
5. *For $\varepsilon > 0$ there exists $K(\varepsilon) > 0$ such that if $u, v \in C^0(M,\mathbb{R})$ and $s,\, t > \varepsilon$, then $\left\| T_s^- u - T_t^- v \right\|_0 \leq \|u - v\|_0 + K(\varepsilon)\, |s - t|$.*

Proof:

1. By proposition 3-4.1, $\lim_{t\to+\infty}\Phi_k(x,y;t)=\pm\infty$ when $k\neq c(L)$, uniformly in x, $y\in M$. This implies that there are no fixed points for $T_t^{\pm}$ when $c\neq c(L)$. The existence of a fixed point is given by item *2*.

2. Let $u\in\mathfrak{S}^-$. For $x\in M$ take $v\in\Gamma^-(u)\cap T_xM$ and let $\gamma(t):=\pi\,\varphi_t(v)$. For $t>0$ we have that

$$\begin{aligned} u(x) &= u(\gamma(-t))+A_{L+c}\big(\gamma|_{[-t,0]}\big)\\ &= u(\gamma(-t))+\Phi_c(\gamma(-t),x;t). \end{aligned}$$

But since $u\prec L+c$, then

$$u(x)\le u(y)+\Phi_c(y,x)\le u(y)+\Phi_c(y,x;t).$$

Hence $T_t^-u=u$ for all $t>0$.

Now suppose that $T_t^-u=u$ for all $t\ge0$. Then

$$u(x)\le u(z)+\Phi_c(z,x;t)\qquad\forall t\ge0,\quad\forall z\in M.$$

Taking the infimum on $t\ge0$, we get that $u\prec L+c$. Moreover,

$$u(x)\le\min_{z\in M}u(z)+\Phi_c(z,x)\le\min_{z\in M}u(z)+h_c(z,x).$$

For $t>0$ let z_t be such that $u(x)=u(z_t)+\Phi_c(z_t,x;t)$. Let $t_n\to+\infty$ be a sequence such that the limit $\lim_n z_{t_n}=y$ exists. Then

$$u(x)=u(y)+\lim_n\Phi_c(z_{t_n},x;t_n)\ge u(y)+h_c(y,x),$$

where the last inequality follows from the definition of h_c and the Lipschitz property for $\Phi_c(x,y;t)$ in proposition 3-4.1.1. Thus $u(x)=\min_{z\in M}u(z)+h_c(z,x)$, by proposition 4-9.2, $u\in\mathfrak{S}^-$.

3. Using proposition 3-4.1, let $K=K(\varepsilon)$ be a Lipschitz constant for $(x,y;t)\mapsto\Phi_c(x,y;t)$ on $M\times M\times[\varepsilon,+\infty[$. Given $u\in C^0(M,\mathbb{R})$,

x, $y \in M$ and $t > \varepsilon$, let $z \in M$ be such that $T_t^- u(y) = u(z) + \Phi_c(z, y; t)$. Then

$$\begin{aligned} T_t^- u(x) &\le u(z) + \Phi_c(z, x; t) \\ &\le u(z) + \Phi_c(z, y; t) + |\Phi_c(z, x; t) - \Phi_c(z, y; t)| \\ &\le T_t^- u(y) + K(\varepsilon)\, d(x, y). \end{aligned}$$

Changing the roles of x and y we get that $T_t^- u \in \mathrm{Lip}_K(M, \mathbb{R})$.

5. Observe that item *5* implies item *4*. Let u, $v \in C^0(M, \mathbb{R})$, s, $t > \varepsilon$ and $x \in M$. Let $K = K(\varepsilon)$ be as in item *3*. Choose $z \in M$ such that $T_s^- u(x) = u(z) + \Phi_c(z, x; s)$. Then

$$\begin{aligned} T_t^- v(x) &\le v(z) + \Phi_c(z, x; t) \\ &\le T_s^- u(x) + |v(z) - u(z)| + |\Phi_c(z, x; t) - \Phi_c(z, x; s)| \\ &\le T_t^- u(x) + \|u - v\|_0 + K(\varepsilon)\, |s - t|. \end{aligned}$$

Changing the roles of u and v we get item *5*. □

4-11.2 Remark. *Fixed Points for the Lax-Oleinik semigroup.*

Another proof of the existence of a fixed point in 4-11.3

We sketch here another proof of existence of a fixed point for the Lax-Oleinik semigroup using properties *3* and *4* of proposition 4-11.1.

Consider the semigroup λT_t^-, with $0 < \lambda < 1$ acting on the space $C^0(M, \mathbb{R})/_{\mathbb{R}}$ of continuous functions modulo an additive constant. Then λT_t^- is a contraction whose image is in the compact space $\mathrm{Lip}_K(M, \mathbb{R})/_{\mathbb{R}}$, independent of λ or t. Let $u_{\lambda,t}$ be a fixed point for λT_t^-. Choosing a sequence $\lambda_n \to 1$ and a subsequence such that $u_{\lambda_n,t}$ converges in $\mathrm{Lip}_K(M, \mathbb{R})/_{\mathbb{R}}$, one obtains a fixed point u_t for T_t^-. Now let $t_n \to 0$ and choose a subsequence such that u_{t_n} converges to some v in $\mathrm{Lip}_K(M, \mathbb{R})/_{\mathbb{R}}$. Fix $s \in \mathbb{R}$ and let $N_n \in \mathbb{Z}$ be such that $\lim_n N_n t_n = s$. Since by the semigroup property, u_{t_n} is also a fixed point for $T_{N_n t_n}^-$, using proposition 4-11.1.*5*, we get that in the C^0-topology,

$$T_s^- v = \lim_n T_{N_n t_n}^- u_{t_n} = \lim_n u_{t_n} = v.$$

So that v is a fixed point for the whole semigroup $\langle T_t^- \rangle_{t \geq 0}$.

4-11.3 Theorem (Fathi [23]). *If M is compact, for all $u \in C^0(M, \mathbb{R})$ the uniform limit $\lim_{t \to +\infty} T_t^- u$ exists.*

To prove this theorem we shall need some lemmas.

4-11.4 Lemma. *If M is compact then $\lim_{t \to +\infty} \frac{1}{t} \Phi_c(x, y; t) = 0$, uniformly on $(x, y) \in M \times M$.*

Proof: Write $h_t(x, y) := \Phi_c(x, y; t)$. Then

$$\inf_{M \times M} h_t + 2 \inf_{M \times M} h_1 \leq h_{t+2} \leq \inf_{M \times M} h_t + 2 \sup_{M \times M} h_1.$$

Then, writing $C = 2\,(\sup h_1 - \inf h_1)$, we have that

$$\forall t > 2, \quad \sup_{M \times M} h_t - \inf_{M \times M} h_t \leq 2\,C.$$

Now let u be a weak KAM solution. Since $T_t^- u = u$, we have that for all $t > 0$,

$$u(y) = \min_{x \in M} u(x) + \Phi_c(x, y; t).$$

Thus, for all $(x, y) \in M \times M$ and $t > 2$,

$$|\Phi_c(x, y; t)| \leq \sup_{x \in M} u(x) - \inf_{x \in M} u(x) + 2\,C.$$

This implies the lemma.

□

4-11.5 Lemma. *For all $\varepsilon > 0$, there exists $T_0 > 0$ such that if $T > T_0$ and $\gamma : [0, T] \to M$ is a Tonelli minimizer, then $|E(\gamma, \dot\gamma) - c(L)| < \varepsilon$.*

Proof: Let $T_n \to +\infty$ and let $\gamma_n : [0, T_n] \to M$ be a Tonelli minimizer. Then by lemma 3-2.2 there is $A > 0$ such that $|\dot\gamma_n| < A$ for all n. Then there exists a subsequence such that the probabilities μ_{γ_n} converge weakly* to a limit μ. Since L is bounded on $|v| \leq A$, and using lemma 4-11.4,

$$A_{L+c}(\mu) = \lim_n A_{L+c}(\mu_{\gamma_n}) = \lim_n \tfrac{1}{T_n}\Phi_c(\gamma_n(0), \gamma_n(T_n); T_n) = 0.$$

Hence μ is a minimizing measure and $E(\operatorname{supp}\mu) \equiv c(L)$. Thus $E(\gamma_n, \dot\gamma_n) \to c(L)$. $\square$

For $f : T^*M \to \mathbb{R}$ measurable and $u : M \to \mathbb{R}$ Lipschitz, define

$$\operatorname*{ess\,sup}_{p\in T^*M} f = \inf_{\substack{A\subseteq T^*M \\ \mathrm{Leb}(T^*M\setminus A)=0}} \sup_{p\in A} f(p),$$

and

$$\mathbb{H}(f) = \operatorname*{ess\,sup}_{x\in M} H(x, d_x f).$$

4-11.6 Lemma. *If $u : M \to \mathbb{R}$ is Lipschitz, then $u \prec L + \mathbb{H}(u)$.*

Proof: Using a convolution argument as in lemma 4-4.5, we can approximate both $\mathbb{H}(u)$ and u in the C^0 topology by a C^∞ function. Hence we can assume that u is C^1. Then for all $(x, v) \in TM$,

$$d_x u \cdot v \leq L(x, v) + H(x, d_x u) \leq L(x, v) + \mathbb{H}(u).$$

If $\gamma \in \mathcal{C}(x, y)$, then

$$u(y) - u(x) \leq \oint_\gamma du \leq \oint_\gamma L + \mathbb{H}(u).$$

$\square$

4-11.7 Lemma. *If $u \in C^0(M, \mathbb{R})$, then $\lim_{t\to+\infty} \mathbb{H}(T_t^- u) = c$.*

Proof: By proposition 4-11.1.*3*, $T_t^- u$ is Lipschitz. If x is a differentiability point for $T_t^- u$, then

$$d_x(T_t^- u) = L_v(\gamma(t), \dot\gamma(t)),$$

where $\gamma : [0,t] \to M$ is a Tonelli minimizer satisfying $\gamma(t) = x$, and $T_t^- u(x) = u(\gamma(0)) + A_{L+c}(\gamma|_{[0,t]})$. Thus $H(d_x(T_t^- u)) = E(\gamma(t), \dot\gamma(t))$. By lemma 4-11.5, $E(\gamma, \dot\gamma)$ converges to $c(L)$ uniformly on t. $\square$

Proof of theorem 4-11.3:

Let $u \in C^0(M, \mathbb{R})$. By proposition 4-11.1.*2* the family $\langle T_t^- u\rangle_{t\geq 1}$ is equilipschitz. By Arzela-Áscoli theorem there exists a sequence $t_n \to +\infty$ such that the uniform limit $u_\infty = \lim_n T_{t_n}^- u$ exists. **By lemma 4-11.6,** $T_t^- u \prec L + \mathbb{H}(T_t^- u)$ and **by lemma 4-11.7,** $\mathbb{H}(T_t^- u) \to c$. This implies that $u_\infty \prec L + c$.

Now we prove that u_∞ is a fixed point of the semigroup T_t^-. Since $u_\infty \prec L + c$, then $u_\infty \leq T_t^- u_\infty$ for all $t \geq 0$. Since T_t^- preserves the order, we get that $u_\infty \leq T_s^- u_\infty \leq T_t^- u_\infty$ for all $s \leq t$. It is enough to show that there is a sequence $s_n \to +\infty$ such that $T_{s_n} u_\infty \to u_\infty$. Write $s_n = t_{n+1} - t_n$, we can assume that $s_n \to +\infty$. Then, using 4-11.1.*4*,

$$\begin{aligned}\left\|T_{s_n}^- u_\infty - u_\infty\right\|_0 &\leq \left\|T_{s_n}^- u_\infty - T_{s_n}^- T_{s_n}^- u\right\|_0 + \|T_{t_{n+1}}^- u - u_\infty\|_0 \\ &\leq \left\|u_\infty - T_{t_n} u\right\|_0 + \|T_{t_{n+1}}^- u - u_\infty\|_0 \to 0.\end{aligned}$$

It remains to show that the fixed point u_∞ of T_t^- is the limit of $T_t^- u$. But for all $s \geq 0$,

$$\left\|T_{t_n+s}^- u - u_\infty\right\|_0 = \left\|T_s^- T_{t_n}^- u - T_s^- u_\infty\right\|_0 \leq \left\|T_{t_n}^- u - u_\infty\right\|_0 \xrightarrow{n} 0.$$

$\square$

The following corollaries are also due to A. Fathi.

4-11.8 Corollary. *If M is compact, then $\lim\limits_{t\to\infty} \Phi_c(x, y; t) = h_c(x, y)$, uniformly in $x \in M$ and also in $y \in M$.*

Proof: The uniform limit on $y \in M$ follows from the equality $\Phi_c(x,y;t) = T^-_{t-1}u(y)$, where $u(z) = \Phi_c(x,z;1)$. And the uniform limit on $x \in M$ follows from $-\Phi_c(x,y;t) = T^+_{t-1}v$, where $v(z) = -\Phi_c(z,y;1)$. $\square$

Recall that a *ray* is a curve $\gamma(t)$ such that it is a Tonelli minimizer for all $t \geq 0$ (resp. $t \leq 0$).

4-11.9 Corollary. *If M is compact then the rays are semistatic.*

Proof: For an a.c. curve γ, and $s \leq t$ in its domain, define

$$\delta(s,t) := A_{L+c}(\gamma_{[s,t]}) - \Phi_c(\gamma(s),\gamma(t)).$$

Observe that $\delta(s,t) \geq 0$. We show that

$$[s_1,t_1] \subseteq [s_2,t_2] \implies \delta(s_1,t_1) \leq \delta(s_2,t_2).$$

Indeed, by the triangle inequality,

$$-\Phi_c(\gamma(s_2),\gamma(s_1)) - \Phi_c(\gamma(s_1),\gamma(t_1)) - \Phi_c(\gamma(t_1),\gamma(t_2)) \leq -\Phi_c(\gamma(s_2,t_2).$$

Adding the equality

$$A_{L+c}(\gamma|_{[s_2,s_1]}) + A_{L+c}(\gamma|_{[s_1,t_1]}) + A_{L+c}(\gamma|_{[t_1,t_2]}) = A_{L+c}(\gamma|_{[s_2,t_2]})$$

and using that $\delta(s,t) \geq 0$, we get that

$$\delta(s_1,t_1) \leq \delta(s_2,s_1) + \delta(s_1,t_1) + \delta(t_1,t_2) \leq \delta(s_2,t_2).$$

Now suppose that $\gamma : [0,+\infty[\to M$ is a ray. Observe that

$$\delta(s,t) = \Phi_c(\gamma(s),\gamma(t);t-s) - \Phi_c(\gamma(s),\gamma(t)). \tag{4.25}$$

By theorem 4-11.3, the map $t \mapsto \delta(0,t)$ is bounded. Define

$$\delta(s,+\infty) := \lim_{t\to+\infty} \delta(s,t) = \sup_{t\geq s} \delta(s,t) \leq \delta(0,t).$$

Since the map $s \mapsto \delta(s,+\infty)$ is increasing, we have that

$$\lim_{s\to+\infty} \delta(s,+\infty) = 0.$$

Now we prove that the ω-limit vectors of a ray are static. Write $\gamma(t) = \pi\varphi_t(v)$, $t \geq 0$. Suppose that $t_n \to +\infty$ and $\dot\gamma(t_n) \to w \in TM$. Let $\eta(t) = \pi\,\varphi_t(w)$. Since γ and η are solutions the Euler-Lagrange equation, then $\gamma|_{[t_n-s,t_n+s]} \xrightarrow[C^1]{} \eta|_{[-s,s]}$. We have that

$$\begin{aligned}
&A_{L+c}\big(\eta|_{[-s,s]}\big) + \Phi_c(\eta(s),\eta(-s)) = \\
&\quad = \lim_n \big\{\, A_{L+c}(\gamma|_{[t_n-s,t_n+s]}) + \lim_m \Phi_c(\gamma(t_n+s),\gamma(t_m-s))\,\big\}\\
&\quad = \lim_n \big\{\, A_{L+c}(\gamma|_{[t_n-s,t_n+s]}) + \lim_m A_{L+c}(\gamma|_{[t_n+s,t_m-s]}) - \delta(t_n+s,t_m-s)\,\big\}\\
&\quad = \lim_n\ \lim_m\ A_{L+c}(\gamma|_{[t_n-s,t_m-s]}) - \delta(t_n+s,t_m-s)\\
&\quad = \lim_n\ \lim_m\ \Phi_c(\gamma(t_n-s),\gamma(t_m-s)) + \delta(t_n-s,t_m+s) - \delta(t_n+s,t_m-s)\\
&\quad = \Phi_c(\eta(-s),\eta(-s)) + \lim_n \delta(t_n-s,+\infty) - \delta(t_n+s,+\infty)\\
&\quad = 0.
\end{aligned}$$

Thus $w \in \widehat{\Sigma}(L)$.

Finally, we prove that γ is semistatic. Let $t_n \to +\infty$ be such that the limit $p = \lim_n \gamma(t_n)$ exists. Then $p \in \mathcal{P} = \pi(\widehat{\Sigma})$. Using (4.25), we have that corollary 4-11.8, implies that

$$\delta(s,+\infty) = \lim_n \delta(s,t_n) = h_c(\gamma(s),p) - \Phi_c(\gamma(s),p) = 0$$

for all $s \geq 0$. Thus $\delta(s,t) \equiv 0$ for all $0 \leq s \leq t$, i.e. γ is semistatic. □

4-12 The extended static classes.

The method in proposition 4-9.7 resembles the construction of Busemann functions in complete manifolds of non-positive curvature. In that case, Ballmann, Gromov and Schroeder [5] proved that the manifold can be compactified adjoining the sphere at infinity that can be defined in terms of Busemann functions.

Here we emulate that construction to obtain a compactification of the manifold that identifies the points in the Peierls set which are in the same static class and adjoins what we call the *extended Peierls set* $\mathfrak{P}^{\mp}$. By definition of Busemann function, the extended static classes in $\mathfrak{B}^{\mp}$ correspond to the α-limits (resp. ω-limits) of semistatic orbits in the compactification. But as we shall see in example 4-12.4 the classes in $\mathfrak{P}^{\mp} \setminus \mathfrak{B}^{\mp}$ do not correspond to α or ω limits of orbits in TM.

On $C^0(M, \mathbb{R})$ we use the topology of uniform convergence on compact subsets. Consider the equivalence relation on $C^0(M, \mathbb{R})$ defined by $f \sim g$ if $f - g$ is constant. Let $\mathcal{F} := C^0(M, \mathbb{R})/_\sim$ with the quotient topology.

Let $\mathfrak{M}^-$ be the closure in $\mathcal{F}$ of $\{\, f(x) = \Phi_c(z, x) \,|\, z \in M \,\}/_\sim$ and $\mathfrak{M}^+$ the closure in $\mathcal{F}$ of $\{\, g(x) = \Phi_c(x, z) \,|\, z \in M \,\}$. Fix a point $0 \in M$. We can identify

$$\mathcal{F} \approx \{\, f \in C^0(M, \mathbb{R}) \,|\, f(0) = 0 \,\}.$$

4-12.1 Lemma. *$\mathfrak{M}^-$ and $\mathfrak{M}^+$ are compact.*

Proof: Observe that the functions in $\mathfrak{M}^-$ and $\mathfrak{M}^+$ are dominated. By lemma 4-2.1.*1* the families $\mathfrak{M}^-$ and $\mathfrak{M}^+$ are equicontinuous. Since M is separable by Arzelá-Ascoli theorem $\mathfrak{M}^-$ and $\mathfrak{M}^+$ are compact in the topology of uniform convergence on compact subsets. □

Then $\mathfrak{M}^-$ is the closure of the classes of the functions

$$f_z(x) := \Phi_c(z, x) - \Phi_c(0, x), \qquad \forall\, z \in M$$

and $\mathfrak{M}^+$ is the closure of the classes of

$$g_z(x) := \Phi_c(x,z) - \Phi_c(0,z), \qquad \forall\, z \in M.$$

4-12.2 Lemma.

1. *If $f_w(x) = f_z(x)$ for all $x \in M$, then $d_c(w,z) = 0$.*
2. *If $g_w(x) = g_z(x)$ for all $x \in M$, then $d_c(w,z) = 0$.*

Proof: We only prove item *1*. Suppose that $f_z = f_w$. From $f_z(z) = f_w(z)$ we get that

$$\Phi_c(w,z) = \Phi_c(w,0) - \Phi_c(z,0),$$

and from $f_z(w) = f_z(w)$ we get

$$\Phi_c(z,w) = -\Phi_c(w,0) + \Phi_c(z,0).$$

Adding these equations we get that $d_c(z,w) = 0$.

Conversely, if $d_c(z,w) = 0$ and $x \in M$, then

$$\Phi_c(w,x) \le d_c(w,z) + \Phi_c(z,x) = \Phi_c(z,x) - \Phi_c(z,w) \le \Phi_c(w,x).$$

Thus $\Phi_c(w,x) = \Phi_c(w,z) + \Phi_c(z,x)$ for all $x \in M$. This implies that $f_z = f_w$. $\square$

Then we have embeddings $M/_{d_c} \hookrightarrow \mathfrak{M}^-$, by $z \mapsto [f_z] \in \mathcal{F}$ and $M/_{d_c} \hookrightarrow \mathfrak{M}^+$ by $z \mapsto [g_z] \in \mathcal{F}$, where $M/_{d_c}$ is the quotient space under the equivalence relation $x \equiv y$ if $d_c(x,y) = 0$. Let $\mathfrak{B}^-$ be the functions defined in proposition 4-9.7.*1* and $\mathfrak{B}^+$ those of 4-9.7.*2*. Let $\boldsymbol{\mathfrak{B}}^+ = \mathfrak{B}^+/_\sim$ and $\boldsymbol{\mathfrak{B}}^- = \mathfrak{B}^-/_\sim$.

4-12.3 Remark. By proposition 4-9.7, if $p \in \mathcal{P} \ne \varnothing$ then $u_-(x) := \Phi_c(p,x) \in \boldsymbol{\mathfrak{B}}^-$ and $u_+(x) := -\Phi_c(x,p) \in \boldsymbol{\mathfrak{B}}^+$ (modulo an additive constant).

Observe that $d_c(z,w)=0$ if and only if $z=w$ or $z,\ w\in\mathcal{P}$ and they are in the same static class. Under the identifications $M\hookrightarrow\mathfrak{M}^{\mp}$ we have that $\mathfrak{B}^{\mp}\cup(M\setminus\mathcal{P})\subseteq\mathfrak{M}^{\mp}$ respectively. But this inclusion may be strict as the following example shows:

4-12.4 Example. $\mathfrak{B}^{-}\cup(M\setminus\mathcal{P})\neq\mathfrak{M}^{-}$.

Let $M=\mathbb{R}$ and $L(x,v):=\frac{1}{2}v^2-\cos(2\pi x)$, corresponding to the universal cover of the simple pendulum lagrangian. Then $c(L)=1$, and the static orbits are the fixed points $(2k+1,0)\in T\mathbb{R}$, $k\in\mathbb{Z}$. Moreover, $H(x,p)=\frac{1}{2}p^2+\cos(2\pi x)$ and the Hamilton-Jacobi equation $H(x,d_xu)=c(L)$ gives $d_xu=\pm 2\sqrt{1-\cos(2\pi x)}$. The function

$$u(x)=\int_0^x 2\sqrt{1-\cos(2\pi s)}\,ds,$$

with $d_xu\equiv +2\sqrt{1-\cos(2\pi x)}$, is in $\mathfrak{S}^{-}$, is the limit of $u_n(x):=\Phi_c(-n,x)-\Phi_c(-n,0)$ but it is not a Busemann function associated to a semistatic orbit γ because if $\gamma(-\infty)=2k+1\in\mathbb{Z}$ is the α-limit of γ, then the Busemann function b_γ associated to γ satisfies

$$d_xb_\gamma=\begin{cases}+2\sqrt{1-\cos(2\pi x)} & \text{if }\ x\geq\gamma(-\infty),\\ -2\sqrt{1-\cos(2\pi x)} & \text{if }\ x\leq\gamma(-\infty).\end{cases}\tag{4.26}$$

Similarly a function $v:\mathbb{R}\to\mathbb{R}$ with $d_xv\equiv -2\sqrt{1-\cos(1\pi x)}$ is in $\mathfrak{S}^{+}$ but it is not a Busemann function.

Observe that in the Busemann function in (4.26), at the point $y=\gamma(-\infty)+3$ the semistatic orbit $\eta(t)$ with $\dot\eta(0)=\Gamma^{-}(u)\cap T_yM$ has α-limit $\eta(-\infty)=\gamma(-\infty)+2\neq\gamma(-\infty)$. Moreover, the Busemann function b_η associated to η satisfies

$$d_xb_\eta=\begin{cases}+2\sqrt{1-\cos(2\pi x)} & \text{if }\ x\geq\gamma(-\infty)+2,\\ -2\sqrt{1-\cos(2\pi x)} & \text{if }\ x\leq\gamma(-\infty)+2;\end{cases}$$

so that $b_\eta\neq b_\gamma$. In fact, there is no semistatic orbit passing through y with α-limit $\gamma(-\infty)$. This implies that the Busemann functions can not be parametrized just by a (semistatic) vector based on a unique point $0 \in M$ as in the riemannian case. In particular, it may not be possible to choose a single point $q_\alpha \equiv 0 \in M$, $\forall \alpha \in \mathfrak{B}^-$ in the construction for theorem 4-12.7.

The functions in $\mathfrak{B}^-$ and $\mathfrak{B}^+$ are special among the weak KAM solutions. They are "directed" towards a single static class and they are the most regular in the following sense:

4-12.5 Lemma.

1. *If $w \in \Sigma^-$ and $u_w \in \mathfrak{B}^-$ is as in proposition 4-9.7.1, then*
$$u_w(x) = \max\{\, u(x) \,|\, u \in \mathfrak{S}^-,\ u(\pi(w)) = 0,\ w \in \Gamma^-(u)\,\}.$$

2. *If $w \in \Sigma^+$ and $u_w \in \mathfrak{B}^+$ is as in proposition 4-9.7.2, then*
$$u_w(y) = \min\{\, u(y) \,|\, u \in \mathfrak{S}^+,\ u(\pi(w)) = 0,\ w \in \Gamma^+(u)\,\}.$$

By the remark 4-12.3, this also holds for the functions $u_-(x) = \Phi_c(p, x)$ and $u_+(x) = -\Phi_c(x, p)$ (modulo an additive constant), for any $p \in \mathcal{P}$.

Proof: We prove item *1.* Let $x := \pi(w)$ and $v \in \mathfrak{S}^-$ with $v(x) = u_w(x) = 0$ and $w \in \Gamma^-(v)$. Let $x_w(t) = \pi(\Phi_t(w))$. Since $v \prec L + c$ and $w \in \Gamma^-(v)$, then for $t < 0$, we have that

$$\begin{aligned} v(y) &\le v(x_w(t)) + \Phi_c(x_w(t), y) \\ &= v(x) - \Phi_c(x_w(t), x) + \Phi_c(x_w(t), y). \end{aligned}$$

Since $v(x) = u_w(x) = 0$, letting $t \downarrow -\infty$, we get that $v(y) \le u_w(y)$ for all $y \in M$. On the other hand, u_w is in the set of such u's, so that the maximum is realized by u_w. $\square$

Define

$$\mathfrak{P}^- := \mathfrak{M}^- \setminus (M - \mathcal{P}) \quad , \qquad \mathfrak{P}^+ := \mathfrak{M}^+ \setminus (M - \mathcal{P}).$$

4-12.6 Proposition.

The functions in $\mathfrak{P}^-$ and $\mathfrak{P}^+$ are weak KAM solutions.

Proof: Let $u \in \mathfrak{M}^- \setminus (M \setminus \mathcal{P})$. Since u is dominated, we only have to prove the condition 4-8.1.*2*. Adding a constant, we can assume that $u(0) = 0$. Then there is a sequence $z_n \in M$ such that $u(x) = \lim_n \Phi_c(z_n, x) - \Phi_c(z_n, 0)$. Let $x \in M$ and let $\gamma_n \in \mathcal{C}_{T_n}(z_n, x)$ be a Tonelli minimizer such that $T_n < 0$, $\gamma_n(0) = x$, $\gamma_n(T_n) = z_n$ and $A_{L+c}(\gamma_n) \le \Phi_c(z_n, x) + \frac{1}{n}$. In particular

$$A_{L+c}(\gamma_n|_{[t,0]}) \le \Phi_c(\gamma_n(t), x) + \tfrac{1}{n}, \qquad \forall\ T_n \le t \le 0.$$

Since $u \in \mathfrak{P}^-$, then we can assume that either $d_M(z_n, x) \to \infty$ or $z_n \to p \in \mathcal{P}$. Since by lemma 3-2.3 $|\dot\gamma_n| < A$ and $h_c(p,p) = 0$ for $p \in \mathcal{P}$, in either case we can assume that $T_n \to -\infty$.

We can assume that $\dot\gamma_n(0) \to v \in T_x M$. Then for $t \le 0$

$$\begin{aligned} u(x_v(x)) - u(x_v(t)) &= \lim_n \Phi_c(z_n, x) - \Phi_c(z_n x_v(t)) \\ &= \lim_n \Phi_c(z_n, x) - \Phi_c(z_n, \gamma_n(t)) + K\, d_c(\gamma_n(t), x_v(t)) \\ &\le \lim_n A_{L+c}\big(\gamma_n|_{[t,0]}\big) + \tfrac{1}{n} \\ &\le A_{L+c}\big(x_v|_{[t,x]}\big), \end{aligned}$$

where K is a Lipschitz constant for Φ_c.

□

For $p \in \mathfrak{B}$ and $z \in M$ let $x \mapsto b_{p,z}(x)$ be the function in the class $p \in \mathfrak{B}$ such that $b_{p,z}(z) = 0$, i.e.

$$b_{p,z}(x) = \lim_{y \to p} \Phi_c(y, x) - \Phi_c(y, z).$$

We now give a characterization of weak KAM solutions similar to that of corollary 4-9.4. For each $\alpha \in \mathfrak{B}^-$ choose $q_\alpha \in M$ such that there is a unique semistatic vector $v \in \Sigma^-$ such that $\pi(v) = q$ and the α-limit of v is in the static class α. This can be done by the graph property 4-8.3.*5*. Moreover, choose them such that the map $\mathfrak{B}^- \ni \alpha \mapsto q_\alpha \in M$ is injective. Let $\mathbb{P} := \{\, q_\alpha \,|\, \alpha \in \mathfrak{B}^- \,\}$. We say that a function $f : \mathbb{P} \to M$ is *strictly dominated* if

$$f(q_a) < f(q_\beta) + b_{\beta,q_\beta}(q_\alpha)$$

for all $a \neq \beta$ in $\mathfrak{B}^-$. And we say that f is *dominated* if $f(q_a) \leq f(q_\beta) + b_{\beta,q_\beta}(q_\alpha)$ for all $a \neq \beta$ in $\mathfrak{B}^-$.

4-12.7 Theorem.
The map $\{\, f : \mathbb{P} \to \mathbb{R} \,|\, f$ *strictly dominated* $\} \to \{u \in \mathfrak{S}^- \,|\, u|_{\mathbb{P}}$ *strictly dominated* $\}$, $f \mapsto u_f$*, given by*

$$u_f(x) := \inf_{\alpha\in\mathfrak{B}^-} f(q_\alpha) + b_{\alpha,q_\alpha}(x),$$

is a bijection.

Proof: We first prove that u_f is bounded below. The domination condition implies that $u_f(q_\alpha) = f(q_\alpha)$ for all $\alpha \in \mathfrak{B}^-$. Then the same argument as in formula (4.13), shows that $u_f \prec L + c$. Fix $\alpha \in \mathfrak{B}^-$, then for all $x \in M$,

$$u_f(x) \geq u_f(q_\alpha) - \Phi_c(q_\alpha, x) = f(q_\alpha) - \Phi_c(q_\alpha, x) > -\infty. \tag{4.27}$$

Since $u_f > -\infty$ and it is an infimum of weak KAM solutions, from lemma 4-9.1 we get that $u_f \in \mathfrak{S}^-$. Since $u_f(q_\alpha) = f(q_\alpha)$ for all $\alpha \in \mathfrak{B}^-$, the map $f \mapsto u_f$ is injective.

We now prove the surjectivity. Suppose that $u \in \mathfrak{S}^-$ and $u|_{\mathbb{P}}$ is strictly dominated. Let

$$v(x) := \min_{\alpha\in\mathfrak{B}^-} u(q_\alpha) + b_{\alpha,q_\alpha}(x).$$

Observe that the domination condition implies that

$$v(q_\alpha) = u(q_a) \quad \text{for all } \alpha \in \mathfrak{B}^-. \tag{4.28}$$

Given $x \in M$, let $\theta \in \Gamma^-(u) \cap T_x M$ and let $\alpha \in \mathfrak{B}^-$ be the α-limit of θ. Then,

$$u(x) = u(x_\theta(s)) + \Phi_c(x_\theta(s), x) \qquad \text{for all } s < 0.$$

Since u is dominated, $u(q_\alpha) \le u(x_\theta(s)) + \Phi_c(x_\theta(s), q_\alpha)$. Hence

$$u(x) \ge u(q_a) - \Phi_c(x_\theta(s), q_a) + \Phi_c(x_\theta(s), x) \qquad \text{for all } s < 0.$$

Taking the limit when $s \to -\infty$, we get that

$$u(x) \ge v(x) \qquad \text{for all } x \in M. \tag{4.29}$$

Now we prove that $u = v$ on the projection of the backward orbits of vectors in $\Gamma^-(u)$ ending at the points q_α, $\alpha \in \mathfrak{B}^-$. Let $\xi \in \Gamma^-(u) \cap T_{q_\alpha} M$ and let $\beta \in \mathfrak{B}^-$ be the α-limit of ξ. From the definition of $v(x)$ for all $\varepsilon > 0$ and $s < 0$ there exists $\gamma = \gamma(s, \varepsilon) \in \mathfrak{B}^-$ such that

$$v(x_\xi(s)) \ge v(q_\gamma) + b_{\gamma, q_\gamma}(x_\xi(s)) - \varepsilon.$$

Since $\xi \in \Gamma^-(u) \cap T_{q_\alpha} M$, then for $s < 0$,

$$\begin{aligned}
u(q_\alpha) &= u(x_\xi(s)) + \Phi_c(x_\xi(s), q_\alpha) && \quad (4.30)\\
&\ge v(x_\xi(s)) + \Phi_c(x_\xi(s), q_\alpha) \qquad \text{by (4.29)} && \quad (4.31)\\
&\ge v(q_\gamma) + b_{\gamma, q_\gamma}(x_\xi(s)) - \varepsilon + \Phi_c(x_\xi(s), q_\alpha)\\
&= \lim_{t \to -\infty} v(q_\gamma) + \Phi_c(x_\xi(t), x_\xi(s)) + \Phi_c(x_\xi(s), q_\alpha) - \Phi_c(x_\xi(t), q_\gamma) - \varepsilon\\
&\ge v(q_\gamma) + \lim_{t \to -\infty} \Phi_c(x_\xi(t), q_\alpha) - \Phi_c(x_\xi(t), q_\gamma) - \varepsilon\\
&\ge v(q_\gamma) + b_{\gamma, q_\gamma}(q_\alpha) - \varepsilon\\
&\ge v(q_\alpha) - \varepsilon && \quad (4.32)\\
&= u(q_\alpha) - \varepsilon. \qquad \text{by (4.28).}
\end{aligned}$$

Letting $\varepsilon \downarrow 0$, from the equality between (4.31) and (4.32) we get that

$$v(q_\alpha) = v(x_\xi(t)) + \Phi_c(x_\xi(t), q_\alpha) \qquad \text{for all } t < 0. \tag{4.33}$$

But then

$$v(q_\beta) \leq v(x_\xi(t)) + \Phi_c(x_\xi(t), q_\beta) = v(q_\alpha) - \Phi_c(x_\xi(t), q_\alpha) + \Phi_c(x_\xi(t), q_\beta).$$

Equivalently

$$v(q_\alpha) \geq v(q_\beta) + \Phi_c(x_\xi(t), q_\alpha) - \Phi_c(x_\xi(t), q_\beta).$$

Taking the limit when $t \to -\infty$, we get that $v(q_\alpha) \geq v(q_\beta) + b_{\beta, q_\beta}(q_\alpha)$. This contradicts the strict domination, hence $\beta = \alpha$. Then, from the equality between (4.30) and (4.31), we have that

$$u(x_\xi(t)) = v(x_\xi(t)) \quad \text{for all } t < 0, \text{ and } \xi \in \Sigma^- \cap T_{q_\alpha}M, \ \alpha\text{-lim}(\xi) = \alpha. \tag{4.34}$$

Now let $x \in M$ and $\alpha \in \mathfrak{B}^-$. Let $\xi \in \Sigma^- \cap T_{q_a}M$ with α-lim$(\xi) = \alpha$. Then for $t < 0$,

$$\begin{aligned} u(x) &\leq u(x_\xi(t)) + \Phi_c(x_\xi(t), x) \\ &= v(x_\xi(t)) + \Phi_c(x_\xi(t), x) && \text{by (4.34)} \\ &= v(q_\alpha) - \Phi_c(x_\xi(t), q_\alpha) + \Phi_c(x_\xi(t), x), && \text{by (4.33)}. \end{aligned}$$

Letting $t \to -\infty$, we have that

$$u(x) \leq v(q_\alpha) + b_{\alpha, q_\alpha}(x) = u(q_\alpha) + b_{\alpha, q_\alpha}(x).$$

Since $\alpha \in \mathfrak{B}^-$ is arbitrary, from the definition of v we get that $u \leq v$. □

4-12.8 Theorem.

Given $u \in \mathfrak{S}^-$, for all $\alpha \in \mathfrak{B}^-(u)$ choose $q_\alpha \in \pi[\Lambda_0^-(\alpha) \cap \Gamma_0^-(u)]$, and let $\mathbb{P}(u) := \{q_\alpha \,|\, \alpha \in \mathfrak{B}^-(u)\}$. Then

$$u(x) = \inf_{q_\alpha \in \mathbb{P}(u)} u(q_\alpha) + b_{\alpha, q_\alpha}(x) \qquad \textit{for all } x \in M.$$

Proof: Let $u \in \mathfrak{S}^-$. For all $\alpha \in \mathfrak{B}^-(u)$, choose $q_\alpha \in \pi(\Lambda_0^-(\alpha) \cap \Gamma_0^-(u))$. Let $\mathbb{P}(u) := \{\, q_\alpha \,|\alpha \in \mathfrak{B}^-(u)\,\}$. We show that $u|_{\mathbb{P}(u)}$ is dominated. Let $\alpha, \beta \in \mathfrak{B}^-(u)$ and let $\theta \in T_{q_\beta}M \cap \Lambda^-(\beta) \cap \Gamma^-(u)$. Then for $t < 0$,

$$\begin{aligned} u(q_\alpha) &\le u(x_\theta(t)) + \Phi_c(x_\theta(t), q_\alpha) \\ &= u(q_\beta) - \Phi_c(x_\theta(t), q_\beta) + \Phi_c(x_\theta(t), q_\alpha). \end{aligned}$$

Letting $t \to -\infty$, we get that $u(q_\alpha) \le u(q_\beta) + b_{\beta, q_\beta}(q_\alpha)$, for all $\alpha, \beta \in \mathfrak{B}^-(u)$.

Let

$$v(x) := \inf_{q_\alpha \in \mathbb{P}(u)} u(q_\alpha) + b_{\alpha, q_\alpha}(x). \tag{4.35}$$

The same arguments as in equation (4.27) show that $v > -\infty$ and by lemma (4-9.1) $v \in \mathfrak{S}^-$.

Given $x \in M$, let $\theta \in \Gamma^-(u) \cap T_x M$ and let $\alpha \in \mathfrak{B}^-(u)$ be the α-limit of θ. Then,

$$\begin{aligned} &u(x) = u(x_\theta(s)) + \Phi_c(x_\theta(s), x) && \text{for all } s < 0, \\ &u(x) \ge u(q_a) - \Phi_c(x_\theta(s), q_a) + \Phi_c(x_\theta(s), x) && \text{because } u \text{ is dominated.} \end{aligned}$$

Since $\alpha \in \mathfrak{B}^=(u)$, taking the limit when $s \to -\infty$, we get that

$$u(x) \ge v(x) \qquad \text{for all } x \in M. \tag{4.36}$$

Now let $x \in M$ and $q_\alpha \in \mathbb{P}(u)$. Let $\xi \in \Lambda(\alpha) \cap \Gamma^-(u) \cap T_{q_\alpha}M$. Then for $s < 0$,

$$\begin{aligned} u(x) &\le u(x_\xi(s)) + \Phi_c(x_\xi(s), x) \\ &= u(q_\alpha) - \Phi_c(x_\xi(s), q_\alpha) + \Phi_c(x_\xi(s), x), \quad \text{because } \xi \in \Gamma^-(u) \cap T_{q_\alpha}M. \end{aligned}$$

Since $\xi \in \Lambda^-(\alpha)$, letting $s \to -\infty$, we have that

$$u(x) \le u(q_\alpha) + b_{\alpha, q_\alpha}(x).$$

Since $q_\alpha \in \mathbb{P}(u)$ is arbitrary, we get that $u \le v$. $\square$

Chapter 5

Examples

5-1 Riemannian Lagrangians.

For a riemannian lagrangian $L(x,v) = \frac{1}{2}\|v\|_x^2$, we have that

$$e_0 = c_u = c_0 = c(L) = 0.$$

The static orbits are the fixed points $(x,0)$, $x \in M$. The only weak KAM solutions are the constant functions and so are the only subsolutions of (H-J) for $k = c(L)$.

Nevertheless, since the α function is superlinear, from (**??**) when we add some closed forms to L we obtain $c(L-\omega) > c(L) = 0$. Moreover, the semistatic static orbits are not fixed points because by **??** they have energy $c(L-\omega) > 0$. By the graph properties, the static set of $L-\omega$, the support of it minimizing measures and the basins of weak KAM solutions are geodesic laminations. Since semistatic geodesics are minimizing, they don't have conjugate points.

5-2 Mechanic Lagrangians.

For a mechanic lagrangian $L(x,v) = \frac{1}{2}\|v_x\|^2 - U(x)$, we have that

$$c(L) = c_0 = c_u = e_0 = \min_{x\in M} E(x,0) = \max_{x\in M} U(x),$$

the maximum of the potential energy. The static orbits are the fixed points $(x,0)$, where $U(x)$ is maximal. If U is non-degenerate at a minimum x (i.e. d_x^2U is non-singular), then x is a hyperbolic saddle point. The constant functions are always subsolutions of (H-J) for $k=c(L)=e_0$.

5-3 Symmetric Lagrangians.

For a symmetric lagrangian $L(x,v)=L(x,-v)$, and then $H(x,p)=H(x,-p)$. Since $v\mapsto L(x,v)$ is convex, it attains its minimum at $v=0$. Thus, if M is compact,

$$L(x,v)+e_0=L(x,v)-\max_{x\in M}L(x,0)\geq 0,$$

and it is 0 exactly at the fixed points $(x,0)$ which maximize $L(x,0)$. This implies that $c(L)=e_0$. In fact, the constant functions are subsolutions of the Hamilton-Jacobi equation (H-J) for $k=e_0$. The static orbits are the fixed points $(x,0)$ where x maximizes $L(x,0)$.

If M is non-compact, the constant functions are still subsolutions of (H-J) for $k=e_0$, and thus

$$c(L)=c_0=c_u=e_0,$$

but there may be no maximizers of $L(x,0)$ and then the static set $\widehat{\Sigma}(L)=\varnothing$ (see example 5-7).

5-4 Simple Pendulum.

The simple pendulum is the mechanic lagrangian $L:TS^1\to\mathbb{R}$ given by

$$L(\theta,\dot\theta)=\tfrac{1}{2}\,|\dot\theta|^2-cos\theta,$$

identifying the circle $S^1=[0,2\pi]\mod 2\pi$. Hence $c(L)=e_0=1$. The static set is $\widehat{\Sigma}(L)=\{(\pi,0)\}$. Since there is a unique static class, there are only one weak KAM solutions on $\mathfrak{S}^+$ and $\mathfrak{S}^-$ modulo an additive constant. But $\mathfrak{S}^- \neq \mathfrak{S}^+$ because the solutions are not differentiable. Their cut locus is $x = 0 \in S^1$. For $u \in \mathfrak{S}^+$ its is the whole $\Sigma^(L)$ and it is shown in figure 5-4.

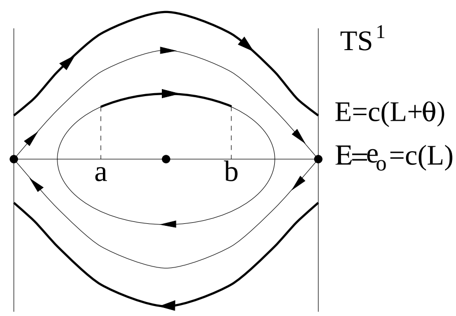


FIG. 1: SIMPLE PENDULUM.

If $\omega \neq 0$ is a closed 1-form, $c(L-\omega) > e_0 = 1$ and the static set $\widehat{\Sigma}(L-\omega)$ is a whole component of the energy level $E = c(L)$, oriented by $\omega > 0$. Again, there is only one static class and $\#\mathfrak{S}^\pm = 1$ modulo constants. Since $\pi(\widehat{\Sigma}(L-\omega)) = S^1$, the functions on $\mathfrak{S}^\pm$ are differentiable.

5-5 The flat Torus $\mathbb{T}^n$.

Also called the harmonic oscillator, it is the riemannian lagrangian for $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ with the riemannian metric induced by the euclidean metric on $\mathbb{R}^n$. The solutions of (E-L) are the projection of straight lines in $\mathbb{R}^n$ parametrized with constant speed. If $\omega(x, v) = \langle A, v\rangle$ with $A \in \mathbb{R}^n$ fixed, then $L - \omega$ is globally minimized at all the vectors (x, A). Since the orbits of this vector field are recurrent, they must be static. These vectors are sent to 0 by the Legendre transform. By the smoothness property 4-8.3.*6*, $du = 0$ and thus any weak KAM solution must be constant. By corollary 4-9.5, there is only one static class.

5-6 Flat domain for the β-function.

This example is a special case of those studied by Carneiro and Lopes [8]. Let $L : T\mathbb{T}^2 \to \mathbb{R}$ be the lagrangian

$$L\big((x,y),(u,v)\big) = \tfrac{1}{2}(u^2+v^2) + v\,\sin(x),$$

where $\mathbb{T}^2 = [0,2\pi]_{\text{ mod } 2\pi}$. Then the lagrangian is minimized at the vectors $x = -\frac{\pi}{2}$, $v = 1$ and $x = \frac{\pi}{2}$, $v = -1$. These vectors are tangent to the closed curves $\gamma_+ \;:\; x \equiv -\frac{\pi}{2}$, $\dot{y} \equiv 1$ and $\gamma_- \;:\; x \equiv \frac{\pi}{2}$, $\dot{y} \equiv -1$. Since the curves γ_+ and γ_- are closed, they must be static curves. Since their tangents are exactly all the vectors which minimize L, the Peierls set is just $\mathcal{P} = \big(\{-\frac{\pi}{2}\} \times [0,2\pi]\big) \cup \big(\{\frac{\pi}{2}\} \times [0,2\pi]\big)$. The invariant measures $\mu_+ := \mu_{\gamma_+}$, $\mu_- := \mu_{\gamma_-}$ supported on the closed orbits $\dot{\gamma}_+$, $\dot{\gamma}_-$ are minimizing measures with homology $\rho_+ = (0, \frac{1}{2\pi})$, $\rho_- = (0, -\frac{1}{2\pi})$.

Since Mather's β-function is convex and attains its minimum at ρ_+ and ρ_-, then the line

$$[\rho_-, \rho+] = \big\{\, (0,t) \in H_1(\mathbb{T}{\cdot}\mathbb{R}) \approx \mathbb{R}^2 \mid -\tfrac{1}{2\pi} \le t \le \tfrac{1}{2\pi} \,\big\}$$

must be a flat domain for the β-function, in fact $\beta|_{[\rho_-,\rho_+]} \equiv c(L)$. Since the static set $\widehat{\Sigma}(L)$ contains only the support of the measures μ_+, μ_-, then for any homology $h \in [\rho_-, \rho_+$, $h = t\rho_- + (1-t)\rho_+$, the unique minimizing measure in homology h is $\mathcal{M}(h) = \{t\mu_- + (1-t)\mu_+\}$.

5-7 A Lagrangian with Peierls barrier $h = +\infty$.

Let $L : T\mathbb{R}^2 \to \mathbb{R}$ be $L(x,v) = \frac{1}{2}\,|v|^2 + \psi(x)$, where $|\cdot|$ is the euclidean metric on $\mathbb{R}^2$ and $\psi(x)$ is a smooth function with $\psi(x) = \frac{1}{|x|}$ for $|x| \ge 2$, $\psi \ge 0$ and $\psi(x) = 2$ for $0 \le |x| \le 1$.

Then

$$c(L) = -\inf \psi = 0,$$

because if γ_n is a smooth closed curve with length $\ell(\gamma_n) = 1$, $|\gamma_n(t)| \geq n$ and energy $E(\gamma_n) = \frac{1}{2}\,\dot\gamma_n^2 - \psi(\gamma_n) \equiv 0$, then

$$\begin{aligned} c(L) \geq -\inf_{n>0} A_L(\gamma_n) &= -\int_0^{T_n} \tfrac{1}{2}\,\dot\gamma_n^2 + \psi(\gamma_n) \\ &= -\int_0^{\frac{1}{|\dot\gamma_n|}} |\dot\gamma_n|^2 = -|\dot\gamma_n| \leq -\sqrt{\tfrac{2}{n}} \longrightarrow 0. \end{aligned}$$

On the other hand,

$$c(L) = -\inf\{A_L(\gamma)\,|\,\gamma \text{ closed}\} \leq 0,$$

because $L \geq 0$.

Observe that since $L > 0$ and on compact subsets of $\mathbb{R}^2$, $L > a > 0$, then we have that

$$d_c(x,y) = \Phi_c(x,y) > 0 \text{ for all } x,y \in \mathbb{R}^2.$$

Hence $\hat\Sigma(L) = \varnothing$.

Suppose that $h(0,0) < +\infty$. Then $u(x) := h(x,0)$ is in $\mathfrak{S}^+$. Let $\xi \in \Gamma^+(u) \cap T_0\mathbb{R}^2$ and write $x_\xi(t) = (r(t), \theta(t))$ in polar coordinates about the origin $0 \in \mathbb{R}^2$. Then $\liminf_{t\to+\infty} r(t) = +\infty$ because otherwise the orbit of ξ would lie on a compact subset $E \equiv 0$ and then $\varnothing \neq \omega\text{-lim}(\xi) \subseteq \Sigma(L) = \varnothing$, then

$$|\dot x_\xi(t)| = \sqrt{\tfrac{2}{r(t)}}$$

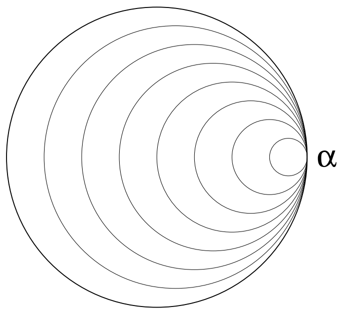
and

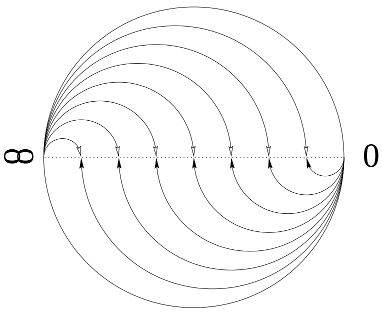
$$L(\varphi_t\,\xi) = |\dot x_\xi(t)|^2 = \sqrt{\tfrac{2}{r(t)}}\;|\dot x_\xi(t)|.$$

Let $T_n \to +\infty$ be such that $r(T_n) \to +\infty$. Since $L + c = L \geq 0$, then

$$\begin{gathered} h(0,0) \geq \int_0^{+\infty} L(\varphi_t(\xi)) + c(L) = \int_0^{+\infty} \sqrt{\tfrac{2}{r(t)}}\,\big[\,|\dot x| + r\,|\dot\theta|\,\big]\,dt \\ \limsup_{T_n} \int_0^{T_n} \sqrt{\tfrac{2}{r}}\,\dot x\,dt = \limsup_n \int_0^{r(T_n)} \sqrt{\tfrac{2}{r}}\,dr = +\infty. \end{gathered}$$

□





5-8 Horocycle flow.

Peierls barrier $0 < h < +\infty$, $\widehat{\Sigma} = \varnothing$ *and differentiable Busemann functions* u *with* $\mathfrak{B}^-(u) = \mathfrak{B}^+(u) = \{\alpha\}$.

Let $\mathbb{H} := \mathbb{R}\times]0, +\infty[$ with the Poincaré metric $ds^2 = \frac{1}{y^2}\,(dx^2 + dy^2)$. Let $L : T\mathbb{H} \to \mathbb{R}$ be a Lagrangian of the form

$$L(x, v) = \tfrac{1}{2}\ \|v\|_x^2 + \eta_x(v),$$

where η_x is a 1-form on $\mathbb{H}$ such that $d\eta(v)$ is the area form an $|\cdot|_x$ is the Poincaré metric. The Euler-Lagrange equation is

$$\tfrac{Dt}{dt}\,\dot{x} = Y_x(\dot{x}) = \dot{x}^{\perp}, \tag{5.1}$$

where $Y_x : T\mathbb{H} \to T\mathbb{H}$ is a bundle map such that

$$d\eta_x(u, v) = \langle Y_x(u), v\rangle.$$

The energy function is $E(x, v) = \frac{1}{2}\ \|v\|_x^2$. On the energy levels $E < \frac{1}{2}$ the solutions of (5.1) are closed curves, and on $E = \frac{1}{2}$ the solutions are the horospheres parametrized by arc length.

Choose the form $\eta(x, y) = \frac{dx}{y}$, where $(x, y) \in \mathbb{H} = \mathbb{R}\times]0, +\infty[$. Then

$$L\big((x, y), (\dot{x}, \dot{y})\big) = \frac{1}{2y^2}\,\big(\dot{x}^2 + \dot{y}^2\big) + \frac{\dot{x}}{y}\cdot$$

Observe that the form η is bounded in the Poincaré metric, so that the Lagrangian is superlinear and satisfies the boundedness condition.

It can be seen directly from the Euler-Lagrange equation that the curves $\dot{x} = -y$, $\dot{y} = 0$ are solutions with

$$L(\dot{x} = -y, \dot{y} = 0) + \tfrac{1}{2} \equiv 0. \tag{5.2}$$

The images of these curves are the stable horospheres associated to the geodesic $x = 0$, $\dot{y} = y$, parametrized by arc length.

We show that $c(L) = \frac{1}{2}$ and $h_c < +\infty$. Observe that if $v = (\dot{x}, \dot{y})$, $\dot{x} < 0$, then

$$L = \tfrac{1}{2}\ \|v\|^2 - \|(\dot{x}, 0)\| \geq \tfrac{1}{2}\ \|v\|^2 - \|v\| \geq -\tfrac{1}{2}. \tag{5.3}$$

Hence $L + \frac{1}{2} \geq 0$ and then $c(L) \leq \frac{1}{2}$.

Now fix $x \in \mathbb{H}$. For $r > 0$ let D_r be a geodesic disc of radius r such that $x \in \partial D_r$. Let γ_r be the curve whose image is the boundary of D_r oriented clockwise and with hyperbolic speed $\|\dot{\gamma}\| \equiv a$. Since $E(\gamma) = \frac{1}{2}\, a^2$, then

$$\begin{aligned}
\int_{\gamma_r} L + \tfrac{1}{2}a^2 &= \int_{\gamma_r} v \cdot L_v = \int_{\gamma_r} \|v\|^2 + \int_{D_r} dA \\
&= a \cdot \text{length}(\gamma_r) - \text{area}(D_r), \\
&= a \cdot 2\pi \sinh(r) - 2\pi \cosh(r), \\
&= 2\pi \big[\tfrac{1}{2}\,(a-1)\, e^r - e^{-r}\big] + 2\pi.
\end{aligned} \tag{5.4}$$

If $a < 1$, for $r > 0$ large, formula (5.4) is negative. Hence $c(L) \geq \frac{1}{2}$ and $c(L) = \frac{1}{2}$. Moreover,

$$h(x, x) \leq \liminf_{r \to +\infty} A_{L+\frac{1}{2}}(\gamma_r) = 2\pi < +\infty.$$

We prove that $\widehat{\Sigma} = \varnothing$. This implies that $h > 0$. First observe that if T is an isometry of $\mathbb{H}$, then $d(T_*\eta)$ is also the area form, so that $T_*\eta$

is cohomologous to η. This implies that given any two points $x, y \in \mathbb{D}$, there is a constant $b = b(x,y) \in \mathbb{R}$ such that for all $\gamma \in \mathcal{C}(x,y)$,

$$A_L(\gamma) = A_L(T \circ \gamma) + b(x,y).$$

In particular, the map dT leaves $\sigma(L)$ and $\widehat{\Sigma}(L)$ invariant. Since a horocycle h_1 can be sent by an isometry to another horocycle h_2 with $h_1 \cap h_2 \neq \varnothing$, then the horocycles can not be static because it would contradict the graph property.

The constant function $u : \mathbb{H} \to \{0\}$ satisfies $u \prec L + \frac{1}{2}$ because $L + \frac{1}{2} \geq 0$ and by (5.2) the vectors $v = (-y, 0) \in \Gamma^+(u) = \Gamma^-(u) \in \Sigma^-$ are semistatic. Its derivative $du = 0$ is sent by the inverse of the Legendre transform $v \mapsto L_v = \langle v, \cdot \rangle_x + \frac{dx}{y}$ to

$$\tfrac{1}{y^2} \langle v, \cdot \rangle_{\text{eucl}} = -\tfrac{dx}{y},$$

that is $v = (-y, 0)$. Also

$$H(du) = \tfrac{1}{2} \left\| du - \tfrac{dx}{y} \right\|^2 = \tfrac{1}{2} \left\| \tfrac{dx}{y} \right\|^2 = \tfrac{1}{2}.$$

Let $T : \mathbb{H} \hookleftarrow$ be an isometry of the hyperbolic metric. Write $\eta = \frac{dx}{y}$. Then $d\eta = A$ is the hyperbolic area 2-form. Since T is an isometry, then

$$d(T^*\eta) = T^*(d\eta) = d\eta.$$

Hence the form $T^*\eta - \eta$ is exact on $\mathbb{H}$ and there is a smooth function $v : \mathbb{H} \to \mathbb{R}$ such that

$$T^*\eta - \eta = -dv.$$

We show that v is a weak KAM solution. Observe that

$$\begin{aligned} L \circ dT(x,v) &= \tfrac{1}{2} \, \|v\|_x^2 + T^*\eta(x,v) \\ &= \tfrac{1}{2} \, \|v\|_x^2 + \eta(x,v) - dv(x,v) \\ &= L(x,v) - dv(x,v). \end{aligned} \tag{5.5}$$

Since by (5.3) $L \circ dT + \frac{1}{2} \geq 0$, then

$$dv \leq L + \tfrac{1}{2}. \tag{5.6}$$

Hence $v \prec L + \frac{1}{2}$. Moreover, the equality in (5.6) holds exactly when $L \circ dT(x, v) + \frac{1}{2} = 0$, i.e. when $dT(v) = (-y, 0) \in T_{(x,y)}\mathbb{H}$.

Since the isometries send horospheres to horospheres, they are self-conjugacies of the hamiltonian flow and hence the curves $\gamma(t) = T^{-1}(x - ty, y)$ realize v, i.e.

$$v(\gamma(t)) - v(\gamma(s)) = \oint_\gamma dv = \oint_\gamma L + \tfrac{1}{2}.$$

Here v is the Busemann weak KAM solution associated to the class $T(\infty) \in \partial\mathbb{H}$, on the sphere at infinity of $\mathbb{H}$.

We now show a picture of a non-Busemann weak KAM solution. We use the isometry $T : \mathbb{H} \hookleftarrow$, $T(z) = -\frac{1}{z}$, $z = x + iy \in \mathbb{C}$. The isometry $T = T^{-1}$ sends the line $t \mapsto -ty + iy$ to a horosphere with endpoint $0 \in \mathbb{C}$, oriented clockwise. Choose $v : \mathbb{H} \to \mathbb{R}$ such that $dv = \eta - T^*\eta$. Since T leaves the line $\operatorname{Re} z = 0$ invariant and $\eta = 0$ on vertical vectors, hence v is constant on $\operatorname{Re} z = 0$.

Now we describe the weak KAM solution

$$w(z) := \min\{\, u(z), v(z)\,\} \in \mathfrak{S}^-.$$

Let $\gamma(t) = -ty + iy$. Then, using (5.5),

$$\begin{aligned}
v(T^{-1}\gamma(t)) &= v(T^{-1}\gamma(0)) + \oint_{T^{-1}\circ\gamma} dv \\
&= 0 + \int_0^t \big[\, L \circ dT \circ \dot\gamma + \tfrac{1}{2}\,\big] - \int_0^t \big[\, L \circ \dot\gamma + \tfrac{1}{2}\,\big] \\
&= 0 + \int_0^t \big[\, L \circ dT \circ \dot\gamma + \tfrac{1}{2}\,\big] - 0.
\end{aligned}$$

Since by (5.6) $L(x,v)+\frac{1}{2}>0$ when $v\neq -y+i0$, then $v(z)>0$ on $\operatorname{Re} z>0$ and $v(z)<0$ on $\operatorname{Re} z<0$. Thus

$$w(z)=\begin{cases}0=u(z) & \text{if } \operatorname{Re} z>0.\\ v(z) & \text{if } \operatorname{Re} z<0.\end{cases} \tag{5.7}$$

The cut locus of w is $\operatorname{Re} z=0$ and the basin of w is $\Gamma^-(w)=A\cup dT(A)$ where A is the set of vectors $(y,0)\in T_{x+iy}\mathbb{H}$.

Chapter 6

Generic Lagrangians.

In [38], Mañé introduced the concept of *generic property of a lagrangian* L. A property P is said to be generic for the lagrangian L if there exists a residual set $\mathcal{O}$ on $C^\infty(M,\mathbb{R})$ such that if ψ is in $\mathcal{O}$ then $L+\psi$ has the property P.

A set is called *residual* if it contains a countable intersection of open and dense subsets. We recall which topology is used in $C^\infty(M,\mathbb{R})$. Given $u \in C^\infty(M,\mathbb{R})$, denote by $|u|_k$ its C^k-norm. Define

$$\|u\|_\infty := \sum_{k\in\mathbb{N}} \frac{\arctan(\|u\|_k)}{2^k}.$$

Note that $\|\cdot\|_\infty$ is not a norm. Endow $C^\infty(M,\mathbb{R})$ with the translation-invariant metric $\|u-v\|_\infty$. This metric is complete, hence the Baire property holds: any residual subsets of $C^\infty(M,\mathbb{R})$ is dense.

One of Mañé's objectives was to show that Mather's theory of minimizing measures becomes much more accurate and stronger if one restricts it to generic lagrangians. In this chapter we shall prove

6-0.1 Theorem.

For every lagrangian L *there exist a residual set* $\mathcal{O} \subset C^\infty(M,\mathbb{R})$ *such that if* $\psi \in \mathcal{O}$

1. *The lagrangian $L+\psi$ has a unique minimizing measure μ, and this measure is uniquely ergodic.*

2. $\operatorname{supp}(\mu)=\hat{\Sigma}(L+\psi)=\Sigma(L+\psi)$.

3. *When μ is supported on a periodic orbit or a fixed point, this orbit (point) Γ is hyperbolic and if its stable and unstable manifolds intersect then they do it transversally.*

Mañé conjectures in [35] that there exists a generic set $\mathcal{O}$ such that this unique minimizing measure is supported on a periodic orbit or an equilibrium point.

Item *1* of theorem 6-0.1 was proved by Mañé in [34]. Item *2* is proved in [16] and item *3* in [13]. Th proof of item *1* presented here is extracted from a more general result in [6] about families of lagrangian systems.

We will prove only the first part and give only the ideas of the proofs of the last part.

6-1 Generic Families of Lagrangians.

In this chapter the manifold M is compact. Denote by $\mathfrak{M}(L)$ the set of minimizing measures of the lagrangian L:

$$\mathfrak{M}(L):=\{\,\mu\in\mathcal{M}(L)\;:\;A_L(\mu)+c(L)=0\,\}.$$

Observe that $\mathfrak{M}(L)$ is a simplex whose extremal points are the ergodic minimizing measures (see exercises 6-1.2). In general $\mathfrak{M}(L)$ may be infinite dimensional.

In this section we shall prove the following

6-1.1 Theorem. *Let A be a finite dimensional convex family of lagrangians. Then there exists a residual subset $\mathcal{O}$ of $C^\infty(M,\mathbb{R})$ such that*

$$u \in \mathcal{O}, \quad L \in A \quad \Longrightarrow \quad \dim \mathfrak{M}(L-u) \leq \dim A.$$

In other words, there exist at most $1+\dim A$ ergodic minimizing measures for $L-u$.

6-1.2 Exercises:

1. Let V be a real vector space. A set $Y \subset V$ is *affinely independent* if for any finite subset $F=\{v_0,v_1,\cdots,v_n\}\subset Y$, every point in the convex hull of F is uniquely expressible as a convex combination $\lambda_0\, v_0+\cdots+\lambda_n\, v_n$, $0\leq\lambda_i\leq 1$ of the elements of F. Show that a finite set F is affinely independent if and only if $\{v_1-v_0,\cdots,v_n-v_0\}$ is linearly independent, and in this case the dimension of the convex hull $\dim(\operatorname{conv}(F))=n$.
2. Let $K\subset V$ be a closed convex subset. An extreme or extremal point of K is a point in K which is not in the (relative) interior of any segment contained in K. Prove that K is the convex closure of its extremal points. A *simplex* is a subset $\Sigma\subset V$ such that the set of its extremal points is affinely independent.
3. Show that if μ and ν are ergodic invariant Borel probabilities of a dynamical system, then they are either equal or mutually singular (i.e. there is a Borel set A such that $\mu(A)=1$ and $\nu(A)=0$).
4. Prove any set of ergodic invariant Borel probabilities is affinely independent.
5. Prove that the set of minimizing measures $\mathfrak{M}(L)$ of a lagrangian L is a simplex in the vector space of signed Borel measures on TM whose extremal points are the ergodic minimizing measures.

Fix $B>0$ and let $\mathcal{H}_B$ be the set of holonomic measures supported on $[|v|\leq B]$:

$$\mathcal{H}_B := \{\, \mu\in\overline{\mathcal{C}(M)} \;:\; \mu([\,|v|>B\,])=0 \,\}.$$

Consider a lagrangian L as a functional $L : \mathcal{H}_B \to \mathbb{R}$ given by $L(\mu) = A_L(\mu)$. Let

$$\mathfrak{M}_{\mathcal{H}_B}(L) := \arg\min_{\mathcal{H}_B} L$$

be the set of measures $\mu \in \mathcal{H}_B$ which minimize $L|_{\mathcal{H}_B}$. Let $\mathcal{P} = \mathcal{P}(M)$ be the set of probability measures on M endowed with the weak* topology. A compatible metric (cf. on $\mathcal{P}(M)$ is defined as follows: fix a countable dense set $\{ f_n \mid n \in \mathbb{N} \}$ in $C^\infty(M, \mathbb{R})$ and let

$$d(\mu, \nu) = \sum_n \frac{1}{2^n} \frac{1}{c_n} \left| \int f_n \, d\mu - \int f_n \, d\nu \right| .$$

Since M is compact, under this topology the set $\mathcal{P}(M)$ is compact Let $\pi_* : \mathcal{H}_B \to \mathcal{P}$ be the push-forward induced by the projection $\pi : TM \to M$. Let

$$\mathfrak{M}_{\mathcal{P}}(L, B) := \pi_*\big(\mathfrak{M}_{\mathcal{H}_B}(L)\big).$$

6-1.3 Proposition. *Let A be a finite dimensional convex family of lagrangians on M and let $B > 0$. There is a residual subset $\mathcal{O}(A, B) \subset C^\infty(M, \mathbb{R})$ such that*

$$L \in A, \quad u \in \mathcal{O}(A, B) \quad \Longrightarrow \quad \dim \mathfrak{M}_{\mathcal{P}}(L, B) \le \dim A. \tag{6.1}$$

Proof of theorem 6-1.1:

The set $\mathcal{O}(A) := \cap_{B \in \mathbb{N}} \mathcal{O}(A, B)$ is residual in $C^\infty(M, \mathbb{R})$. By corollary 3-6.3, $E(\mathfrak{M}(L)) = \{c(L)\}$. Then inequality 1.6 implies that there is $B_0(L) > 0$ such that $\mathfrak{M}(L) \subset \mathcal{H}_B$ for all $B > B_0(L)$. From theorem 2-4.1.2

$$\mathfrak{M}(L) = \arg\min_{\mathcal{H}_B} L = \mathfrak{M}_{\mathcal{H}_B}(L) \qquad \text{for all} \quad B > B_0(L).$$

By the graph property 3-8.1 and 3-6.1

$$\dim \mathfrak{M}(L) = \dim \pi_*\big(\mathfrak{M}(L)\big) = \dim \mathfrak{M}_{\mathcal{P}}(L, B) \quad \text{for all } B > B_0(L).$$

These remarks, together with proposition 6-1.3 prove theorem 6-1.1.

□

Proof of proposition 6-1.3:

Define the ε-neighbourhood W_ε of a subset $W \subset \mathcal{P}(M)$ as the union of all the open balls in $\mathcal{P}(M)$ which have radius ε and are centered in W. Given $D \subset A$, $k \in \mathbb{N}$, $\varepsilon > 0$, denote by $\mathcal{O}(D, \varepsilon, k)$ the set of potentials $u \in C^\infty(M, \mathbb{R})$ such that for all $L \in D$, the convex set $\mathfrak{M}_{\mathcal{P}}(L, B)$ is contained in the ε-neighbourhood of some k-dimensional convex subset of $\mathcal{P}(M)$.

We shall prove that the proposition holds with

$$\mathcal{O}(A, B) = \bigcap_{\varepsilon > 0} \mathcal{O}(A, \varepsilon, \dim A).$$

Indeed, if $u \in \mathcal{O}(A, B)$ then the inequality in (6.1) holds. Because otherwise for some $L \in A$ the convex set $\mathfrak{M}_{\mathcal{P}}(L, B)$ would contain a ball of dimension $\dim A + 1$, and, if ε is small enough, such a ball is not contained in the ε-neighbourhood of any convex set of dimension $\dim A$.

So we have to prove that $\mathcal{O}(A, B)$ is residual. It is enough to prove that $\mathcal{O}(A, \varepsilon, \dim A)$ is open and dense for any compact subset $D \subset A$ and any $\varepsilon > 0$. In 6-1.a we prove that it is open and in 6-1.b that it is dense.

□

6-1.a Open.

We prove that for all $k \in \mathbb{N}$, $\varepsilon > 0$ and any compact $D \subset A$, the set $\mathcal{O}(D, \varepsilon, k)$ is open. We need a lemma:

6-1.4 Lemma.

The set valued map $(L, u) \mapsto \mathfrak{M}_{\mathcal{H}_B}(L - u)$ is upper semi-continuous on $A \times C^\infty(M, \mathbb{R})$. This means that for any open subset U of $\mathcal{H}_B$, the set

$$\{(L, u) \in A \times C^\infty(M, \mathbb{R}) \;:\; \mathfrak{M}_{\mathcal{H}_B}(L - u) \subset U\} \subset A \times C^\infty(M, \mathbb{R})$$

is open in $A \times C^\infty(M,\mathbb{R})$*. Consequently, the set-valued map* $(L,u) \mapsto \mathfrak{M}_{\mathcal{P}}(L-u,B)$ *is also upper semi-continuous.*

Proof: The lemma is a consequence of the continuity of the map

$$A \times C^\infty(M,\mathbb{R}) \times \mathcal{H}_B \ni (L,u,\mu) \longmapsto (L-u)(\mu) = L(\mu) - \textstyle\int u\,d\mu.$$

□

Now let $u_0 \in \mathcal{O}(D,\varepsilon,k)$. For each $L \in D$ there is a k-dimensional convex subset $V \subset \mathcal{P}(M)$ such that $\mathfrak{M}_{\mathcal{P}}(L-u_0,B) \subset V_\varepsilon$. Then the open sets in $D \times C^\infty(M,\mathbb{R})$ of the form

$$\{\,(L,u) \in D \times C^\infty(M,\mathbb{R}) \mid \mathfrak{M}_{\mathcal{P}}(L-u,B) \subset V_\varepsilon\,\},$$

where V is a k-dimensional convex subset of $\mathcal{P}(M)$, cover the compact subset $D \times \{u_0\}$. This implies that there is a finite subcover of $D \times \{u_0\}$ by open sets of the form $\Omega_i \times U_i$, where Ω_i is an open set in A and $U_i \subset \mathcal{O}(\Omega_i,\varepsilon,k)$ is an open set in $C^\infty(M,\mathbb{R})$ containing u_0. We conclude that the open set $\cap U_i$ is contained in $\mathcal{O}(D,\varepsilon,k)$ and contains u_0.

□

6-1.b Dense.

We prove the density of $\mathcal{O}(D,\varepsilon,\dim A)$ in $C^\infty(M,\mathbb{R})$ for any $\varepsilon > 0$. Let $w \in C^\infty(M,\mathbb{R})$. We want to prove that w is in the closure of $\mathcal{O}(D,\varepsilon,\dim A)$. We consider a function $w \in C^\infty(M,\mathbb{R})$ as a linear functional $w : \mathcal{P}(M) \to \mathbb{R}$ as $w(\nu) = \int w\,d\nu$ or $w : \mathcal{H}_B \to \mathbb{R}$ as $w(\mu) = \int (w \circ \pi)\,d\mu$.

6-1.5 Lemma. *There is an integer* m *and a continuous map*

$$T_m = (w_1,\ldots,w_m) : \mathcal{P}(M) \to \mathbb{R}^m,$$

with $w_i \in C^\infty(M,\mathbb{R})$ *such that*

$$\forall x \in \mathbb{R}^m \qquad \operatorname{diam} T_m^{-1}\{x\} < \varepsilon. \tag{6.2}$$

Proof: To each function $w \in C^\infty(M,\mathbb{R})$ we associate the open set

$$U_w = \left\{ (\nu,\mu) \in \mathcal{P}(M)\times\mathcal{P}(M) \;\middle|\; \int w\,d\nu \neq \int w\,d\mu \right\}.$$

The open sets U_w cover the complement of the diagonal in $\mathcal{P}(M)\times\mathcal{P}(M)$. One can extract a countable subcover U_{w_k}, $k \in \mathbb{N}$. This amounts to say that the sequence w_k separates $\mathcal{P}(M)$. Defininig $T_m = (w_1,\dots,w_m)$, we have to prove that (6.2) holds for m large enough. Otherwise, we would have some $\varepsilon > 0$ and two sequences ν_m and μ_m in $\mathcal{P}(M)$ such that

$$T_m(\nu_m) = T_m(\mu_m) \qquad \text{and} \qquad d(\nu_m,\mu_m) \geq \varepsilon.$$

By extracting a subsequence, we can assume that the sequences ν_m and μ_m have different limits ν and μ, which satisfy $d(\nu,\mu) \geq \varepsilon$. Take m large enough so that $T_m(\nu) \neq T(\mu)$. Such m exists because the sequence w_k separates $\mathcal{P}(M)$. We have that

$$T_m(\nu_k) = T_m(\mu_k) \quad \text{for} \quad k \geq m.$$

Hence at the limit $T_m(\nu) = T_m(\mu)$. This is a contradiction.

□

Define the convex function $F_m : A \times \mathbb{R}^m \to \mathbb{R}\cup\{+\infty\}$ as

$$F_m(L,x) = \min_{\substack{\mu\in\mathcal{H}_B \\ T_m\circ\pi(\mu)=x}} (L-w)(\mu),$$

when $x \in T_m(\mathcal{H}_B)$ and $F_m(L,x) = +\infty$ if $x \notin T_m(\mathcal{H}_B)$. For $y = (y_1,\dots,y_m) \in \mathbb{R}^m$ let

$$M_m(L,y) := \arg\min_{x\in\mathbb{R}^m} [F(L,x) - y\cdot x] \subset \mathbb{R}^m$$

be the set of points wich minimize the function $x \mapsto F(L,x) - y \cdot x$. We have that

$$\mathfrak{M}_{\mathcal{P}}(L - w - y_1 w_1 - \cdots - y_m w_m, B) \subset T_m^{-1}(M_m(L,y)).$$

Let

$$\mathcal{O}_m(w) := \{\, y \in \mathbb{R}^m \,|\, \forall L \in A \,:\, \dim M_m(L,y) \le \dim A \,\}.$$

From lemma 6-1.5 it follows that

$$y \in \mathcal{O}_m(w) \quad \Longrightarrow \quad w + y_1\, w_1 + \cdots + y_m\, w_m \in \mathcal{O}(A, \varepsilon, \dim A).$$

Therefore, in order to prove that w is in the closure of $\mathcal{O}(A, \varepsilon, \dim A)$ it is enough to prove that 0 is in the closure of $\mathcal{O}_m(w)$, which follows from the next proposition.

6-1.6 Proposition. *The set $\mathcal{O}_m(w)$ is dense in $\mathbb{R}^m$.*

Proof: Consider the Legendre transform of F_m with respect to the second variable

$$\begin{aligned} G_m(L,y) &= \max_{x\in\mathbb{R}^m}\; y\cdot x - F_m(L,x) && (6.3)\\ &= \max_{\mu\in\mathcal{H}_B} \int \big(\, w + y_1 w_1 + \cdots + y_m w_m - L \,\big)\, d\mu && (6.4) \end{aligned}$$

It follows from the second expession that the function G_m is convex and finite-valued, hence continuous on $A \times \mathbb{R}^m$.

Let ∂G_m be the subdifferential of G_m and let

$$\widetilde{\Sigma} = \{\, (L,y) \in A \times \mathbb{R}^m \;|\; \dim \partial G_m(L,y) \ge \dim A + 1 \,\}\,.$$

By proposition E.1 the Hausdorff dimension

$$HD(\widetilde{\Sigma}) \le (m + \dim A) - (\dim A + 1) = m - 1.$$

Consequently, the projection Σ of the set $\tilde{\Sigma}$ on the second factor $\mathbb{R}^m$ also has Hausdorff dimension at most $m-1$. Therefore, the complement of Σ is dense in $\mathbb{R}^m$. So it is enough to prove that

$$y \notin \Sigma \quad \Longrightarrow \quad \forall L \in A \;: \quad \dim M_m(L,y) \le \dim A.$$

Since we know by definition of Σ that if $y \notin \Sigma$, $\dim \partial G_m(L, y) \leq \dim A$, it is enough to observe that

$$\dim M_m(L, y) \leq \dim \partial G_m(L, y).$$

The last inequality follows from the fact that the set $M_m(L, y)$ is contained in the subdifferential of the convex function

$$\mathbb{R}^m \ni z \mapsto G_m(L, z)$$

at the point y.

$\square$

Chapter 7

Generic Lagrangians.

In [34], Mañé introduced the concept of *generic property of a lagrangian* L. A property P is said to be generic for the lagrangian L if there exists a generic set $\mathcal{O}$ (in the Baire sense) on the set $C^\infty(M,\mathbb{R})$ such that if ψ is in $\mathcal{O}$ then $L+\psi$ has the property P. One of Mañé's objectives was to show that Mather's theory of minimizing measures becomes much more accurate and stronger if we restrict ourselves to generic lagrangians. The main purpose of this chapter is to proof the following

7-0.1 Theorem. *For every lagrangian L there exists a generic set $\mathcal{O} \subseteq C^\infty(M,\mathbb{R})$ such that*

(A) If ψ is in $\mathcal{O}$ then $L+\psi$ has a unique minimizing measure, μ and this measure is uniquely ergodic.

(B)Moreover supp$(\mu) = \hat{\Sigma}(L+\psi) = \Sigma(L+\psi)$.

(C) When μ is supported on a periodic orbit or a fixed point, this orbit (point) Γ is hyperbolic and its stable and unstable manifolds if intersect they do it transversally.

On Mañé [35], it is conjectured that there exists a generic set $\mathcal{O}$ such that this unique minimizing measure is supported on a periodic orbit or an equilibrium point.

The first statement of this was proved by Mañé in [34]. The second statement is proved in [16] and the third one in [13]. We will prove only the first part and give only the ideas of the proofs of the last part.

7-1 Generic Lagrangians.

Proof of (A)

Given a potential ψ on $C^\infty(M,\mathbb{R})$ define

$$\begin{aligned} m(\psi) &= \min_{\nu\in\overline{C}} \int L+\psi d\nu, \\ M(\psi) &= \{\nu\in\overline{C} : \int L+\psi d\nu = m(\psi)\} \end{aligned}$$

Where $\overline{C}$ is the set of holonomic measures. For $\epsilon>0$ let

$$\mathcal{O}_\epsilon = \{\psi : \text{diam } \ M(\psi)<\epsilon\}$$

This set is open, in fact if ν_n is in $M(\psi_n)$ then

$$\int L+\psi d\nu_n \leq m(\psi)+2||\psi-\psi_n||_{C^0} \tag{7.1}$$

So by theorem 2-4.1 in chapter 2 if $\psi_n\to\psi$ then the sequence ν_n is precompact and the limit is in $M(\psi)$. From this follows that $\mathcal{O}_\epsilon$ is open.

It remains to prove that it is also dense.

Given a compact convex set K_0 on $\overline{C}$ and potential ψ on $C^\infty(M,\mathbb{R})$ define

$$\begin{aligned} m_0(\psi) &= \min_{\nu\in K_0} \int \psi d\nu, \\ M_0(\psi) &= \{\nu\in K_0 : \int \psi d\nu = m_0(\psi)\} \end{aligned}$$

7-1.1 Lemma. *Let K_0 as before then if μ is an extremal point of K_0, for all $\epsilon > 0$there exists ψ on $C^\infty(M,\mathbb{R})$ such that*

$$\begin{aligned} \text{diam } M_0(\psi) &< \epsilon \\ d(\mu, M_0(\psi)) &< \epsilon \end{aligned}$$

Proof

Denote by D the diagonal of $K_0 \times K_0$ for each pair (μ,ν) in $K_0 \times K_0 - D$ take a potential $\psi_{(\mu,\nu)}$ such that $\int \psi_{(\mu,\nu)} d\mu \neq \int \psi_{(\mu,\nu)} d\nu$, then there is a neigbourhood $U(\mu,\nu)$ contained on $K_0 \times K_0$ such that $\int \psi_{(\mu,\nu)} d\mu' \neq \int \psi_{(\mu,\nu)} d\nu'$ for every (μ',ν') in $U(\mu,\nu)$.

Take a covering $\{U(\mu_n,\nu_n)\}$ of $K_0 \times K_0 - D$ and set $\psi_n = \psi_{(\mu_n,\nu_n)}$ then if (μ,ν) in $K_0 \times K_0 - D$ there exist n such that

$$\int \psi_n d\mu \neq \int \psi_n d\nu \tag{7.2}$$

Define $T_n : \overline{C} \to \mathbb{R}^n$ as

$$T_n(\mu) = (\int \psi_1 d\mu, ..., \int \psi_n d\mu)$$

Using (7.2) and the compactness of $K_0 \times K_0$ it is easy to see that given ϵ there exist $\delta > 0$ and $n > 0$ such that

$$S \subset \mathbb{R}^n, \text{diam } S < \delta \Rightarrow \text{diam } T^{-1} S < \epsilon \tag{7.3}$$

Let $B = T_n(k_0)$ then B is a compact convex set, let $f : \mathbb{R}^n \to \mathbb{R}$ a linear function such that its minimum restricted to B is attained in only one point p.

Define $\psi = \sum_i \lambda_i \psi_i$ where $f = \sum_i \lambda_i p_i$, then

$$f \circ T_n = \sum_i \lambda_i \int \psi_i$$

and so

$$M_0(\psi) = T_n^{-1}(p)$$

Then by (7.3) we get

$$\text{diam } M_0(\psi) < \epsilon$$

The following lemma proves the density of $\mathcal{O}_\epsilon$ and hence the first part of (A).

7-1.2 Lemma. *If ψ is on $C^\infty(M, \mathbb{R})$ and μ is an extremal point of*

$M(\psi)$ then for every neighbourhood U of ψ and every $\epsilon > 0$ there exists ψ_1 on U such that

$$\text{diam } M(\psi_1) = \epsilon$$

Proof

For $K_0 = M(\psi)$ applying the previous lemma, we can find given ϵ a ψ_1 such that $\int \psi_1 d\nu$ attains its minimum, say m_1 for all measures ν on $K_0 = M(\psi)$ on a set $S = M_0(\psi_1)$ such that $d(\mu, S) < \epsilon$. Set

$$\begin{aligned} m_0 &= m(\psi) \\ f_0(\nu) &= \int L + \psi - m_0 d\nu \\ f_1(\nu) &= \int \psi_1 - m_1 d\nu \end{aligned}$$

Then

$$f_1(\nu) = f_0(\nu) \text{ if } \nu \in S \tag{7.4}$$

$$f_1(\nu) \geq 0 \text{ if } \nu \in M(\psi) \tag{7.5}$$

$$\nu \in M(\psi), f_1(\nu) = 0 \Rightarrow \nu \in S \tag{7.6}$$

$$\nu \in \overline{C}, f_0(\nu) = 0 \Rightarrow \nu \in M(\psi) \tag{7.7}$$

For $\lambda > 0$ define

$$f_\lambda = f_0 + \lambda f_1$$

and set

$$\begin{aligned} m(\lambda) &= \min_{\nu \in \overline{C}} f_\lambda(\nu), \\ M(\lambda) &= \{\nu \in \overline{C} : f_\lambda(\nu) = m(\lambda)\}. \end{aligned}$$

We claim that

$$\lim_{\lambda \to 0} \text{diam } (M(\lambda), \{\mu\}) \leq \epsilon \tag{7.8}$$

This proves the lemma since

$$M(\lambda) = M(\psi + \lambda \psi_1)$$

Proof of the claim

Suppose otherwise that there exist $\lambda_n \to 0$ and $\mu_{\lambda_n}, \nu_{\lambda_n}$ on $M(\lambda_n) = M(\psi + \lambda_n \psi_1)$ such that $d(\mu_{\lambda_n}, \nu_{\lambda_n}) > \epsilon$. Then by (7.1) $\{\mu_{\lambda_n}\}$ and $\{\nu_{\lambda_n}\}$ are precompact and as in the proof of the open property we may assume that $\mu_{\lambda_n} \to \mu \in M(\psi)$ and $\nu_{\lambda_n} \to \nu \in M(\psi)$. Naturally $d(\mu, \nu) \geq \epsilon$.

Now Because of (7.4) we have that $m(\lambda) \leq 0$

$$\begin{aligned} 0 &\geq m(\lambda_n) = \\ &= f_0(\mu_{\lambda_n}) + \lambda f_1(\mu_{\lambda_n}) \\ &\geq \lambda_n f_1(\mu_{\lambda_n}) \end{aligned}$$

So $f_1(\mu_{\lambda_n}) \leq 0$ and hence $f_1(\mu) \leq 0$, since $\mu \in M(\psi)$ by (7.5) $f_1(\mu) = 0$ and then by (7.6) μ is in S. Similarly ν is in S. This is a contradiction with the fact that the diameter of S is less than ϵ.

The fact that μ is uniquely ergodic follows from the fact that ergodic components of a minimizing measure are also minimizing. And the proof of (A) is complete.

It is worth to remark that the proof presented here is a particular case of Mañé's original [38] more general setting:

Let E, F be real convex spaces, K contained on F a metrizable convex subset and $\phi : E \to F'$, $L : F \to \mathbb{R}$ linear maps satisfying

- (a) The map $E \times F \to \mathbb{R}$ defined by $(w, x) \mapsto \phi(w)(x)$ is continuous.
- (b) For any $x \neq y$ in K there exists w in E such that $\phi(w)(x) \neq \phi(w)(y)$.
- (c) For all w in E and c in $\mathbb{R}$ the set
$$\{x \in K : L(x) + \phi(w)(x) \leq c\}$$
is compact.

Denote by

$$m(w) = \min_{x \in K} L(x) + \phi(w)(x)$$

which exists by (c). And

$$M(w) = \{x \in K : L(x) + \phi(w)(x) = m(w)$$

7-1.3 Proposition. *If E is a Frechet space then there exists a residual set $\mathcal{O}$ contained on E such that if w is on E then $M(w)$ has only one element.*

The reader can verify that with the following choices, we get the desired result.

- (1) Let E be the Banach space $C^\infty(M, \mathbb{R})$
- (2) As in section 2-3 let C^0_ℓ be the set of continuous functions f on TM such that $\sup \frac{f(x,v)}{1+|v|} \leq \infty$, and $\overline{C}$ the set of holonomic probabilities. Let K be $\overline{C}$ and F be the subspace of $(C^0_\ell)^*$ spanned by $\overline{C}$.

- (3) Finally let $L : F \to \mathbb{R}$ is the linear map such that if μ is in $\overline{C}$ then $L(\mu) = \int L d\mu$; and for ψ in $C^\infty(M,\mathbb{R})$ $\phi(\psi)$ is the restriction to F of the linear map on $(C^0_\ell)^*$ such that $w \mapsto < w, \psi >$.

This general setting has some other applications see theorems A, C D in [35] and also [11]

Proof of (C)

Let $\mathcal{O}$ be the residual given by (A). Let $\mathcal{A}$ be the subset of $\mathcal{O}$ of potentials ψ for which the measure on $\mathcal{M}(L+\psi)$ is supported on a periodic orbit. Let $\mathbb{B} := \mathcal{O} \setminus \mathcal{A}$ and let $\mathcal{A}_1$ be the subset of $\mathcal{A}$ on which the minimizing periodic orbit is hyperbolic. We prove that $\mathcal{A}_1$ is relatively open on $\mathcal{A}$. For, let $\psi \in \mathcal{A}_1$ and

$$\mathcal{M}(L+\psi) = \{\mu_\gamma\}$$

where μ_γ is the invariant probability measure supported on the hyperbolic periodic orbit γ for the flow of $L+\psi$. We claim that if $\phi_k \in \mathcal{A}$, $\phi_k \to \psi$ and $\mathcal{M}(L+\phi_k) = \{\mu_{\eta_k}\}$, then $\eta_k \to \gamma$. Indeed, since L is superlinear, the velocities in the support of the minimizing measures $\mu_k := \mu_{\eta_k}$ are bounded (cf. corollary 3-6.3 and inequality 1.6), and hence there exists a subsequence $\mu_k \to \nu$ converging weakly* to a some invariant measure ν for $L+\psi$. Then if $\nu \neq \mu_\gamma$,

$$\lim_k S_{L+\phi_k}(\mu_k) = S_{L+\psi}(\nu) > S_{L+\psi}(\mu_\gamma)\,. \tag{7.9}$$

Thus if δ_k is the analytic continuation of the hyperbolic periodic orbit γ to the flow of $L+\phi_k$ in the original energy level $c(L+\psi)$, since $\lim_k S_{L+\psi_k}(\mu_{\delta_k}) = S_{L+\psi}(\mu_\gamma)$, for k large we have that,

$$S_{L+\phi_k}(\mu_{\delta_k}) < S_{L+\phi_k}(\mu_{\eta_k})\,,$$

which contradicts the choice of η_k. Therefore $\nu = \mu_\gamma$. For energy levels h near to $c(L+\psi)$ and potentials ϕ near to ψ, there exist hyperbolic periodic orbits $\gamma_{\phi,h}$ which are the continuation of γ. Now, on a small neighbourhood of a hyperbolic orbit there exists a unique invariant measure supported on it, and it is in fact supported in the periodic orbit. Thus, since $\eta_k \to \gamma$, then η_k is hyperbolic. Hence $\phi_k \in \mathcal{A}_1$ and $\mathcal{A}_1$ contains a neighbourhood of ϕ in $\mathcal{A}$.

Let $\mathcal{U}$ be an open subset of $C^\infty(M,\mathbb{R})$ such that $\mathcal{A}_1 = \mathcal{U} \cap \mathcal{A}$. We shall prove below that $\mathcal{A}_1$ is dense in $\mathcal{A}$. This implies that $\mathcal{A}_1 \cup \mathbb{B}$ is generic. For, let $\psi :=$ int $(C^\infty(M,\mathbb{R}) \setminus \mathcal{U})$, then $\mathcal{U} \cup \psi$ is open and dense in $C^\infty(M,\mathbb{R})$. Moreover, $\psi \cap \mathcal{A} = \varnothing$ because $\mathcal{A} \subseteq \overline{\mathcal{A}_1} \subseteq \overline{\mathcal{U}}$ and $\psi \cap \mathcal{A} \subseteq \mathcal{A} \setminus \overline{\mathcal{U}} = \varnothing$. Since $\mathcal{O} = \mathcal{A} \cup \mathbb{B}$ is generic and

$$\begin{aligned}(\mathcal{U} \cup \psi) \cap (\mathcal{A} \cup \mathbb{B}) &= (\mathcal{U} \cap \mathcal{A}) \cup \big((\mathcal{U} \cup \psi) \cap \mathbb{B}\big) \\ &\subseteq \mathcal{A}_1 \cup \mathbb{B}\,,\end{aligned}$$

then $\mathcal{A}_1 \cup \mathbb{B}$ is generic.

The perturbation to achieve hyperbolicity in a fixed point is easy. Is very much as the mechanic case: $L = \frac{1}{2} < v, v >_x -U(x)$. The reader can verify that if $x_0 = \max U$ then the Dirac measure supported on the point $(x_0, 0)$ is minimizing. And it is well known that this critical point is hyperbolic if and only if the maximum has

non degenerate quadratic form.

In fact, from the Euler-Lagrange equation (E-L) we get that $L_x(x_0, 0) = 0$. Differentiating the energy function (1.3) we see that $(x_0, 0)$ is a singularity of the energy level $c(L)$. Moreover, the minimizing property of μ implies that x_0 is a minimum of the function $x \mapsto L_{xx}(x, 0)$. In particular, $L_{xx}(x_0, 0)$ is positive semidefinite in linear coordinates in $T_{x_0}M$. And it is hyperbolic if and only if it is positive definite. So to achieve hyperbolicity we must just add a small quadratic form.

The perturbation needed in the case of a periodic orbit the same spirit; Because of the graph property the projection of the orbit Γ, $\pi(\Gamma)$ is a simple closed curve. We add a C^∞-small non negative potential ψ, which is zero if and only if x is on $\pi(\Gamma)$ that is nondegenerate in the transversal direction. It follows that Γ is also a minimizing solution of the perturbed lagrangian $L + \psi$.

To prove that it actually is hyperbolic is much more difficult. The reason is that the linearization of the flow (the Jacobi equation) is along the periodic orbit and hence non autonomous as in the case of a singularity.

To explain the idea of the proof we need some definitions. Let H be the associated hamiltonian by the Legendre transformation on T^*M and ψ its flow. Denote by $\pi : T^*M \to M$ be the canonical projection and define the *vertical subspace* on $\theta \in T^*M$ by $\psi(\theta) = \ker(d\pi)$. Two points θ_1, $\theta_2 \in T^*M$ are said to be *conjugate* if $\theta_2 = \psi_\tau(\theta_1)$ for some $\tau \neq 0$ and $d\psi_\tau(\psi(\theta_1)) \cap \psi(\theta_2) \neq \{0\}$.

A basic property of orbits without conjugate points is given by the following

7-1.4 Proposition. *Suppose that the orbit of $\theta \in T^*M$ does not contain conjugate points and $H(\theta) = e$ is a regular value of H. Then there exist two φ-invariant lagrangian subbundles $\mathbb{E}$, $\mathbb{F} \subset T(T^*M)$ along the orbit of θ given by*

$$
\begin{aligned}
\mathbb{E}(\theta) &= \lim_{t\to+\infty} d\psi_{-t}\big(\psi(\psi_t(\theta))\big) \ , \\
\mathbb{F}(\theta) &= \lim_{t\to+\infty} d\psi_{t}\big(\psi(\psi_{-t}(\theta))\big) \ .
\end{aligned}
$$

Moreover, $\mathbb{E}(\theta)\cup\mathbb{F}(\theta) \subset T_\theta\Sigma$, $\mathbb{E}(\theta)\cap\psi(\theta) = \mathbb{F}(\theta)\cap\psi(\theta) = \{0\}$, $\langle X(\theta)\rangle \subseteq \mathbb{E}(\theta)\cap\mathbb{F}(\theta)$ and $\dim\mathbb{E}(\theta) = \dim\mathbb{F}(\theta) = \dim M$, where $X(\theta) = (H_p, -H_q)$ is the hamiltonian vector field and $\Sigma = H^{-1}\{e\}$.

These bundles where constructed for disconjugate geodesics of riemannian metrics by Green [27] and of Finsler metrics by Foulon [24]. In the general case where constructed in [13]

We will only sketch the proof;

Fix a riemannian metric on M and the corresponding induced metric on T^*M. Then $T_\theta T^*M$ splits as a direct sum of two lagrangian subspaces: the vertical subspace $\psi(\theta) = \ker(d\pi(\theta))$ and the horizontal subspace $H(\theta)$ given by the kernel of the connection map. Using the isomorphism $K : T_\theta T^*M \to T_{\pi(\theta)}M \times T^*_{\pi(\theta)}M$, $\xi \mapsto (d\pi(\theta)\,\xi\,,\, \nabla_\theta(\pi\,\xi))$, we can identify $H(\theta) \approx T_{\pi(\theta)}M \times \{0\}$ and $\psi(\theta) \approx \{0\} \times T^*_{\pi(\theta)}M \approx T_{\pi(\theta)}M$. If we choose local coordinates along $t \mapsto \pi\,\psi_t(\theta)$ such that $t \mapsto \frac{\partial}{\partial q_i}(\pi\,\psi_t(\theta))$ are parallel vector fields, then this identification becomes $\xi \leftrightarrow (dq(\xi),\, dp(\xi))$. Let $E \subset T_\theta T^*M$ be an n-dimensional subspace such that $E \cap \psi(\theta) = \{0\}$. Then E is a graph of some linear map $S : H(\theta) \to \psi(\theta)$. It can be checked that E is lagrangian if and only if in symplectic coordinates S is symmetric.

Take $\theta \in T^*M$ and $\xi = (h, v) \in T_\theta T^*M = H(\theta) \oplus \psi(\theta) \approx T_{\pi(\theta)}M \oplus T_{\pi(\theta)}M$. Consider a variation

$$\alpha_s(t) = \big(q_s(t), p_s(t)\big)$$

such that for each $s \in]-\varepsilon, \varepsilon[$, α_s is a solution of the hamiltonian H such that $\alpha_0(0) = \theta$ and $\frac{d}{ds}\alpha_s(0)|_{s=0} = \xi$.

Writing $d\psi_t(\xi) = \big(h(t), v(t)\big)$, we obtain the hamiltonian Jacobi equations

$$\begin{aligned} \dot h &= \quad H_{pq}\, h + H_{pp}\, v\,, \\ \dot v &= -H_{qq}\, h - H_{qp}\, v\,, \end{aligned} \tag{7.10}$$

where the covariant derivatives are evaluated along $\pi(\alpha_o(t))$, and H_{qq}, H_{qp}, H_{pp} and H_{qq} are linear operators on $T_{\pi(\theta)}M$, that in local coordinates coincide with the matrices of partial derivatives $\Big(\frac{\partial^2 H}{\partial q_i \partial q_j}\Big)$, $\Big(\frac{\partial^2 H}{\partial q_i\, \partial p_j}\Big)$, $\Big(\frac{\partial^2 H}{\partial p_i\, \partial p_j}\Big)$ and $\Big(\frac{\partial^2 H}{\partial p_i\, \partial q_j}\Big)$. Moreover, since the hamiltonian H is convex, then H_{pp} is positive definite.

We derive now the Riccati equation. Let E be a lagrangian subspace of $T_\theta T^*M$. Suppose that for t in some interval $]-\varepsilon, \varepsilon[$ we have that $d\psi_t(E) \cap \psi(\psi_t(\theta)) = \{0\}$. Then we can write $d\psi_t(E) = \text{graph}\ S(t)$, where $S(t) : H(\psi_t\,\theta) \to \psi(\psi_t\,\theta)$ is a symmetric map. That is, if $\xi \in E$ then

$$d\psi_t(\xi) = \big(h(t), S(t)\, h(t)\big)\,.$$

Using equation (7.10) we have that

$$\dot{S}h + S(H_{pq}h + H_{pp}Sh) = -H_{qq}h - H_{qp}Sh\,.$$

Since this holds for all $h \in H(\psi_t(\theta))$ we obtain the Riccati equation:

$$\dot{S} + SH_{pp}S + SH_{pq} + H_{qp}S + H_{qq} = 0\,. \tag{7.11}$$

Let $K_c(\theta) : H(\theta) \to \psi(\theta)$ be the symmetric linear map such that $\operatorname{graph}(K_c(\theta)) = d\psi_{-c}(\psi(\psi_c(\theta)))$. Define a partial order on the symmetric isomorphisms of $T_{\pi(\theta)}M$ by writing $A \succ B$ if $A - B$ is positive definite.

The following proposition based essentially on the convexity of H proves 7-1.4.

7-1.5 Proposition. *For all $\varepsilon > 0$,*

(a) If $d > c > 0$ then $K_{-\varepsilon} \succ K_d \succ K_c$.

(b) If $d < c < 0$ then $K_\varepsilon \prec K_d \prec K_c$.

(c) $\lim_{d \to +\infty} K_d = \mathbb{S}$, $\quad \lim_{d \to -\infty} K_d = \mathbb{U}$.

(d) $\mathbb{S} \preccurlyeq \mathbb{U}$.

(e) The graph of $\mathbb{S}$ is the stable green bundle $\mathbb{E}$ and the graph of $\mathbb{U}$ is the unstable green bundle $\mathbb{F}$

An example of the relationship between the transversality of the Green subspaces and hyperbolicity appears in the following

7-1.6 Proposition. *Let Γ be a periodic orbit of ψ_t without conjugate points. Then Γ is hyperbolic (on its energy level) if and only if $\mathbb{E}(\theta) \cap \mathbb{F}(\theta) = \langle X(\theta) \rangle$ for some $\theta \in \Gamma$, where $\langle X(\theta) \rangle$ is the 1-dimensional subspace generated by the hamiltonian vector field $X(\theta)$. In this case $\mathbb{E}$ and $\mathbb{F}$ are its stable and unstable subspaces.*

This proposition follows ideas of Eberlein [18] and Freire, [25].

It is known that minimizing orbits do not have conjugate points. So by proposition 7-1.4 and 7-1.6 to prove the density of hyperbolicity it is enough to perturb to make the Green bundles transverse. This is done using two formulas for the index. One in the lagrangian setting and another one in the hamiltonian setting.

Let Ω_T be the set of continuous piecewise C^2 vector fields ξ along $\gamma_{[0,T]}$. Define the *index form* on Ω_T by

$$I(\xi,\eta)=\int_0^T \Big(\dot\xi\, L_{vv}\,\dot\eta+\dot\xi\, L_{vx}\,\eta+\xi\, L_{xv}\,\dot\eta+\xi\, L_{xx}\,\eta\Big)\; dt\,, \tag{7.12}$$

which is the second variation of the action functional for variations $f(s,t)$ with $\frac{\partial f}{\partial s}\in\Omega_T$. For general results on this form see Duistermaat [17].

From this formula it is easy to compare the index of the original and the perturbed lagrangian along the same solution Γ.

Finally we use the following transformation of the index form. It is taken from Hartman [29] and originally due to Clebsch [10] see also [13]. Let $\theta\in T^*M$ and suppose that the orbit of θ, $\psi_t(\theta)$, $0\le t\le T$ does not have conjugate points. Let $E\subset T_\theta T^*M$ be a lagrangian subspace such that $d\psi_t(E)\cap\psi(\psi_t(\theta))=\{0\}$ for all $0\le t\le T$.

Let $E(t):=d\psi_t(E)$ and let $H(t)$, $\psi(t)$ be a matrix solution of the hamiltonian Jacobi equation (7.10) such that $\det H(t)\neq 0$ and $E(t)=\text{Image}\,(H(t),\psi(t))\subset T_{\psi_t(\theta)}(T^*M)$ is a lagrangian subspace. For $\xi=H\zeta\in\Omega_T$, $\eta=H\rho\in\Omega_T$, we obtain (see [13])

$$I(\xi,\eta)=\int_0^T (H\zeta')^*\,(H_{pp})^{-1}\,(H\rho')\,dt+(H\zeta)^*(V\rho)|_0^T. \tag{7.13}$$

Define $\mathbf{N}(\theta):=\{\,w\in T_{\pi(\theta)}M\,|\,\langle w,\dot\gamma\rangle=0\,\}$. Then $\mathbf{N}(\theta)$ is the subspace of $T_{\pi(\theta)}M$ generated by the vectors $\frac{\partial}{\partial x_2},\ldots,\frac{\partial}{\partial x_n}$. Let $v_o\in\mathbf{N}(\theta)$, $|v_o|=1$ and let $\xi^T(t):=\widetilde Z_T(t)\,v_o$. Denote by $\tilde I_T$ and I_T be the index forms on $[0,T]$ for $\widetilde L$ and L respectively. Using the solution $(\widetilde Z_T,\,\widetilde V_T)$ on formula (7.13), we obtain that

$$\tilde{I}_T(\xi^T,\, \xi^T) = -(\widetilde{Z}_T(0)\, v_o)^*\, (\widetilde{V}_T(0)\, v_o) = -{v_o}^*\, \tilde{K}_T(0)\, v_o\,. \tag{7.14}$$

Moreover, in the coordinates $(x_1, \dots, x_n; \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$ on TU we have that

$$\begin{aligned}\tilde{I}_T(\xi^T,\, \xi^T) &= \int_0^T \big(\dot{\xi}^T\, \widetilde{L}_{vv}\, \dot{\xi}^T + 2\, \dot{\xi}^T\, \widetilde{L}_{xv}\, \xi^T + \xi^T\, \widetilde{L}_{xx}\, \xi^T\big)\, dt \\ &= \int_0^T \big(\dot{\xi}^T\, L_{vv}\, \dot{\xi}^T + 2\, \dot{\xi}^T\, L_{xv}\, \xi^T + \xi^T\, L_{xx}\, \xi^T\big)\, dt \\ &\qquad\qquad + \int_0^T \varepsilon \sum_{i=2}^n \left|\xi_i^T\right|^2\, dt\,.\end{aligned} \tag{7.15}$$

We have that $\widetilde{Z}_T(0) = I$ and for all $t > 0$, $\lim_{t\to\infty} \widetilde{Z}_T(t) = \widetilde{\mathbf{h}}(t)$ with $\widetilde{\mathbf{h}}(t)$ the solution of the Jacobi equation for $\tilde{H}$ corresponding to the stable Green bundle. Writing $\pi_N(\xi) = (\xi_2, \xi_3, \dots \xi_n)$ then $\left|\pi_N \widetilde{\mathbf{h}}(0)\, v_0\right| = |v_0| = 1$ because $v_0 \in \mathbf{N}(\theta)$. Hence there exists $\lambda > 0$ and $T_0 > 0$ such that $\left|\pi_N \xi^T(t)\right| = \left|\pi_N \widetilde{Z}_T(t)\, v_0\right| > \frac{1}{2}$ for all $0 \le t \le \lambda$ and $T > T_0$. Therefore

$$\tilde{I}_T(\xi^T,\, \xi^T) \ge I_T(\xi^T,\, \xi^T) + \frac{\varepsilon\lambda}{4}\,. \tag{7.16}$$

Let $(\boldsymbol{h}(t),\, \boldsymbol{v}(t)) = d\psi_t \circ \big(d\pi|_{\mathbb{E}(\theta)}\big)^{-1}$ be the solution of the Jacobi equation for H corresponding to the stable Green subspace $\mathbb{E}$ and let $\mathbb{S}(\psi_t(\theta)) = \boldsymbol{v}(t)\, \boldsymbol{h}(t)^{-1}$ be the corresponding solution of the Riccati equation, with graph$[\mathbb{S}(\psi_t(\theta))] = \mathbb{E}(\psi_t(\theta))$. Using formula (7.13), and writing $\xi^T(t) = \boldsymbol{h}(t)\, \zeta(t)$, we have that

$$\begin{aligned}I_T(\xi^T,\, \xi^T) &= \int_0^T (\boldsymbol{h}\, \dot{\zeta})^*\, H_{pp}^{-1}\, (\boldsymbol{h}\, \dot{\zeta})\, dt + 0 - \big(\boldsymbol{h}(0)\, \zeta(0)\big)^* \big(\boldsymbol{v}(0)\, \zeta(0)\big) \\ I_T(\xi^T,\, \xi^T) &\ge -v_o^*\, \mathbb{S}(\theta)\, v_o\,.\end{aligned} \tag{7.17}$$

From (7.14), (7.16) and (7.17), we get that

$$v_o^*\, \mathbb{S}(\theta)\, v_o \ge v_o^*\, \tilde{K}_T\, v_o + \frac{\varepsilon\lambda}{4}\,.$$

From proposition 7-1.5, we have that $\lim_{T\to+\infty}\tilde{K}_T(0)=\widetilde{\mathbb{S}}(\theta)$, where $\operatorname{graph}(\widetilde{\mathbb{S}}(\theta))=\widetilde{\mathbb{E}}(\theta)$, the stable Green bundle for $\tilde{H}$. Therefore

$$v_o^*\,\mathbb{S}(\theta)\,v_o\geq v_0^*\,\widetilde{\mathbb{S}}(\theta)\,v_o+\frac{\varepsilon\lambda}{4}\,. \tag{7.18}$$

Similarly, for the unstable Green bundles we obtain that

$$v_o^*\,\mathbb{U}(\theta)\,v_o+\lambda_2\leq v_o^*\,\widetilde{\mathbb{U}}(\theta)\,v_o \qquad \text{for } v_o\in\mathbf{N}(\theta),|v_o|=1. \tag{7.19}$$

for some $\lambda_2>0$ independent of v_o.

From proposition 7-1.5 we have that $\mathbb{U}(\theta)\succeq\mathbb{S}(\theta)$. From (7.18) and (7.19) we get that $\widetilde{\mathbb{U}}|_{\mathbf{N}}\succ\mathbb{U}|_{\mathbf{N}}\succeq\mathbb{S}|_{\mathbf{N}}\succ\widetilde{\mathbb{S}}|_{\mathbf{N}}$. Since $\widetilde{\mathbb{E}}(\theta)=\operatorname{graph}(\widetilde{\mathbb{S}}(\theta))$ and $\widetilde{\mathbb{F}}(\theta)=\operatorname{graph}(\widetilde{\mathbb{U}}(\theta))$, we get that $\widetilde{\mathbb{E}}(\theta)\cap\widetilde{\mathbb{F}}(\theta)\subseteq\langle\widetilde{X}(\theta)\rangle$. Then proposition B shows that Γ is a hyperbolic periodic orbit for $L+\phi$.

This proves that $\mathcal{A}_1$ is dense in $\mathcal{A}$.

Let $\mathcal{A}_2$ be the subset of $\mathcal{A}_1$ of potentials ψ for which the minimizing hyperbolic periodic orbit Γ has transversal intersections. The proof that $\mathcal{A}_2$ is open and dense in $\mathcal{A}_1$ is similar to the previous proof, see [13].

7-2 Homoclinic Orbits.

Assume in this section that $\widehat{\Sigma}$ contains only *one* static class. By theorem 7-0.1.(A), this is true for generic lagrangians. By proposition 3-11.4, the static classes are always connected, thus if we assume that there is only one static class, $\widehat{\Sigma}$ must be connected.

Given $\varepsilon>0$, let U_ε be the ε-neighbourhood of $\pi(\widehat{\Sigma})$. Since $\widehat{\Sigma}$ is connected, the open set U_ε is connected for ε sufficiently small. Let $H_1(M,U_\varepsilon,\mathbb{R})$ denote the first relative singular homology group of the pair (M,U_ε) with real coefficients.

We shall say that an orbit of L is *homoclinic* to a closed invariant set $K\subset TM$ if its α and ω-limit sets are contained in K.

Observe that to each homoclinic orbit $x:\mathbb{R}\to M$ to the set of static orbits $\widehat{\Sigma}$ we can associate a homology class in $H_1(M,U_\varepsilon,\mathbb{R})$. Indeed, since there exists $t_0 > 0$ such that for all t with $|t| \geq t_0$, $x(t) \in U_\varepsilon$, the class of $x|_{[-t_0,t_0]}$ defines an element in $H_1(M, U_\varepsilon, \mathbb{R})$. Let us denote by $\mathcal{H}$ the subset of $H_1(M, U_\varepsilon, \mathbb{R})$ given by all the classes corresponding to homoclinic orbits to $\widehat{\Sigma}$.

7-2.1 Theorem. *Suppose that $\widehat{\Sigma}$ contains only one static class. Then for any ε sufficiently small the set $\mathcal{H}$ generates over $\mathbb{R}$ the relative homology $H_1(M, U_\varepsilon, \mathbb{R})$. In particular, there exist at least* $\dim H_1(M, U_\varepsilon, \mathbb{R})$ *homoclinic orbits to the set of static orbits $\widehat{\Sigma}$.*

Let U_ε be an ε-neighbourhood of $\mathrm{supp}(\mu)$. From theorems 7-2.1 and 7-0.1 we obtain:

7-2.2 Corollary. *Given a lagrangian L there exists a generic set $\mathcal{O} \subset C^\infty(M, \mathbb{R})$ such that if $\psi \in \mathcal{O}$ the lagrangian $L + \psi$ has a unique minimizing measure μ in $\mathcal{M}^0(L + \psi)$ and this measure is uniquely ergodic. For any ε sufficiently small the set $\mathcal{H}$ of homoclinic orbits to* $\mathrm{supp}(\mu)$ *generates over $\mathbb{R}$ the relative homology $H_1(M, U_\varepsilon, \mathbb{R})$. In particular, there exist at least* $\dim H_1(M, U_\varepsilon, \mathbb{R})$ *homoclinic orbits to* $\mathrm{supp}(\mu)$.

To prove theorem 7-2.1 we consider finite coverings M_0 of M whose group of deck transformations is given by the quotient of the torsion free part of $H_1(M, U_\varepsilon, \mathbb{Z})$ by a finite index subgroup. Using that the lifted lagrangian L_0 has the same critical value as L, we conclude that the number of static classes of L_0 must be finite. Hence we can apply theorem 3-11.1 to L_0 to deduce that the group generated by the homoclinic orbits to the set of static orbits of L coincides with $H_1(M, U_\varepsilon, \mathbb{R})$.

We note that the homoclinic orbits that we obtain in theorem 7-2.1 and corollary 7-2.2 have energy c but they are not semistatic orbits of L. However, they are semistatic for lifts of L to suitable finite covers.

Combining corollary 7-2.2, theorem 7-0.1 and lemma 7-2.4, we obtain

7-2.3 Corollary. *Let M be a closed manifold with first Betti number ≥ 2. Given a lagrangian L there exists a generic set $\mathcal{O} \subset C^\infty(M, \mathbb{R})$ such that if $\psi \in \mathcal{O}$ the lagrangian $L+\psi$ has a unique minimizing measure* *in $\mathcal{M}^0(L+\psi)$ and this measure is uniquely ergodic. When this measure is supported on a periodic orbit, this orbit is hyperbolic and the stable and unstable manifolds have transverse homoclinic intersections.*

7-2.4 Lemma. *Let M be a closed manifold with first Betti number $b_1(M,\mathbb{R}) \geq 2$. Then if $A \subset M$ is a closed submanifold diffeomorphic to S^1 and U_ε denotes the ε neighborhood of A, we have that $H_1(M,U_\varepsilon,\mathbb{R})$ is non zero for all ε sufficiently small.*

Proof: Since A is diffeomorphic to a circle, the singular homology of the pair (M,U_ε) coincides with the singular homology of the pair (M,A) and therefore the vector space $H_1(M,U_\varepsilon,\mathbb{R})$ must have dimension $\geq b_1(M,\mathbb{R})-1 \geq 1$. □

For the proof of theorem 7-2.1 we shall need the following lemma:

7-2.5 Lemma. *Let $p: M_1 \to M_2$ be a covering such that $c(L_1)=c(L_2)$. Then any lift of a semistatic curve of L_2 is a semistatic curve of L_1. Also the projection of a static curve of L_1 is a static curve of L_2. If in addition, p is a finite covering, then any lift of a static curve of L_2 is a static curve of L_1.*

Proof: Observe first that for any $k \in \mathbb{R}$ we have that

$$\Phi_k^1(x,y) \geq \Phi_k^2(px,py),$$

for all x and y in M_1. Hence if we write $c=c(L_1)=c(L_2)$ we have

$$\Phi_c^1(x,y) \geq \Phi_c^2(px,py), \tag{7.20}$$

for all x and y in M_1.

Suppose now that $x_2 : \mathbb{R} \to M_2$ is a semistatic curve of L_2 and let $x_1 : \mathbb{R} \to M_1$ be any lift of x_2 to M_1. Using (7.20) and the fact that x_2 is semistatic we have for $s \leq t$,

$$\begin{aligned}\Phi_c^1(x_1(s), x_1(t)) \leq A_{L_1+c}(x_1|_{[s,t]}) &= A_{L_2+c}(x_2|_{[s,t]}) \\ &= \Phi_c^2(x_2(s), x_2(t)) \leq \Phi_c^1(x_1(s), x_1(t)).\end{aligned}$$

Hence x_1 is semistatic for L_1.

Suppose now that $x_1 : \mathbb{R} \to M_1$ is a static curve of L_1 and let $x_2 : \mathbb{R} \to M_2$ be $p \circ x_1$. Using (7.20) and the fact that x_1 is static we have for $s \leq t$,

$$\begin{aligned}-\Phi_c^1(x_1(t), x_1(s)) = \Phi_c^1(x_1(s), x_1(t)) &= A_{L_1+c}(x_1|_{[s,t]}) = A_{L_2+c}(x_2|_{[s,t]}) \\ &\geq \Phi_c^2(x_2(s), x_2(t)) \geq -\Phi_c^2(x_2(t), x_2(s)) \geq -\Phi_c^1(x_1(t), x_1(s)).\end{aligned}$$

Hence x_2 is static for L_2.

Suppose now that p is a *finite* covering and let $x_2 : \mathbb{R} \to M_2$ be a static curve of L_2. Let $x_1 : \mathbb{R} \to M$ be any lift of x_2 to M_1. Since x_2 is static, given $s \leq t$ and $\varepsilon > 0$, there exists a curve $\alpha : [0, T] \to M_2$ with $\alpha(0) = x_2(t)$, $\alpha(T) = x_2(s)$ such that

$$A_{L_2+c}(x_2|_{[s,t]}) + A_{L_2+c}(\alpha) \leq \varepsilon.$$

Since p is a finite covering, there exists a positive integer n, bounded from above by the number of sheets of the covering, such that the n-th iterate of $x_2|_{[s,t]} * \alpha$ lifts to M_1 as a closed curve. Hence, there exists a curve β joining $x_1(t)$ to $x_1(s)$ such that

$$A_{L_1+c}(x_1|_{[s,t]}) + A_{L_1+c}(\beta) \leq n\varepsilon,$$

and thus x_1 is static for L_1. □

Proof of theorem 7-2.1:

Let $U \overset{\text{def}}{=} U_\varepsilon$ denote the ε-neighborhood of $\pi(\widehat{\Sigma}(L))$, where $\widehat{\Sigma}(L)$ is the set of static vectors of L. Since we are assuming that $\widehat{\Sigma}(L)$ contains only one static class, the set U is also connected for small ε. Let $i : U \to M$ be the inclusion map. The vector space $H_1(M, U, \mathbb{R})$ is isomorphic to the quotient of $H_1(M, \mathbb{R})$ by $i_*(H_1(U, \mathbb{R}))$.

Let H denote the torsion free part of $H_1(M,\mathbb{Z})$ and let K denote the torsion free part of $i_*(H_1(U,\mathbb{Z}))$. Let us write $G \overset{\text{def}}{=} H/K = \mathbb{Z}\oplus \overset{k}{\ldots} \oplus \mathbb{Z}$ where $k = \dim H_1(M,U,\mathbb{R})$. Let J be a finite index subgroup of G. There is a surjective homomorphism $j : G \to G/J$ given by the projection.

If we take the Hurewicz map

$$\pi_1(M) \mapsto H_1(M,\mathbb{Z}),$$

and we compose it with the projections $H_1(M,\mathbb{Z}) \mapsto H$, $H \mapsto G$ and $j : G \to G/J$, we obtain a surjective homomorphism

$$\pi_1(M) \mapsto G/J,$$

whose kernel will be the fundamental group of a finite covering M_0 of M with covering projection map $p : M_0 \to M$ and group of deck transformations given by the finite abelian group G/J.

Since J is a subgroup of $G = H/K$, G/J acts transitively and freely on the set of connected components of $p^{-1}(U)$ which coincides with the set of connected components of $p^{-1}(\pi(\widehat{\Sigma}(L)))$. Therefore we have

7-2.6 Lemma. *There is a one to one correspondence between elements in G/J and connected components of $p^{-1}(\pi(\widehat{\Sigma}(L)))$.*

Observe that to each homoclinic orbit $x : \mathbb{R} \to M$ to $\widehat{\Sigma}(L)$ we can associate a homology class in H/K. Indeed, since there exists $t_0 > 0$ such that for all t with $|t| \geq t_0$, $x(t) \in U$, the class of $x|_{[-t_0,t_0]}$ defines an element in $H_1(M,U,\mathbb{Z})$. Let us denote by $\mathcal{H}$ the subset of H/K given by all the classes corresponding to homoclinic orbits to $\widehat{\Sigma}(L)$.

7-2.7 Lemma. *For any J as above, the image of $\langle\mathcal{H}\rangle$ under j is precisely G/J.*

Proof: Let L_0 denote the lift of the lagrangian L to M_0. Observe first that by proposition 2-7.2, $c(L) = c(L_0)$ and therefore by lemma 7-2.5 we have

$$\pi_0(\widehat{\Sigma}(L_0)) = p^{-1}(\pi(\widehat{\Sigma}(L))), \tag{7.21}$$

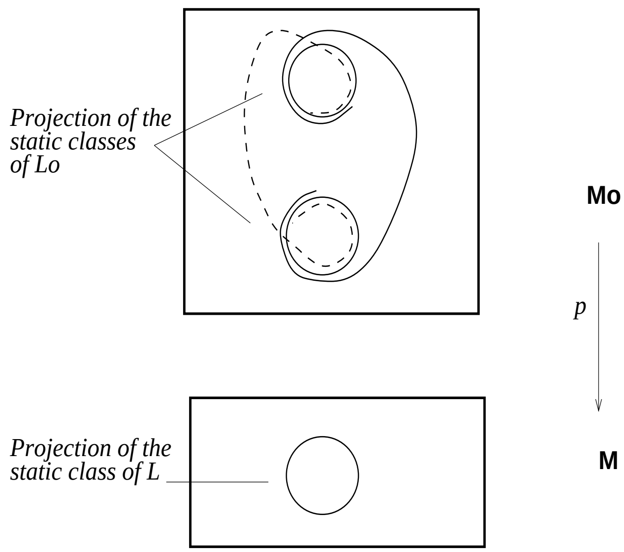


FIG. 1: Creating homoclinic connections with finite coverings.

where $\pi_0 : TM_0 \to M_0$ is the canonical projection of the tangent bundle TM_0 to M_0.

Let us prove now that L_0 satisfies the hypothesis of theorem 3-11.1, that is, the number of static classes of L_0 is finite. In fact, we shall show that the projection to M_0 of a static class of L_0 coincides with a connected component of $p^{-1}(\pi(\widehat{\Sigma}(L)))$. Using (7.21) and proposition 3-11.4 we see that the projection of a static class of L_0 to M_0 must be contained in a single connected component of $p^{-1}(\pi(\widehat{\Sigma}(L)))$. Hence, it suffices to show that if x and y belong to a connected component of $p^{-1}(\pi(\widehat{\Sigma}(L)))$ then $d_c^0(x,y) = 0$. Since we are assuming that $\widehat{\Sigma}(L)$ contains only one static class we have that $d_c(px, py) = 0$. Since $p : M_0 \to M$ is a finite covering there are lifts x_1 of px and y_1 of py such that $d_c^0(x_1, y_1) = 0$. Since static classes are connected x_1 and y_1 must belong to the same connected component of $p^{-1}(\pi(\widehat{\Sigma}(L)))$ and thus there is a covering transformation taking x_1 into x and y_1 into y which implies that $d_c^0(x,y) = 0$ as desired.

Now theorem 3-11.1 and (7.21) imply that every covering transformation in G/J can be written as the composition of covering transformations that arise from elements in $\mathcal{H}$, that is, $j(\langle\mathcal{H}\rangle) = G/J$. $\square$

We shall need the following algebraic lemma.

7-2.8 Lemma. *Let $G = \mathbb{Z} \oplus \overset{k}{\dots} \oplus \mathbb{Z}$. Given a finite index subgroup $J \subset G$ let us denote by $j : G \to G/J$ the projection homomorphism.*

Let A be a subgroup of G. If A has the property that for all J as above $j(A) = G/J$, then $A = G$.

Proof: The hypothesis readily implies that

$$A/A \cap J \text{ is isomorphic to } G/J \tag{7.22}$$

- If the rank of A is strictly less than the rank of G, one can easily construct a subgroup $J \subset G$ with finite index such that $A \subseteq J$ and $G/J \neq \{0\}$. But this contradicts (7.22) because $A/A \cap J = \{0\}$.
- If the rank of A equals the rank of G, then A has finite index in G and by (7.22) $G/A = \{0\}$ and thus $G = A$.

$\square$

Observe now that any set $\mathcal{H}$ of a free abelian group G of rank k such that the group generated by $\mathcal{H}$ is G must have at least k elements. Therefore if we combine lemma 7-2.7 and lemma 7-2.8 with $\langle\mathcal{H}\rangle = A$ we deduce that the set $\mathcal{H}$ of classes corresponding to homoclinic orbits generates G and must have at least k elements thus concluding the proof of theorem 7-2.1. $\square$

Appendix.

A Absolutely continuous functions.

A.1 Definition. A function $f : [a,b] \to \mathbb{R}$ is *absolutely continuous*, if $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ such that

$$\sum_{i=1}^{N} |t_i - s_i| < \delta \quad \Longrightarrow \quad \sum_{i=1}^{N} |f(t_i) - f(s_i)| < \varepsilon,$$

whenever $]s_1, t_1[, \ldots,]s_N, t_N[$ are disjoint intervals in $[a,b]$.

A.2 Proposition.

The function $f : [a,b] \to \mathbb{R}$ is absolutely continuous if and only if

(i) The derivative $f'(t)$ exists for a.e. $t \in [a,b]$.

(ii) $f' \in \mathcal{L}^1([a,b])$.

(iii) $f(t) = f(a) + \int_a^t f'(s)\, ds$.

Proof: Define

$$\mu([s,t]) := f(t) - f(s).$$

We claim that μ defines a finite signed Borel measure on $[a,b]$. Indeed, let $\mathcal{A}$ be the algebra of finite unions of intervals. The function μ can be extended to a σ-additive function on $\mathcal{A}$. Moreover, if B is a Borel set and $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ is a family with $A_n \downarrow B$, then $\mu(B) := \lim_n \mu(A_n)$ exists because $\mu(B_n \setminus B_m) \to 0$ when $n, m \to +\infty$.

Observe that the properties of external measures and the absolute continuity of f imply that $\mu \ll m$, where m is the Lebesgue measure. Let $g = \frac{d\mu}{dm}$ be the Radon-Nikodym derivative. Then $g \in \mathcal{L}^1$ and

$$f(t) - f(a) = \mu([a,t]) = \int_a^t g(s)\, ds.$$

By the Lebesgue differentiation theorem

$$\lim_{h\to 0} \frac{f(t+h) - f(t)}{h} = \lim_{h\to 0} \frac{1}{h} \int_t^{t+h} g = g(t) \qquad \text{for a.e. } t \in [a,b].$$

Conversely, suppose that (i)-(iii) hold. Using (ii), let $\mu(A) = \int_A f'\, dm$. Then by (iii),

$$\mu([s,t]) = f(t) - f(s) \qquad \text{for } s,\, t \in [a,b].$$

Then $\mu \ll m$ implies[1] that f is absolutely continuous. $\square$

The Lebesgue differentiation theorem gives the following characterization.

A.3 Corollary.

The function $f : [a,b] \to \mathbb{R}$ is absolutely continuous if and only if there exists $g \in \mathcal{L}^1([a,b])$ such that $f(t) = f(a) + \int_a^t g(s)\, ds$.

[1] If μ is finite, then $\mu \ll m$ is equivalent, using the Borel-Cantelli lemma, to

$$\forall\, e > 0 \;\; \exists\, \delta > 0, \;\; m(A) < \delta \Longrightarrow |\mu|(A) < \varepsilon.$$

B Measure Theory

B.1 Riesz Theorem. *[28]*

Let X be a locally compact Hausdorff topological space and let $C_b(X)$ the vector space of continuous functions $f : X \to \mathbb{R}$ with compact support. Then any positive linear functional $I : P(X) \to \mathbb{R}$ defines a unique Borel measure μ on X such that $I(f) = \int f \, d\mu$ for all $f \in C_b(X)$.

C Convex functions.

A function $f:\mathbb{R}^n \to \mathbb{R}$ is said *convex* if

$$f\big(\lambda\, x + (1-\lambda)\, y\big) \le \lambda\, f(x) + (1-\lambda)\, f(y)$$

for all $0 \le \lambda \le 1$ and $x,\, y \in \mathbb{R}^n$. Equivalently, if the set $\{\,(x,r) \in \mathbb{R}^n \times \mathbb{R} \,|\, r \ge f(x)\,\}$ is convex.

For $x_0 \in \mathbb{R}^n$ the *subdifferential* of f at x_0 is the set

$$\partial f(x_0) := \{\, p \in \mathbb{R}^{n^*} \,|\, f(x) \ge p\,(x - x_0) + f(x_0)\,\}.$$

Its elementst are called *subderivatives* or *subgradient* of f at x_0, and the planes

$$\{\,(x,r) \in \mathbb{R}^n \times \mathbb{R} \,|\, r = p\,(x - x_0) + f(x_0)\,\}$$

are called *supporting hyperplanes* for f at x_0. The functional $p \in \mathbb{R}^{n^*}$ is called the *slope* of the hyperplane.

For the proof of the following proposition see Rockafellar [62].

C.1 Proposition.

(a) $\partial f(x) \neq \varnothing$ *for every* $x \in \mathrm{Dom}(f)$.

(b) A finite convex function is continuous and Lebesgue almost everywhere differentiable.

(c) If $\partial f(x) = \{p\}$ *then* f *is differentiable at* x *and* $f'(x) = p$.

D The Fenchel and Legendre Transforms.

Given a convex function $f : \mathbb{R}^n \to \mathbb{R}$ the *Fenchel Transform* (or the *convex dual* of f is the function $f : (\mathbb{R}^n)^* \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f^*(p) = \max_{x \in \mathbb{R}^n} [\, p\, x - f(x)\,] \tag{D.1}$$

The function f admits a supporting hyperplane with slope $p \in \mathbb{R}^{n^*}$ if and only if $f^*(p) \neq +\infty$. If f is superlinear, then f^* is finite on all $\mathbb{R}^n$.

D.1 Proposition.

1. *If f is convex then f^* is convex.*
2. *If f and f^* are superlinear then $f^{**} = f$.*
3. *f is superlinear if and only if f^* is bounded on balls, more explicitly,*

   $$f(x) \geq A\ |x| - B(A)\,, \quad \forall x \in \mathbb{R}^n \iff f^*(p) \leq B(|p|)\,, \quad \forall p \in \mathbb{R}^{n^*}$$

4. *If f is superlinear, the maximum D.1 is attained at some point $x \in \mathbb{R}^n$.*

Proof:

1. Given $0 \leq \lambda \leq 1$ and $p_1, p_2 \in \mathbb{R}^{n^*}$ we have that

   $$\begin{aligned} f^*\big(\lambda\, p_1 + (1-\lambda)\, p_2\big) &= \max_{x \in \mathbb{R}^n} \big[\big(\lambda\, p_1 + (1-\lambda)\, p_2\big)\, x - f(x) \big] \\ &\leq \lambda \max_{x \in \mathbb{R}^n} [\, p_1\, x - f(x)\,] + (1-\lambda) \max_{x \in \mathbb{R}^n} [\, p_2\, x - f(x)\,] \\ &= \lambda\, f^*(p_1) + (1-\lambda)\, f^*(p_2). \end{aligned}$$

2. From (D.1) we get that

   $$f(x) \geq p\, x - f^*(p) \qquad \text{for all } x \in \mathbb{R}^n,\ p \in (\mathbb{R}^n)^*$$

Hence,

$$f(x) \geq \sup_{p \in \mathbb{R}^{n^*}} [\, p\, x - f^*(p)\,] = f^{**}(x).$$

Let $p_x \in \partial f(x) \neq \varnothing$. Then

$$f(y) \geq f(x) + p_x\,(y - x), \qquad \forall\, y \in \mathbb{R}^n.$$

Hence

$$f^*(p_x) = \max_{y \in \mathbb{R}^n} \big[\, p_x\, y - f(y)\, \big] = p_x\, x - f(x).$$

And

$$f(x) = p_x\, x - f^*(p_x) \leq \max_{p \in \mathbb{R}^{n^*}} \big[\, p\, x - f(x)\, \big] = f^{**}(x).$$

3. We have that

$$\begin{aligned} f_x^*(p) &= \max_{v \in \mathbb{R}^n} [\, p\, v - f_x(v)\,] \\ &\leq \max_{v \in \mathbb{R}^n} [\, p\, v - |p|\, v\,] + B(|p|) \\ &= B(|p|). \end{aligned}$$

Conversely, suppose that $f_x^*(p) \leq B(|p|)$. Given $A \in \mathbb{R}$ and $x \in \mathbb{R}^n$ there exists $p_x \in \mathbb{R}^{n^*}$ such that $|p_x| = A$ and $p_x\, v = |p_x|\,|x| = A\,|x|$. Then

$$\begin{aligned} f(x) &= \max_{p \in \mathbb{R}^{n^*}} [\, p\, x - f^*(p)\,] \\ &\geq p_x\, v - B(|p_x|) = A\,|x| - B(A). \end{aligned}$$

4. By item *3*, f^* is finite. Let $p \in \mathbb{R}^{n^*}$. If $b > 0$ is such that $f(x) > (|p| + 1)\,|x| - b$, then

$$p\, x - f(x) < b - |x| < f^*(p) - 1 \quad \text{ for } |x| > b + 1 - f^*(p).$$

Hence

$$f^*(p) = \max_{|x| \leq b+1-f^*(p)} [\, p\, x - f(x)\,],$$

and the maximum is attained at some interior point x_p in the closed ball $|x| \leq b + 1 - f^*(p)$.

$\square$

D.2 Corollary.
If $f : \mathbb{R}^n \to \mathbb{R}$ is convex and superlinear then so is $f^* : \mathbb{R}^{n^*} \to \mathbb{R}$. In this case $f^{**} = f$.

Observe that in this case we have

$$f^*(0) = -\min_{x \in \mathbb{R}^n} f(x) \qquad \text{and} \qquad f(0) = -\min_{p \in \mathbb{R}^{n^*}} f^*(p).$$

If $f : \mathbb{R}^n \to \mathbb{R}$ is convex and superlinear we define the *Legendre Transform* $\mathcal{L} : \mathbb{R}^n \to 2^{\mathbb{R}^{n^*}}$ of f, by

$$\mathcal{L}(x) = \{\, p \in \mathbb{R}^{n^*} \mid p\, x = f(x) + f^*(p) \,\}, \tag{D.2}$$

D.3 Proposition. *If $f : \mathbb{R}^n \to \mathbb{R}$ is C^2 and there is $a > 0$, such that*

$$y \cdot f''(x) \cdot y \geq a\, |y|^2 \quad \text{for all } x,\, y \in \mathbb{R}^n,$$

then the Legendre transform $\mathcal{L} : \mathbb{R}^n \to \mathbb{R}^{n^*}$ is a C^1 diffeomorphism given by $\mathcal{L}(x) = d_x f$.

Proof: The function f is convex and it is superlinear because

$$\begin{aligned} f(x) &= f(0) + \int_0^1 f'(sx)\, ds \\ &= f(0) + \int_0^1 \int_0^1 sx\, f''(ts\, x)\, x\, dt\, ds \\ &\geq f(0) + \tfrac{1}{2}\, a\, |x|^2. \end{aligned}$$

From (D.1) we get that

$$p\,x \le f(x) + f^*(p) \qquad \text{for all } x \in \mathbb{R}^n,\ p \in \mathbb{R}^{n^*}. \tag{D.3}$$

By proposition D.2, f^* is superlinear. Then item *4* in proposition D.2 implies that

$$\mathcal{L}(x) = \arg\max_{p\in\mathbb{R}^{n^*}} \{\, p\,x - f^*(p)\,\} \ne \varnothing.$$

Moreover, from (D.3), if $p \in \mathcal{L}(x)$ then $x = \arg\max_{x\in\mathbb{R}^n}\{\, p\,x - f(x)\,\}$. Thus $p = d_x f = \mathcal{L}(x)$. This proves that $\mathcal{L}$ differentiable and singled valued. Moreover since $d_x\mathcal{L} = f''(x)$ is non-singular, then $\mathcal{L}$ is a local C^1 diffeomorphism.

Since

$$(y-x)\cdot[\, d_y f - d_x f\,] = \int_0^1 f''\big(sx + (1-s)y\big)\, ds > 0,$$

then $x \mapsto d_x f = \mathcal{L}(x)$ is injective. We now prove that $\mathcal{L}$ is surjective. By item *4* in proposition D.1 the maximum

$$f^*(p) = \max_{x\in\mathbb{R}^n} [\, p\,x - f(x)\,]$$

is attained at some $x_p \in \mathbb{R}^n$. Then $p \in \mathcal{L}(x_p)$. $\square$

E Singular sets of convex funcions.

Let $f:\mathbb{R}^n\to\mathbb{R}$ be a convex function. Recall that its subdifferential at $x\in\mathbb{R}^n$ is the set

$$\partial f(x) := \{\, \ell:\mathbb{R}^n\to\mathbb{R} \ \text{ linear} \mid f(y)\ge f(x)+\ell\,(y-x)\,,\ \forall y\in\mathbb{R}^n\,\}.$$

Then the sets $\partial f(x)\subset\mathbb{R}^n$ are convex. If $k\in\mathbb{N}$, let

$$\Sigma_k(f) := \{\, x\in\mathbb{R}^n \mid \dim\partial f(x)\ge k\,\}.$$

E.1 Proposition. *If $f:\mathbb{R}^n\to\mathbb{R}$ is a convex function then for all $0\le k\le n$ the Hausdorff dimension $HD(\Sigma_k(f))\le n-k$.*

We recall here an elegant proof due to Ambrosio and Alberti, see [3]. More can be said on the structure of Σ_k, see [2, 66] for example.

By adding $|x|^2$ if necessary (which does not change Σ_k) we can assume that f is superlinear and that

$$f(y)\ge f(x)+\ell(y-x)+\tfrac12\,|y-x|^2 \quad \forall x,y\in\mathbb{R}^n, \quad \forall \ell\in\partial f(x). \tag{E.4}$$

E.2 Lemma. $\qquad \ell\in\partial f(x), \quad \ell'\in\partial f(x') \quad\Longrightarrow\quad |x-x'|\le\|\ell-\ell'\|.$

Proof: From inequality (E.4) we have that

$$\begin{aligned} f(x') &\ge f(x)+\ell(x'-x)\ +\ \tfrac12\left|x'-x\right|^2,\\ f(x) &\ge f(x')+\ell'(x-x')+\tfrac12\left|x-x'\right|^2. \end{aligned}$$

Then

$$0\ge(\ell'-\ell)(x-x')+\left|x-x'\right|^2 \tag{E.5}$$

$$\left\|\ell-\ell'\right\|\,\left|x-x'\right|\ge(\ell-\ell')(x-x')\ge\left|x-x'\right|^2. \tag{E.6}$$

Therefore $\|\ell-\ell'\|\ge|x-x'|$. ☐

Since f is superlinear, the subdifferential ∂f is surjective and we have:

E.3 Corollary.

There exists a Lipschitz function $F : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\ell \in \partial f(x) \quad \Longrightarrow \quad x = F(\ell).$$

Proof of Proposition E.1:

Let A_k be a set with $HD(A_k) = n - k$ such that A_k intersects any convex subset of dimension k. For example

$$A_k = \{x \in \mathbb{R}^n \,|\, x \text{ has at least } k \text{ rational coordinates } \}.$$

Observe that

$$x \in \Sigma_k \quad \Longrightarrow \quad \partial f(x) \text{ intersects } A_k \quad \Longrightarrow \quad x \in F(A_k).$$

Therefore $\Sigma_k \subset F(A_k)$. Since F is Lipschitz, we have that $HD(\Sigma_k) \leq HD(A_k) = n - k$.

$\square$

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