A GENERIC PROPERTY OF FAMILIES OF LAGRANGIAN SYSTEMS

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Abstract. We prove that a generic lagrangian has finitely many minimizing measures for every cohomology class.

1. Introduction

Let $M$ be a compact boundaryless smooth manifold.
Let $T$ be either the group $(\mathbb{R}/\mathbb{Z}, +)$ or the trivial group $\{0\}$.
A Tonelli Lagrangian is a $C^2$ function $L : T \times TM \to \mathbb{R}$ such that
- The restriction to each fiber of $T \times TM \to T \times M$ is a convex function.
- It is fiberwise superlinear:
  $\lim_{|\theta|\to +\infty} L(t, \theta)/|\theta| = +\infty, \quad (t, \theta) \in T \times TM$.
- The Euler-Lagrange equation
  $\frac{d}{dt} L_u = L_x$
  defines a complete flow $\varphi : \mathbb{R} \times (T \times TM) \to T \times TM$.
We say that a Tonelli Lagrangian $L$ is strong Tonelli if $L + u$ is a Tonelli Lagrangian for each $u \in C^\infty(T \times M, \mathbb{R})$. When $T = \{0\}$ we say that the lagrangian is autonomous.

Let $\mathcal{P}(L)$ be the set of Borel probability measures on $T \times TM$ which are invariant under the Euler-Lagrange flow $\varphi$. The action functional $A_L : \mathcal{P}(L) \to \mathbb{R} \cup \{+\infty\}$ is defined as

$A_L(\mu) := \langle L, \mu \rangle := \int_{T \times TM} L \, d\mu$.

The functional $A_L$ is lower semi-continuous and the minimizers of $A_L$ on $\mathcal{P}(L)$ are called minimizing measures. The ergodic components of a minimizing measure are also minimizing, and they are mutually singular, so that the set $\mathcal{M}(L)$ of minimizing measures is a simplex whose extremal points are the ergodic minimizing measures.

In general, the simplex $\mathcal{M}(L)$ may be of infinite dimension. The goal of the present paper is to prove that this is a very exceptional phenomenon. The first results in that direction were obtained by Mañé in [4]. His paper has been very influential to our work.

We say that a property is generic in the sense of Mañé if, for each strong Tonelli Lagrangian $L$, there exists a residual subset $\mathcal{O} \subset C^\infty(T \times M, \mathbb{R})$ such that the property holds

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for all the Lagrangians $L - u, u \in \mathcal{O}$. A set is called residual if it is a countable intersection of open and dense sets. We recall which topology is used on $C^\infty(\mathbb{T} \times M, \mathbb{R})$. Denoting by $\|u\|_k$ the $C^k$-norm of a function $u : \mathbb{T} \times M \rightarrow \mathbb{R}$, define

$$\|u\|_\infty := \sum_{k \in \mathbb{N}} \frac{\arctan(\|u\|_k)}{2^k}.$$  

Note that $\|\cdot\|_\infty$ is not a norm. Endow the space $C^\infty(\mathbb{T} \times M, \mathbb{R})$ with the translation-invariant metric $\|u - v\|_\infty$. This metric is complete, hence the Baire property holds: any residual subset of $C^\infty(\mathbb{T} \times M, \mathbb{R})$ is dense.

**Theorem 1.** Let $A$ be a finite dimensional convex family of strong Tonelli Lagrangians. Then there exists a residual subset $\mathcal{O}$ of $C^\infty(\mathbb{T} \times M, \mathbb{R})$ such that,

$$u \in \mathcal{O}, \quad L \in A \quad \Rightarrow \quad \dim \mathfrak{M}(L - u) \leq \dim A.$$  

In other words, there exist at most $1 + \dim A$ ergodic minimizing measures of $L - u$.

The main result of Mañe in [4] is that having a unique minimizing measure is a generic property. This corresponds to the case where $A$ is a point in our statement. Our generalization of Mañe’s result is motivated by the following construction due to John Mather:

We can view a 1-form on $M$ as a function on $TM$ which is linear on the fibers. If $\lambda$ is closed, the Euler-Lagrange equation of the Lagrangian $L - \lambda$ is the same as that of $L$. However, the minimizing measures of $L - \lambda$, are not the same as the minimizing measures of $L$. Mather proves in [5] that the set $\mathfrak{M}(L - \lambda)$ of minimizing measures of the lagrangian $L - \lambda$ depends only on the cohomology class $c$ of $\lambda$. If $c \in H^1(M, \mathbb{R})$ we write $\mathfrak{M}(L - c) := \mathfrak{M}(L - \lambda)$, where $\lambda$ is a closed form of cohomology $c$.

It turns out that important applications of Mather theory, such as the existence of orbits wandering in phase space, require understanding not only of the set $\mathfrak{M}(L)$ of minimizing measures for a fixed or generic cohomology classes but of the set of all Mather minimizing measures for every $c \in H^1(M, L)$. The following corollaries are crucial for these applications.

**Corollary 2.** The following property is generic in the sense of Mañe:

For all $c \in H^1(M, \mathbb{R})$, there are at most $1 + \dim H^1(M, \mathbb{R})$ ergodic minimizing measures of $L - c$.

We say that a property is of infinite codimension if, for each finite dimensional convex family $A$ of strong Tonelli Lagrangians, there exists a residual subset $\mathcal{O}$ in $C^\infty(\mathbb{T} \times M, \mathbb{R})$ such that none of the Lagrangians $L - u, L \in A, u \in \mathcal{O}$ satisfy the property.
Corollary 3. The following property is of infinite codimension:

There exists $c \in H^1(M, \mathbb{R})$, such that $L - c$ has infinitely many ergodic minimizing measures.

Another important issue concerning variational methods for Arnold diffusion questions is the total disconnectedness of the quotient Aubry set. John Mather proves in [7, §3] that the quotient Aubry set $A$ of any Tonelli lagrangian on $\mathbb{T} \times TM$ with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $\dim M \leq 2$ (or with $\mathbb{T} = \{0\}$ and $\dim M \leq 3$) is totally disconnected. See [7] for its definition.

The elements of the quotient Aubry set are called static classes. They are disjoint subsets of $TM$ and each static class supports at least one ergodic minimizing measure. We then get

Corollary 4. The following property is generic in the sense of Mañé:

For all $c \in H^1(M, \mathbb{R})$ the quotient Aubry set $A_c$ of $L - c$ has at most $1 + \dim H^1(M, \mathbb{R})$ elements.

2. Abstract Results

Assume that we are given

- Three topological vector spaces $E$, $F$, $G$.
- A continuous linear map $\pi : F \to G$.
- A bilinear pairing $\langle u, \nu \rangle : E \times G \to \mathbb{R}$.
- Two metrizable convex compact subsets $H \subset F$ and $K \subset G$ such that $\pi(H) \subset K$.

Suppose that

(i) The map

$$E \times K \ni (u, \nu) \longmapsto \langle u, \nu \rangle$$

is continuous.

We will also denote $\langle u, \pi(\mu) \rangle$ by $\langle u, \mu \rangle$ when $\mu \in H$. Observe that each element $u \in E$ gives rise to a linear functional on $F$

$$F \ni \mu \longmapsto \langle u, \mu \rangle$$

which is continuous on $H$. We shall denote by $H^*$ the set of affine and continuous functions on $H$ and use the same symbol $u$ for an element of $E$ and for the element $u \longmapsto \langle u, \mu \rangle$ of $H^*$ which is associated to it.

(ii) The compact $K$ is separated by $E$. This means that, if $\eta$ and $\nu$ are two different points of $K$, then there exists a point $u$ in $E$ such that $\langle u, \eta - \nu \rangle \neq 0$.

Note that the topology on $K$ is then the weak topology associated to $E$. A sequence $\eta_n$ of elements of $K$ converges to $\eta$ if and only if we have $\langle u, \eta_n \rangle \longrightarrow \langle u, \eta \rangle$ for each $u \in E$. We shall, for notational conveniences, fix once and for all a metric $d$ on $K$. 
(iii) $E$ is a Frechet space. It means that $E$ is a topological vector space whose topology is defined by a translation-invariant metric, and that $E$ is complete for this metric.

Note then that $E$ has the Baire property. We say that a subset is residual if it is a countable intersection of open and dense sets. The Baire property says that any residual subset of $E$ is dense.

Given $L \in H^*$ denote by

$$M_H(L) := \arg \min_H L$$

the set of points $\mu \in H$ which minimize $L|_H$, and by $M_K(L)$ the image $\pi(M_H(L))$. These are compact convex subsets of $H$ and $K$.

Our main abstract result is:

**Theorem 5.** For every finite dimensional affine subspace $A$ of $H^*$, there exists a residual subset $\mathcal{O}(A) \subset E$ such that, for all $u \in \mathcal{O}(A)$ and all $L \in A$, we have

$$\dim M_K(L - u) \leq \dim A.$$

**Proof:** We define the $\varepsilon$-neighborhood $V_\varepsilon$ of a subset $V$ of $K$ as the union of all the open balls in $K$ which have radius $\varepsilon$ and are centered in $V$. Given a subset $D \subset A$, a positive number $\varepsilon$, and a positive integer $k$, denote by $\mathcal{O}(D, \varepsilon, k) \subset E$ the set of points $u \in E$ such that, for each $L \in D$, the convex set $M_K(L - u)$ is contained in the $\varepsilon$-neighborhood of some $k$-dimensional convex subset of $K$.

We shall prove that the theorem holds with

$$\mathcal{O}(A) = \bigcap_{\varepsilon > 0} \mathcal{O}(A, \varepsilon, \dim A).$$

If $u$ belongs to $\mathcal{O}(A)$, then (1) holds for every $L \in A$. Otherwise, for some $L \in A$, the convex set $M_K(L - u)$ would contain a ball of dimension $\dim A + 1$, and, if $\varepsilon$ is small enough, such a ball is not contained in the $\varepsilon$-neighborhood of any convex set of dimension $\dim A$.

So we have to prove that $\mathcal{O}(A)$ is residual. In view of the Baire property, it is enough to check that, for any compact subset $D \subset A$ and any positive $\varepsilon$, the set $\mathcal{O}(D, \varepsilon, \dim A)$ is open and dense. We shall prove in 2.1 that it is open, and in 2.2 that it is dense.

\[\square\]

2.1. Open.

We prove that, for any $k \in \mathbb{Z}^+$, $\varepsilon > 0$ and any compact $D \subset A$, the set $\mathcal{O}(D, \varepsilon, k) \subset E$ is open. We need a Lemma.

**Lemma 6.** The set-valued map $(L, u) \mapsto M_H(L - u)$ is upper semi-continuous on $A \times E$. This means that for any open subset $U$ of $H$, the set

$$\{(L, u) \in A \times E : M_H(L - u) \subset U\} \subset A \times E$$

is open in $A \times E$. Consequently, the set-valued map $(L, u) \mapsto M_K(L - u)$ is also upper semi-continuous.
Proof: This is a standard consequence of the continuity of the map
\[ A \times E \times H \ni (L, u, \mu) \mapsto (L - u)(\mu) = L(\mu) - \langle u, \mu \rangle. \]

Now let \( u_0 \) be a point of \( O(D, \varepsilon, k) \). For each \( L \in D \), there exists a \( k \)-dimensional convex set \( V \subset K \) such that \( M_K(L - u_0) \subset V \). In other words, the open sets of the form
\[ \{ (L, u) \in D \times E : M_H(L - u) \subset V \} \subset D \times E, \]
where \( V \) is some \( k \)-dimensional convex subset of \( K \). So there exists a finite subcovering of \( D \times \{ u_0 \} \) by open sets of the form \( \Omega_i \times U_i \), where \( \Omega_i \) is an open set in \( A \) and \( U_i \subset O(\Omega_i, \varepsilon, k) \) is an open set in \( E \) containing \( u_0 \). We conclude that the open set \( \cap U_i \) is contained in \( O(D, \varepsilon, k) \), and contains \( u_0 \). This ends the proof.

2.2. Dense.

We prove the density of \( O(A, \varepsilon, \dim A) \) in \( E \) for \( \varepsilon > 0 \). Let \( w \) be a point in \( E \). We want to prove that \( w \) is in the closure of \( O(A, \varepsilon, \dim A) \).

Lemma 7. There exists an integer \( m \) and a continuous map
\[ T_m = (w_1, \ldots, w_m) : K \to \mathbb{R}^m, \]
with \( w_i \in E \) such that
\[ \forall x \in \mathbb{R}^m \quad \text{diam } T_m^{-1}(x) < \varepsilon, \]
where the diameter is taken for the distance \( d \) on \( K \).

Proof: In \( K \times K \), to each element \( w \in E \) we associate the open set
\[ U_w = \{ (\eta, \mu) \in K \times K : \langle w, \eta - \mu \rangle \neq 0 \}. \]
Since \( E \) separates \( K \), the open sets \( U_w, w \in E \) cover the complement of the diagonal in \( K \times K \). Since this complement is open in the separable metrizable set \( K \times K \), we can extract a countable subcovering from this covering. So we have a sequence \( U_{w_k} \), with \( w_k \in E \), which covers the complement of the diagonal in \( K \times K \). This amounts to say that the sequence \( w_k \) separates \( K \). Defining \( T_m = (w_1, \ldots, w_m) \), we have to prove that (2) holds for \( m \) large enough. Otherwise, we would have two sequences \( \eta_m \) and \( \mu_m \) in \( K \) such that
\[ T_m(\mu_m) = T_m(\eta_m) \quad \text{and} \quad d(\mu_m, \eta_m) \geq \varepsilon. \]

By extracting a subsequence, we can assume that the sequences \( \mu_m \) and \( \eta_m \) have different limits \( \mu \) and \( \eta \), which satisfy \( d(\eta, \mu) \geq \varepsilon \). Take \( m \) large enough, so that \( T_m(\eta) \neq T_m(\mu) \). Such a value of \( m \) exists because the linear forms \( w_k \) separate \( K \). We have that
\[ T_m(\mu_k) = T_m(\eta_k) \quad \text{for} \quad k \geq m. \]
Hence at the limit \( T_m(\eta) = T_m(\mu) \). This is a contradiction.

□
Define the function $F_m : A \times \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ as

$$F_m(L, x) := \min_{\mu \in H} (L - w)(\mu),$$

when $x \in T_m(\pi(H))$ and $F_m(L, x) = +\infty$ if $x \in \mathbb{R}^m \setminus T_m(\pi(H))$. For $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$, let

$$M_m(L, y) := \arg\min_{x \in \mathbb{R}^m} \left[ F_m(L, x) - y \cdot x \right] \subset \mathbb{R}^m$$

be the set of points which minimize the function $x \mapsto F_m(L, x) - y \cdot x$. We have that

$$M_K(L - w - y_1 w_1 - \cdots - y_m w_m) \subset T_m^{-1}(M_m(L, y)).$$

Let

$$O_m(A, \dim A) := \{ y \in \mathbb{R}^m \mid \forall L \in A : \dim M_m(L, y) \leq \dim A \}.$$ 

From Lemma 7 it follows that

$$y \in O_m(A, \dim A) \implies w + y_1 w_1 + \cdots + y_m w_m \in \mathcal{O}(A, \varepsilon, \dim A).$$

Therefore, in order to prove that $w$ is in the closure of $\mathcal{O}(A, \varepsilon, \dim A)$, it is enough to prove that 0 is in the closure of $\mathcal{O}_m(A, \dim A)$, which follows from the next proposition.

**Proposition 8.** The set $\mathcal{O}_m(A, \dim A)$ is dense in $\mathbb{R}^m$.

**Proof:** Consider the Legendre transform of $F_m$ with respect to the second variable,

$$G_m(L, y) = \max_{x \in \mathbb{R}^m} y \cdot x - F_m(L, x)$$

$$= \max_{\mu \in H} \langle w + y_1 w_1 + \cdots + y_m w_m, \mu \rangle - L(\mu).$$

It follows from this second expression that the function $G_m$ is convex and finite-valued, hence continuous on $A \times \mathbb{R}^m$.

Consider the set $\Sigma \subset A \times \mathbb{R}^m$ of points $(L, y)$ such that $\dim \partial G_m(L, y) \geq \dim A + 1$, where $\partial G_m$ is the subdifferential of $G_m$. It is known, see the appendix, that this set has Hausdorff dimension at most

$$(m + \dim A) - (\dim A + 1) = m - 1.$$ 

Consequently, the projection $\Sigma$ of the set $\hat{\Sigma}$ on the second factor $\mathbb{R}^m$ also has Hausdorff dimension at most $m - 1$. Therefore, the complement of $\Sigma$ is dense in $\mathbb{R}^m$. So it is enough to prove that

$$y \notin \Sigma \implies \forall L \in A : \dim M_m(L, y) \leq \dim A.$$ 

Since we know by definition of $\Sigma$ that $\dim \partial G_m(L, y) \leq \dim A$, it is enough to observe that

$$\dim M_m(L, y) \leq \dim \partial G_m(L, y).$$

The last inequality follows from the fact that the set $M_m(L, y)$ is the subdifferential of the convex function

$$\mathbb{R}^m \ni z \mapsto G_m(L, z)$$

at the point $y$. 
3. Application to Lagrangian dynamics

Let $C$ be the set of continuous functions $f : \mathbb{T} \times TM \to \mathbb{R}$ with linear growth, i.e.

$$\|f\|_\ell := \sup_{(t, \theta) \in \mathbb{T} \times TM} \frac{|f(t, \theta)|}{1 + |\theta|} < +\infty,$$

endowed with the norm $\| \cdot \|_\ell$.

We apply Theorem 5 to the following setting:

- $F = C^*$ is the vector space of continuous linear functionals $\mu : C \to \mathbb{R}$ provided with the weak-$*$ topology. Recall that

  $$\lim_n \mu_n = \mu \iff \lim_n \mu_n(f) = \mu(f), \quad \forall f \in C.$$

- $E = C_\infty(T \times M, \mathbb{R})$ provided with the $C_\infty$ topology.

- $G$ is the vector space of finite Borel signed measures on $TM$, or equivalently the set of continuous linear forms on $C^0(T \times M, \mathbb{R})$, provided with the weak-$*$ topology.

- The pairing $E \times G \to \mathbb{R}$ is given by integration:

  $$\langle u, \nu \rangle = \int_{\mathbb{T} \times M} u \, d\nu.$$

- The continuous linear map $\pi : F \to G$ is induced by the projection $\mathbb{T} \times TM \to \mathbb{T} \times M$.

- The compact $K \subset G$ is the set of Borel probability measures on $\mathbb{T} \times M$, provided with the weak-$*$ topology. Observe that $K$ is separated by $E$.

- The compact $H_n \subset F$ is the set of holonomic probability measures which are supported on

  $$B_n := \{(t, \theta) \in \mathbb{T} \times TM \mid |\theta| \leq n\}.$$

Holonomic probabilities are defined as follows: Given a $C^1$ curve $\gamma : \mathbb{R} \to M$ of period $T \in \mathbb{N}$ define the element $\mu_\gamma$ of $F$ by

$$\langle f, \mu_\gamma \rangle = \frac{1}{T} \int_0^T f(s, \gamma(s), \dot{\gamma}(s)) \, ds$$

for each $f \in C$. Let

$$\Gamma := \{ \mu_\gamma \mid \gamma \in C^1(\mathbb{R}, M) \text{ is periodic of integral period } \} \subset F.$$ 

The set $\mathcal{H}$ of holonomic probabilities is the closure of $\Gamma$ in $F$. One can see that $\mathcal{H}$ is convex (cf. Mañé [4, prop. 1.1(a)]). The elements $\mu$ of $\mathcal{H}$ satisfy $\langle 1, \mu \rangle = 1$ therefore we have $\pi(\mathcal{H}) \subset K$.

Note that each Tonelli Lagrangian $L$ gives rise to an element of $H_n^*$. 

Let $\mathfrak{M}(L)$ be the set of minimizing measures for $L$ and let $\text{supp} \mathfrak{M}(L)$ be the union of their supports. Recalling that we have defined $M_{H_n}(L)$ as the set of measures $\mu \in H_n$ which minimize the action $\int L \, d\mu$ on $H_n$, we have:
Lemma 9. If $L$ is a Tonelli lagrangian then there exists $n \in \mathbb{N}$ such that

$$\dim \pi(M_{H_n}(L)) = \dim \mathcal{M}(L).$$

Proof:

Birkhoff theorem implies that $\mathcal{M}(L) \subset \mathcal{H}$ (cf. Mañé [4, prop. 1.1.(b)]). In [5, Prop. 4, p. 185] Mather proves that $\text{supp} \mathcal{M}(L)$ is compact, therefore $\mathcal{M}(L) \subset H_n$ for some $n \in \mathbb{N}$.

In [4, §1] Mañé proves that minimizing measures are also all the minimizers of the action functional $A_L(\mu) = \int L \, d\mu$ on the set of holonomic measures, therefore $\mathcal{M}(L) = M_{H_n}(L)$ for some $n \in \mathbb{N}$.

In [5, Th. 2, p. 186] Mather proves that the restriction $\text{supp} \mathcal{M}(L) \to M$ of the projection $TM \to M$ is injective. Therefore the linear map $\pi : \mathcal{M}(L) \to \mathbb{R}$ is injective, so that $\dim \pi(M_{H_n}(L)) = \dim \pi(\mathcal{M}(L)) = \dim \mathcal{M}(L).$
\textbf{Proof:} From inequality (3) we have that
\[ f(x') \geq f(x) + \ell(x' - x) + \frac{1}{2} |x' - x|^2, \]
\[ f(x) \geq f(x') + \ell'(x - x') + \frac{1}{2} |x - x'|^2. \]
Then
\[ 0 \geq (\ell' - \ell)(x - x') + |x - x'|^2 \]
(4)
\[ \|\ell - \ell'\| \geq (\ell - \ell')(x - x') \geq |x - x'|^2. \]
(5)
Therefore \(\|\ell - \ell'\| \geq |x - x'|.\) \qed

Since \(f\) is superlinear, the subdifferential \(\partial f\) is surjective and we have:

\textbf{Corollary 12.} There exists a Lipschitz function \(F : \mathbb{R}^n \rightarrow \mathbb{R}^n\) such that
\[ \ell \in \partial f(x) \implies x = F(\ell). \]

\textbf{Proof of Proposition 10:}

Let \(A_k\) be a set with \(HD(A_k) = n - k\) such that \(A_k\) intersects any convex subset of dimension \(k\). For example
\[ A_k = \{x \in \mathbb{R}^n \mid x \text{ has at least } k \text{ rational coordinates} \}. \]

Observe that
\[ x \in \Sigma_k \implies \partial f(x) \text{ intersects } A_k \implies x \in F(A_k). \]
Therefore \(\Sigma_k \subset F(A_k)\). Since \(F\) is Lipschitz, we have that \(HD(\Sigma_k) \leq HD(A_k) = n - k.\) \qed

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