Recovering a Theorem of Poincaré

Gonzalo Contreras

CIMAT
Guanajuato, Mexico

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\( C^2 \)-densely, the 2-sphere has an elliptic closed geodesic.

Gonzalo Contreras  
CIMAT, Guanajuato  
México

Fernando Oliveira  
UFMG, Belo Horizonte  
Brasil

M. Herman’s memorial issue  
1905  Tr. AMS.  H. Poincaré claims:
Any convex surface in $\mathbb{R}^3$ has an
elliptic or degenerate simple closed geodesic.

1979  A.I. Grjuntal: Counterexample.
$M$ riemannian surface.

Geodesic = curve that locally minimizes length.

\[ \frac{D}{dt} \dot{\gamma} = 0 \]

curve with no acceleration “inside the surface”.
**Example:**

Embedded surface $M \subset \mathbb{R}^3$

$\gamma \subset M$ geodesic $\iff \ddot{\gamma} \perp T\gamma M$

Recovering a Theorem of Poincaré
Geodesic Flow

\[ \varphi_t : TM \to TM \]
\[ \varphi_t(x, v) = (\gamma(t), \dot{\gamma}(t)) \]
\[ \gamma \text{ geodesic} \implies \| \dot{\gamma}(t) \| \text{ constant.} \]

\[ \implies \text{ unit tangent bundle.} \]

\[ SM = \{ (x, v) \in TM \mid \| v \| = 1 \} \]

is invariant under \( \varphi_t : SM \to SM \).

- On \([\| v \| = a]\), \(a \neq 1\), \(\varphi_t\) is a reparametrization of \(\varphi_t|_{SM}\).
- \(\dim M = 2 \implies \dim SM = 3\).
A closed geodesic $\gamma$ corresponds to a periodic orbit $\Gamma = (\gamma, \dot{\gamma})$ for the geodesic flow. The 1st return map is called the "Poincaré map" $P : \Sigma \to \Sigma$. It maps $\Sigma$ to $\Sigma$ and preserves area, leading to eigenvalues $\lambda, \frac{1}{\lambda}$. The dimension of $SM$ is $3$. Recovering a Theorem of Poincaré.
\( \gamma \) or \( \Gamma \) is *degenerate* \iff \( d_\theta P \) has an eigenvalue 1.

*hyperbolic* \iff \( d_\theta P \) has no eigenvalue of modulus 1.

*elliptic* \iff \( d_\theta P \) has eigenvalues of modulus 1.
In 3rd vol. of *New Methods of Celestial Mechanics* (1899) Poincaré exclaimed, “If one attempts to imagine the figure formed by these two curves and their infinitely many intersections, each of which corresponds to a doubly asymptotic solution, these intersections form something like a lattice or fabric or a net with infinitely tight loops. None of these loops can intersect itself, but it must wind around itself in a very complicated fashion in order to intersect all the other loops of the net infinitely many times. One is struck by the complexity of this figure, which I shall not even attempt to draw. Nothing gives us a better idea of the complicated nature of the three-body problem and the problems of dynamics in general, in which there is no unique integral and in which the Bohlin series diverge.”
**ELLIPTIC CLOSED GEODESIC:**

If it is generic:

Poincaré map is a generic twist map

$$(r, \theta) \mapsto (R, \Theta)$$

$$\frac{\partial \Theta}{\partial r} > 0$$
1. KAM theorem $\implies$ 
   $\exists$ $\textit{+ve}$ measure set of invariant circles where the Poincaré map is conjugated to a rotation.

2. Between invariant circles periodic orbits $\left\{\text{elliptic}, \text{hyperbolic with intersections}\right\}$
3. Separation of phase space $\Rightarrow$ non ergodicity
Idea of Poincaré

1. Study bifurcations of/b by simple closed geodesics & show that
   \[ \# \text{elliptic} - \# \text{hyperbolic} = \text{constant}. \]

2. Ellipsoid

   3 simple closed geodesics
   \[
   \begin{align*}
   2\text{ elliptic} \quad & \quad 1\text{ hyperbolic} \\
   \implies & \quad 2 - 1 = 1 \neq 0.
   \end{align*}
   \]

THE PROBLEM IN $K \neq 0$:
Blue sky catastrophe.
A (simple) closed geodesic may disappear (or appear) when its period \( \to +\infty \).
Topogonov Thm $\Rightarrow$ in bounded $K > 0$ length of simple closed geodesics is bounded.

A geodesic can not touch itself $\Rightarrow$ continuation of simple closed geod. are simple.

Anosov: Proves that under bifurcations (in $K > 0$) 

$\#$(simple closed geod.) remains odd.

$\exists$ metrics on $\mathbb{S}^2$ with simple geodesics with arbitrary large length: large simple closed curve in $\mathbb{R}^2 + \mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$. 

Blue sky catastrophe $\Rightarrow$ No way to follow these arguments in $K \neq 0$ case.
Comparison with other theorems

1988 V. Donnay
Burns, Donnay, $C^\infty$

$\exists \ C^\infty$ riemannian metric on $S^2$ whose geodesic flow is ergodic and has $+$ve metric entropy.

All closed geod. but finite (3) (which are degenerate) are hyperbolic.

**NOT KNOWN:**

- Donnay’s thm in $+$ve curvature $K>0$.
- If $\exists \ C^\infty$ riem. metric on $S^2$ with all closed geod. hyperbolic.
Theorem

Any riemannian metric on $S^2$ or $\mathbb{RP}^2$ can be $C^2$ approximated by a $C^\infty$ metric with an elliptic closed geodesic.

$\implies \exists$ open and dense set of riemannian metrics in $S^2$ or $\mathbb{RP}^2$ (in $C^2$ topology) whose geodesic flow has an elliptic closed geodesic.

2000 IMPA Michel Herman

- announced this theorem when $K > 0$.
- conjectured it for arbitrary $K$.

Ballmann, Thorbergsson, Ziller

- Pinching conditions on $K > 0$ to have an elliptic closed geodesic on $S^n$. 
1977 Newhouse Theorem

$H : (M, \omega) \rightarrow \mathbb{R}$ smooth hamiltonian on a symplectic manifold. If the energy level $H^{-1}\{0\}$ is compact

$\implies \exists C^2$ perturbation $H_1$ of $H$

s.t. its hamiltonian flow either

- is Anosov.
- has a 1-elliptic closed orbit.

Rmks:

- Newhouse Thm uses the $C^2$ Closing Lemma (not known for geodesic flows).
- Corresponds to stability conjecture for hamiltonian flows.
- Main Thm above is a version of Newhouse Thm for geod. flows in $S^2$ or $\mathbb{RP}^2$ because

  $\nexists$ Anosov geodesic flow on $S^2$ [or $\mathbb{RP}^2$].

- Newhouse thm is not known in any other compact mfld.
Anosov geodesic flow on $S^2$ [or $\mathbb{RP}^2$].

2 proofs:

1. Anosov flow on $N = T^1S^2 = \mathbb{RP}^3$
   \[\implies \pi_1(N) \text{ has exponential growth.} \ (\Rightarrow \Leftarrow)\]

2. Anosov geodesic flow for $M$
   \[\implies \text{(Klingenberg)} \implies \text{No conjugate points} \implies \tilde{M} = \mathbb{R}^n\]
   but \[\tilde{S}^2 = S^2 \not\approx \mathbb{R}^2 \text{ re} \ (\Rightarrow \Leftarrow)\]
Klingenberg-Takens-Anosov Theorem

Given a closed geodesic one can perturb the Riemannian metric in the $C^\infty$ topology s.t.

1. does not move the closed geodesic.
2. makes any $k$-jet of the Poincaré map generic.

Klingenberg-Takens: perturbation for a single periodic orbit.
Anosov: Bumpy metric theorem & $\implies$ countable periodic orbits.
Applications of the Main Theorem

1. Make the Poincaré map of the elliptic geodesic $C^4$ generic

$\implies$ KAM Thm (Moser) $\implies$

$\exists$ invariant circle which separates the phase space

$\therefore \exists C^2$-dense set of $(C^\infty)$ riemannian metrics on $S^2$ or $\mathbb{RP}^2$
such that the geodesic flow is not ergodic.
Recall Lazutkin:
A billiard map in the interior of a $C^\infty$ embedded curve in $\mathbb{R}^2$ with $+ve$ curvature is not ergodic.
In higher dimensions:

Kobachev & Popov:
Billiard map in a strictly convex domain in $\mathbb{R}^n$ with $C^\infty$ boundary has a set of positive measure of invariant quasi-epriodic tori provided that the geodesic flow on the boundary has an elliptic periodic geodesic which is $k$-elementary, $k \geq 5$, (in particular the billiard is not ergodic).

Main Thm $\implies$ For $M \approx S^2 \subset \mathbb{R}^3$, $(n = 3)$, this condition is $C^2$ generic.
Case $K > 0$

There exists a simple closed geodesic. ▶ proof

Birkhoff section

$F(x, \theta) = (x', \theta')$
\[ \text{vol}(\varphi_{[0,\varepsilon]}(A)) = \text{area}(A) \]

\( F \): return map is smooth and preserves area.

- \( \text{int}(A) \cap \text{geodesic vector field.} \)
- \( \partial A = \Gamma \cup (-\Gamma), \quad \Gamma = (\gamma, \dot{\gamma}). \)
- any orbit \( \neq \{\Gamma, -\Gamma\} \) intersects \( A \).
- return times uniformly bounded \( 0 < T(x, \theta) < T_0. \)
- (can extend \( F \) to \( \partial A \) by \( \theta \mapsto 2\text{nd conjugate pt. to } \theta \))
\[ \exists \text{ simple closed geodesic on } S^2 \]

e.g. Birkhoff minimax closed geodesic.

Family of closed curves covering the sphere.
\( \gamma : [0, 1] \rightarrow S^2 \)

\[ E(\gamma) := \int_0^1 |\dot{\gamma}|^2 \, dt \]

\[ c := \inf_{F} \max_{s \in [0,1]} E(F(\cdot, s)) > 0 \]

\( c \) is a critical value of the energy functional with a critical point \( \gamma \) called the Birkhoff minimax geodesic.
$\mathcal{H}^2(S^2) := \{ C^2 \text{ riem. metrics on } S^2 \text{ without elliptic closed geodesics} \}$

$\mathcal{F}^2(S^2) := \text{int}_{C^2}(\mathcal{H}^2(S^2))$

JDG 2002: G. Paternain & G. Contreras

$g \in \mathcal{F}^2(S^2)$

$g \in C^4$ \implies $\text{Per}(g)$ is uniformly hyperbolic.

sketch:

- Prove perturb. (“Franks”) lemma for geod. flows in dim2.
- The periodic orbits are stably hyperbolic.
- Use Mañé-Liao theory on dominated splittings \implies $\text{Per}(g)$ has dominated splitting.
- Preserves area \implies [dom. splitting \implies uniform hyp.]
Uniform Hyperbolicity

\[ N = T^1\mathbb{S}^2 \]

\( \phi_t : \Lambda \rightarrow \Lambda \) invariant subset, is hyperbolic if

\[ T_\Lambda N = E^s \oplus \langle \mathbf{X} \rangle \oplus E^u, \quad \exists C, \lambda > 0 \]

\[ \| d\phi_t | E^s \| < C e^{-\lambda t}, \quad t > 0. \]

\[ \| d\phi_t | E^u \| < C e^{-\lambda t}, \quad t > 0. \]
\( \mathcal{H}^2(\mathbb{S}^2) := \{ C^2 \text{ riem. metrics on } \mathbb{S}^2 \text{ without elliptic closed geodesics} \} \)

\( \mathcal{F}^2(\mathbb{S}^2) := \text{int}_{C^2}(\mathcal{H}^2(\mathbb{S}^2)) \)

\[
g \in \mathcal{F}^2(\mathbb{S}^2) \quad \text{JDG 2002} \quad \overline{\text{Per}(g)} \text{ is uniformly hyperbolic.}
\]

- want \( \mathcal{F}^2(\mathbb{S}^2) = \emptyset \).
- Assume \( \neq \emptyset \), take \( g \in \mathcal{F}^2(\mathbb{S}^2) \).
- \( \mathcal{F}^2(\mathbb{S}^2) \) is open \( \implies \) \( \left\{ \begin{array}{l} \text{can assume } g \in C^\infty, \\ \text{can assume } g \text{ is Kupka-Smale,} \\ \text{[also in JDG 2002].} \end{array} \right. \)
Bangert + Franks: Any riem. metric on $\mathbb{S}^2$ has $\infty$-many closed geodesics.

+ Smale Spectral Decomposition Thm.

\[ \Rightarrow \quad \text{Per}(g) \text{ contains a non-trivial hyperbolic basic set.} \]

$\Lambda := \text{homoclinic class = hyperbolic basic set.}$

\[ g \in C^3 \quad \Rightarrow \quad F : \Lambda \leftrightarrow \text{is } C^3 \]

\[ \Rightarrow \quad \begin{cases} 
F \text{ Anosov} \quad \Rightarrow \quad g \text{ Anosov} \quad (\Rightarrow \Leftarrow) \\
\Lambda \text{ has measure } 0.
\end{cases} \]

Poincaré recurrence \[ \Rightarrow \quad \text{meas}[W^s(\Lambda) \setminus \Lambda] = 0 \]

similarly $W^u$ \[ \Rightarrow \quad \text{meas}[W^s(\Lambda) \cap W^u(\Lambda)] = 0. \]
Prove using the hyperbolicity:

- $B$ small closed ball, $\bar{B} \cap \Lambda \neq \emptyset$
- $Q := \overline{W^s(\Lambda) \cap W^u(\Lambda)}$

$\implies \quad Q \cap B \subset Q$

Take a “hole” $D$ of $Q$ in $B$, i.e.:

- $D = \text{a connected compo. of } \Lambda \setminus Q \text{ contained in } B$

  - $\text{meas}(D) > 0$
  - Poincaré recurrence $\exists N > 0 \quad F^N(D) \cap D \neq 0$

  but $\begin{cases} Q \text{ invariant} \\ D \text{ compo. of } \Lambda \setminus Q \end{cases} \implies F^N(D) \subset D.$
Brower Translation Theorem

\[ f : \mathbb{R}^2 \to \mathbb{R}^2 \text{ homeo. without fixed points} \]

\[ \implies \text{it has a “translation domain”}, \]

(i.e. it is semiconjugate to a translation in \( \mathbb{R} \)).
But

\[ F^N(D) = D \approx \mathbb{R}^2, \quad F \text{ preserves finite measure} \]
\[ \implies (\text{Poincaré recurrence}) \implies \text{no translation domain} \]
\[ \implies F^N : D \leftarrow \text{has fixed pt. } x \]

which is not in \( Q = \overline{W^s(\Lambda) \cap W^u(\Lambda)} \).

Uniform hyperbolicity
\[ \implies W^s(\Lambda), W^u(\Lambda) \]
are large.
\[ \implies x \in \text{homo. class } \Lambda \cap D. \]

\[ [ (\Rightarrow \Leftrightarrow ) \quad D \subset A \setminus D ] \]

\[ \therefore \quad \mathcal{F}^2(S^2) = \text{int}_{C^2} \mathcal{H}^2(S^2) = \emptyset. \]

\[ \text{Recovering a Theorem of Poincaré} \]
GENERAL CASE

Want the same for local transversal sections.

PROBLEMS:

1. Return time is $C^0$ only locally: it may tend to $\infty$.
2. Return map may be discontinuous.
3. Some wandering orbits may tend to some unknown wild strange set.

Recovering a Theorem of Poincaré
HOFER - WYSOCKI - ZEHNDER theory will say that non-returning points can only go to periodic orbits

[i.e. they must be in $W^s(\text{periodic})$]
END OF PART I
HOFER - WYSOCKI - ZEHNDER: Theory for tight contact forms in $S^3$.

$\dim M = 2n + 1$.

$\lambda$: contact 1-form in $M$: if $\lambda \wedge (d\lambda)^n$ is volume form.

$X$: Reeb vector field for $\lambda$: \[
\begin{cases}
    i_X(d\lambda) \equiv 0 \\
    \lambda(X) \equiv 1
\end{cases}
\]

$\varphi_t$: Reeb flow preserves $\lambda$.

Geodesic case: $\lambda_{(x,v)} = \langle v, dx \rangle_x$ Liouville form on $T^1M$.

geodesic flow $\equiv$ Reeb flow of $\lambda$. 

Recovering a Theorem of Poincaré
2 KINDS OF CONTACT FORMS IN $S^3$:

1. Overtwisted.

2. Tight: Canonical contact form in $S^3$:

$$\eta|_{S^3} = \frac{1}{2} [x \, dy - y \, dx]|_{S^3}$$

$S^3 \subset \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2 \ni (x, y)$

$\theta$ is tight $\iff \theta = f(x, y) \cdot \eta$

some $f : S^3 \to \mathbb{R}$. 
FROM $S^2$, $RP^2$ TO $S^3$

\[ TS^2 = RP^3 \xleftarrow{2 \times} S^3 \quad \text{double cover} \]

\[ RP^2 \xleftarrow{2 \times} S^2 \]

canonical contact form on $S^3$ $\implies$ Reeb flow $=$ Hopf fibration of $S^3$ $\xrightarrow{2 \times} T^1S^2$

geod. flow of “round sphere” $K = \text{constant}$

+ all riemannian metrics on $S^2$

are conformally equivalent (Beltrami eqs.)

$\implies$ Liouville forms of any riemannian metric on $S^2$

lift to tight contact forms on $S^3$. 
Hofer - Wysocki - Zehnder theory
is for “generic” tight contact forms in $\mathbb{S}^3$.

“generic” = all per. orbits non-degenerate
(i.e. no eigenvalue 1).

True for $C^\infty$ generic geodesic flows by Anosov
(Bumpy metric Thm).
Hofer - Wysocki - Zehnder:

- ∃ kind of “open book decomposition” of $\mathbb{S}^3$ by “surfaces of section” ⨂ Reeb flow.

- Each surface $\Sigma \approx \mathbb{S}^2 \setminus \{\text{finite points}\}$.

- $\partial \Sigma \subset \{\text{finite periodic orbits}\} = \text{Biding orbits} =: \mathbb{B}$.

- $d\lambda$-area of each surface is finite.

- ∃ finite set of those surfaces (“rigid surfaces”) which intersect all orbits except those in $\mathbb{B}$.
Figure 3. Stable finite energy foliation of $S^3$. 

H. Hofer, K. Wysocki, and E. Zehnder
Figure 31. A family of surfaces $C_\tau$ decomposes into the broken trajectory $(C^+, C^-)$. 
As before $\text{Per}(g)$ uniformly hyperbolic.

$\Lambda$ = homoclinic class = hyperbolic basic set.

$\Sigma$ finite set of surfaces of section.

$D$ = small hole in $\Sigma \setminus Q$, $Q = W^s(\Lambda) \cup W^u(\Lambda)$.

$F: \Sigma \to \Sigma$ return map where well defined.

Poincaré recurrence $F^N(D) \cap D \neq \emptyset$.

1. If $F^N$ well defined on whole $D$ $\implies F^N(D) = D \implies$ Brower Translation Thm ....

2. If not $\implies$ discontinuity of $F^N$ $\implies W^s(\text{biding orbits}) \cap D \neq \emptyset$. (biding orbits = $\partial \Sigma$)
**Lemma:**
A return of an arbitrarily small piece of $W^u$(biding orbit) must be large [$\text{diam} > a$].

- $B$ a connected compo. of $D \cap [\tau_N < +\infty]$

But $B$ connected
$B \cap Q = \emptyset$, $Q$ conn., invar.
$Q = W^s(\Lambda) \cup W^u(\Lambda)$
$\implies F^N(B) \subset D$ (⇒⇐)

Recovering a Theorem of Poincaré
**Lemma:** A return of an arbitrarily small piece of $W^u$(biding orbit) must be large.

**Proof:**

1. Return of $W^s$ is $W^u$.

Return of a small transversal to $W^s$ accumulates on whole compo. of $W^u$
Recovering a Theorem of Poincaré

$W^s(P)$

$W^s(P_1)$

$W^u(P_1)$
2. If it is 1st return.

2.a Return circle contains a hole (biding orbit) in $\partial \Sigma \implies \text{Large.}$

$\Sigma \approx S^2 \setminus (\text{finit. pts.})$
2.b Return does not contain holes of $\Sigma$:

\[
0 = \int_{\text{cylinder}} d\lambda = \int_{\text{per. orbit}} \lambda - \int_{\text{return to } \Sigma} \lambda
\]

$W^u$ lagrangian

or \[
\{ i_\mathcal{X} d\lambda = 0 \}
\]

$\mathcal{X} \in T W^u$

integral is large $\Rightarrow$ return large

Stokes

Recovering a Theorem of Poincaré
Following returns always accumulate on a complete 1st return $\Rightarrow$ Large!