POSITIVE TOPOLOGICAL ENTROPY FOR GENERIC GEODESIC FLOWS

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GEODESIC FLOW

M closed $\mathcal{C}^\infty$ manifold [compact, connected, $\partial M = \emptyset$]

$g = \langle \cdot , \cdot \rangle_x$ $\mathcal{C}^\infty$ riemannian metric on $M$.

Unit tangent bundle = sphere bundle of $(M, g)$

$SM = \{ (x, v) \in TM \mid ||v||_x = 1 \}$

$\pi : SM \rightarrow M$

$(x, v) \mapsto x$

$(x, v) \in SM$

$\gamma : \mathbb{R} \rightarrow M$
geodesic s.t. $\gamma(0) = x$, $\dot{\gamma}(0) = v$

"locally length minimizing curve with $||\dot{\gamma}|| = 1$"

Geodesic Flow

$\phi_t : SM \rightarrow SM$

$(x, v) \mapsto (\gamma(t), \dot{\gamma}(t))$
**Topological Entropy**

Measures the "complexity" of the orbit structure of the flow. Measures the difficulty in predicting the position of an orbit given an approximation of its initial state.

**Dynamic Ball**: \( \Theta \in SM, \epsilon, T > 0 \)

\[ \mathcal{B}(\Theta, \epsilon, T) = \{ \omega \in SM \mid d(\phi^t \Theta, \phi^t \omega) \leq \epsilon, \forall t \in [0, T] \} \]

Points whose orbit stay near the orbit of \( \Theta \) for times in \( [0, T] \)

\[ N(\epsilon, T) = \min \left\{ \# \mathcal{E} \mid \mathcal{E} \text{ cover of SM by} \ (\epsilon, T) \text{-dyn. balls} \right\} \]

\[ h_{\text{top}}(g) = \lim_{\epsilon \to 0} \limsup_{T \to \infty} \frac{1}{T} \log N(\epsilon, T) \]

\[ N(\epsilon, T) \sim e^{h_{\text{top}} \cdot T} \]

If \( h_{\text{top}} > 0 \) some dynamic balls must contract exponentially at least in one direction.
For $C^\infty$ riemannian metrics

M
c\[ h_{top}(g) = \lim_{T \to \infty} \frac{1}{T} \log \int_{M \times M} n_T(x, y) \, dx \, dy \]

$n_T(x, y) := \# \{ \text{geod. arcs } x \to y \text{ of length } \leq T \}$

$h_{top} > 0 \Rightarrow$ positive measure of $(x, y)$

s.t. $n_T(x, y)$ is exponentially large.

**Topology**

Some manifolds have always $h_{top}(g) > 0$

- Dinaburg: $\Pi_1(M)$ exponential growth
  $\Rightarrow h_{top} > 0$
  [# dyn balls growths expo]

- Paternain - Petean: If $H^1(\text{Loop Space, } x)$
  growths exponentially $\Rightarrow h_{top} > 0$.

**Geometry**

Sectional curvatures $K < 0 \Rightarrow \emptyset$ Anosov $\Rightarrow h_{top} > 0$

$K > 0$ not clear.
If the geod. flow $\phi_t$ contains a "horseshoe"

$\Rightarrow h_{\text{top}}(g) > 0$.

$\exists$ hyperbolic periodic orbit
with transversal homoclinic point.

$\exists$ horseshoe.

$R^2(M) := C^\infty$ riemannian metrics on $M$
with the $C^2$ topology

**THEOREM**

$\dim M \geq 2$,

$\exists U \subset R^2(M)$ open and dense s.t.

$g \in U \Rightarrow \phi_t g$ has a horseshoe.

*Previous Work:*
- Proved for $\dim M = 2$: Paternain & C. JDC 2002
- $\dim M = 2$ & $C^\infty$ topology: Knieper & Weiss JDG 2002

**Application:**

A. Delshams, R. de la Hare, T. Seara:

Initial system that allows Arnold's diffusion
by perturbation with generic non-autonomous potentials.

mp_arc
Comparison with other systems:

1. General Hamiltonian Systems

Newhouse: \((M^{2n}, \omega)\) closed symplectic manifold
\[ \exists \mathcal{R} \subset C^2(M, \mathbb{R}) \] residual s.t.

\[ \mathcal{R} \Rightarrow \text{Hamiltonian flow of } H \]
\[ \text{Has a generic } 1\text{-elliptic periodic orbit} \]

1-elliptic = \(2\) (elliptic) eigenvalues of modulus 1
1 eigenvalue \(\lambda=1\) (direct. of Ham. vect. field)
1 eigenvalue \(\lambda=1\) (TH direct. to energy level)
2n-4 hyperbolic eigenvalues.

In this case:

Poincaré map restricted to energy level
is Twist map \(\times\) normally hyperbolic.

\[ \Rightarrow \text{homoclinic orbits} \]
Newhouse Thm uses the closing lemma.

**Closing lemma** is not known for geodesic flows.

*Reason:* Proof uses local perturbations.

Perturbations of Riemannian metrics \( g_i^{(x)} \) are never local in phase space \( = SM \).

"The orbit to close could have passed through the cylinder before coming back."

Newhouse theorem for geodesic flows is only known for \( M = S^2 \) or \( \mathbb{RP}^2 \).

*General Finsler Metrics*

\[ = \text{norm } \|l_x \| \text{ on tangent spaces } T_x M \]

*Unit sphere does not need to be symmetric (or a level set of a quadratic form)*

- Closing Lemma holds
- Newhouse theorem should hold.
INGREDIENTS OF THE PROOF

1. Kupka-Smale Theorem (for Geod. Flows)

\[ M^{n+1} \]

\[ J^k_s(n) = \{ k\text{-Jets of symplectic } \text{diffeos } f: (\mathbb{R}^{2n}, 0) \rightarrow \mathbb{R}^{2n} \} \]

\[ Q \subset J^k_s(n) \text{ is invariant iff} \]

\[ \forall Q = Q^{-1} = Q \quad \forall \sigma \in J^k_s(n) \]

\[ \mathcal{R}^r(M) = \text{C}^\infty \text{ riem. metrics on } M \text{ with } C^r \text{ topology} \]

Theorem

If \( Q \subset J^k_s(n) \) is open, dense and invariant

\[ \Rightarrow \forall r > k + 1 \exists \mathcal{G} \subset \mathcal{R}^r(M) \text{ residual s.t.} \]

(a) [Anosov, Klingenberg-Takens]

Poincaré maps of all periodic orbits of \( \varnothing \) are in \( Q \).

(b) All heteroclinic intersections are transversal.

OBS:

(a) Also holds for \( Q \) residual and invariant.

(b) Donnay for \( n=2 \), Petroll \( n > 2 \) show how to perturb a single non-transverse intersection. But perhaps this is not enough.
Simple Proof of (b) \[ \mathcal{M} \text{ inter.} \]:

- \( W^s \) is Lagrangian in \((T^*M, \omega_0)\)
- Choose place where is locally a Lagrangian graph
- Deform to another Lagrangian graph (by adding a \( df \))
  \[ \omega_0 = dp \wedge dx \text{ fixed canonical sympl. form.} \]
- Change metric s.t.
  \[ H(\text{new } W^s) = 1 \]

\[ \Rightarrow \text{New } W^s \text{ is invariant.} \]

2. **Elliptic Fixed Points**

Symplectic Diffeomorphism \( F : (\mathbb{R}^{2n}, 0) \to \) will be Poincaré map of closed orbit.

Elliptic periodic point := non-hyperbolic.

If \( q \)-elliptic \( \Rightarrow \exists 2q \)-dim. central manifold which is normally hyperbolic.
We choose $Q \subset J^3_s(n)$. 3-Jets of sympl. $C^\infty$ diffeos $F : (T\mathbb{R}^{2n}, 0) \rightarrow$ s.t. map restricted to central manifold is "weakly monotonous" twist map.

i.e. (a) Elliptic eigenvalues $\rho_1, \ldots, \rho_q, \bar{\rho}_1, \ldots, \bar{\rho}_q$ are 4-elementary:

$$1 \leq \sum_{i=1}^q |\nu_i| \leq 4 \Rightarrow \prod_{i=1}^q \rho_{\nu_i} \neq 1.$$

(b) Birkhoff normal form

$$z_k = e^{2\pi i \phi_k} + f_k(z)$$

$$\phi_k = a_k + \sum_{l=1}^{q_k} b_{kl} |z|^{2l}$$

satisfies $\det [b_{kl}] \neq 0$.

Using techniques of Moser, Herman, M.C. Arnaud

**Theorem:** If $F : (T\mathbb{R}^{2n}, 0) \rightarrow$ germ of sympl. diffeo s.t. (a) $F$ is $Q$-Kupka-Smale.

(b) 0 is elliptic fixed point.

$\Rightarrow$ $F$ has a 1-elliptic periodic point.

In particular, $F$ has a $\mathbb{T}$ homoclinic orbit.
3. **Rademacher Theorem**

\[ \exists \mathcal{D} \subseteq C^\infty(M) \text{ Residual set s.t.} \]
\[ g \in \mathcal{D} \Rightarrow (M, g) \text{ has infinitely many prime closed geodesics.} \]

Moreover, one can take
\[ \mathcal{D} = \text{bumpy metrics} = \text{eigenvalues of Poincaré maps are not roots of 1}. \]

4. **Theory of Dominated Splittings** [Mañé]

"If one cannot perturb in \( C^2 \) topology to create an elliptic periodic orbit, then closure of hyperbolic per. orbits is uniformly hyperbolic."

\[ \Rightarrow (\text{spectral decomposition}) \text{ contains a horseshoe} \]
Theory of Dominated Splittings

\( \text{Sp}(n) := \text{symplectic linear isom. of } \mathbb{R}^{2n} \)

sequence \( \varpi : \mathbb{Z} \to \text{Sp}(n) \) is periodic if
\[ \exists m \quad \varpi_{i+m} = \varpi_i \quad \forall i \in \mathbb{Z} \]

A Periodic sequence \( \varpi \) is hyperbolic if
\[ \prod_{i=1}^{m} \varpi_i \text{ is hyperbolic.} \]

Family of periodic sequences \( \varpi = \{ \varpi_i \}_{i \in \mathbb{Z}} \)

is bounded if \( \exists B > 0 \quad \| \varpi_i \| < B \quad \forall i \in \mathbb{Z}, \forall \varpi \in \mathbb{R} \)

is hyperbolic if \( \varpi^x \) is hyp. \( \forall \varpi \in \mathbb{R} \).

Families \( \varpi = \{ \varpi_i \}_{i \in \mathbb{Z}} \), \( \eta = \{ \eta_i \}_{i \in \mathbb{Z}} \)

are periodically equivalent iff \( \forall \varpi \in \mathbb{R} \) \( \varpi^x, \eta^x \) have same periods.

Families \( \varpi, \eta \) period. equiv. define
\[ \| \varpi - \eta \| := \sup \{ \| \varpi^x_n - \eta^x_n \| : \varpi \in \mathbb{R}, n \in \mathbb{Z} \} \]

This determines how to perturb:
up to a fixed amount in each
time \( 1 - \text{Poincare' map} \).

\[ \Rightarrow \text{Following theorem would be useful only in } C^1 \text{-topology of flow} \]
\[ = C^2 \text{-topology of metric (or Hamiltonian)} \]
Family $\mathcal{F}$ is \underline{stably hyperbolic} iff
\[ \exists \epsilon > 0 \text{ s.t. } \forall \mathcal{F}, \mathcal{F}' \text{ family period. equiv. to } \mathcal{F} \]
\[ \| \mathcal{F} - \mathcal{F}' \| < \epsilon \implies \mathcal{F} \text{ is hyperbolic.} \]

Family $\mathcal{F}$ is \underline{uniformly hyperbolic} iff
\[ \exists M > 0 \text{ s.t.} \]
\[ \| \prod_{i=0}^{M} \frac{1}{2^{i+j}} 1_{E^s_i(2^j)} \| < \frac{1}{2}, \| (\prod_{i=0}^{M} \frac{1}{2^{i+j}} 1_{E^u_i(2^j)})^{-1} \| < \frac{1}{2} \]
\[ \forall \mathcal{F} \in \mathcal{A}, \forall j \in \mathbb{Z}. \]

\underline{Theorem}

$\mathcal{F}$ \underline{Bounded periodic family}

is \underline{stably hyperbolic}

$\implies$ $\mathcal{F}$ \underline{uniformly hyperbolic}.

\underline{Remark:}

- Families in $\text{Sp}(n)$: stably hyp $\implies$ unif. hyp.
- Families in $\text{GL}(\mathbb{R}^n)$: stably hyp $\implies$ dominated splitting

i.e.,
\[ \| \prod_{i=0}^{M} \frac{1}{2^{i+j}} 1_{E^s} \| - \| (\prod_{i=0}^{M} \frac{1}{2^{i+j}} 1_{E^u})^{-1} \| < \frac{1}{2} \]
Perturbation Lemma: "Franks Lemma"

Example: Statement for Diffeos $f: M \to M$.

$\exists \varepsilon_0 > 0 \, \forall \varepsilon \in [0, \varepsilon_0] \, \exists \delta > 0 \, \text{s.t.}$ if

$F = \{ x_1, \ldots, x_N \} \subset M$ any finite set

$U$ any neighbourhood of $F$

$A_i \in L(T_{x_i}M, T_{f(x_i)}M)$ "candidate for $Df$"

$\| Df(x_i) - A_i \| < \varepsilon$

$\Rightarrow \exists g \in \text{Diff}(M) \, \text{s.t.}$

$g|_{M \setminus U} = f|_{M \setminus U}$

$g(x_i) = f(x_i) \, \forall x_i \in F$

$Dg(x_i) = A_i$

$\| f - g \|_{C^1} < \delta$

Example: $\varepsilon$ dimension 1

$U$ arbitrary small.
Analogous for geodesic flows:

realize any perturbation in $Sp(n)$ of a fixed distance of the derivative of the Poincaré map of any geodesic segment of length 1

- fixing the geodesic
- with support in an arbitrarily narrow strip $U$
- Outside small neighborhood of given finitely many transversal segments

By a metric which is $C^2$ close.

The perturbation is done on nbhd of one point.

Following result allowed to pass from dim 2 to dim $n > 2$.

Theorem

$\exists \gamma \in C^{\infty}(M)$ residual s.t.

$\forall g \in \gamma \forall \theta \in \mathbb{S}^M \exists t_0 \in [0, \frac{1}{2}]$

s.t. sectional curvature matrix

$K_{ij}(\theta) = \langle R(\theta, e_i)\theta, e_j \rangle$

has no repeated eigenvalues.
The Perturbation Lemma

Derivative of the geodesic flow
\[ d\varphi_t(J(0), \dot{J}(0)) = (J(t), \dot{J}(t)) \]
\[ J(t) = \text{Jacobi field orthogonal to the geodesic } \gamma = \pi_0 \varphi_t(\theta). \]

Jacobi Equation:
\[ \ddot{J} + K(t) J = 0 \]
\[ K(u,v) = \langle R(u,\dot{\gamma})v, \dot{\gamma} \rangle \]

1. Can change the Jacobi eq. at will

Use Fermi coordinates:
\[ e_0 = \dot{\gamma}, e_1, \ldots, e_n = \text{parallel transp. of orthonormal basis along } \gamma. \]

\[ F(t = x_0, x_1, \ldots, x_n) = \exp \left( \sum_{i=1}^{n} x_i e_i(t) \right) \]
\[ \text{exp for a fixed initial metric } g_0 \]
Our general perturbation of the metric \( g^0 \) is:

\[
\mathcal{g}_{00}(t,x) = \left[ g^0(t,x) \right]_{00} + \sum_{i,j=1}^{n} \mathcal{a}_{ij}(t,x) x_i x_j
\]

\[
\mathcal{g}_{ij}(t,x) = \left[ g^0(t,x) \right]_{ij} \quad \text{if } (i,j) \neq (0,0)
\]

This perturbation:

1. Preserves the geodesic \( \gamma \).
2. Preserves the metric along \( \gamma \).
   (orthogonal vector fields along \( \gamma \) are still \( \perp \))
3. Changes the curvature along \( \gamma \) by
   \[
   K(t) = K_0(t) - \alpha(t,x)
   \]
4. If the perturbation term is
   \[
   \gamma \ast \alpha \ast \gamma = \gamma(x) \ast \gamma P(t) \ast \gamma
   \]
   \( \leftarrow \) bump function in \( x_0, \ldots, x_n \)
   and \( \text{supp}(\gamma) \) is sufficiently small

\[
\Rightarrow \quad \| \gamma \ast \alpha \ast \gamma \|_{C^2} \sim \| P(t) \|_{C^0}
\]
We will use

\[ x^* d x = h(t) \varphi(x) x^* I(t) x \]

\[ \varphi(x) = \text{bump function in } x_1, \ldots, x_n \]

\[ h(t) = \text{approximation of characteristic function of } [0,1] \setminus F'(X) \]

i.e. \( 0 \leq h(t) \leq 1 \), \( \text{supp}(h) \cap \{ \text{intersecting points} \} = \emptyset \)

\[ \int_0^1 h \geq 1 - \epsilon \]

[only \( \|h\|_C^0 \leq 1 \) counts if \( \text{supp}(\varphi) \) small]

\[ I(t) = a \delta(t) + b \delta'(t) + c \delta''(t) + d \delta'''(t) \]

\[ a, b, c, d \in \text{Sym}(n \times n) = S(n) \quad \text{d}_{ii} = 0 \quad S^*(n) \]

\( \delta(t) = \text{approximation of Dirac } \delta \text{ at some point } \xi \text{ near } \frac{1}{2} \text{ where } K(t) \text{ has no repeated eigenvalues:} \)

\[ \min_{i \neq j} |\lambda_i - \lambda_j| > \gamma = \gamma(n) > 0. \]

\( \downarrow \text{nbhd of } \gamma. \)
Estimate the perturbation in the solutions of the Jacobi equation.

\[ \dot{J} + K(t) J = 0 \]

\[ \begin{bmatrix} J' \\ \dot{J} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -K & 0 \end{bmatrix} \begin{bmatrix} J \\ \dot{J} \end{bmatrix} \]

\[ X = AX, \quad X \in \mathbb{R}^{n \times n} \]

\[ X(0) = I \Rightarrow X(t) = e^{\lambda t} \]

**OBS:**
- Can only perturb on $K$ not on whole matrix $A$
- Only perturbations $K \mapsto K + \alpha$

because it was $X^* \alpha X$

The solutions $X$ are symplectic linear maps

$Sp(n) = \{ X \in \mathbb{R}^{n \times n} \mid X^* J X = J \}$, \quad $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$

$T_x Sp(n) = \{ x \mid y \mid y^* J + J y = 0 \}$

$= \{ x y \mid y = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}, \alpha, \beta \text{ symmetric} \}$

$\beta$ arbitrary
Strategy:

Think on 1-parameter family of metrics $s \mapsto g_s$

\[ s \mapsto K_s(t) = K(t) + s \alpha(t) \]
\[ s \mapsto X_s(t) = d\phi^s \]

$\alpha(t) = \alpha(t, E)$  \hspace{1cm} $E = (a, b, c, d) \in S(n) \times S^*(n)$

same dim as $T_xSp(n)$

Take the derivative

\[ Z_s = \frac{dX_s}{ds} = \frac{d}{ds} (d\phi^s) \]

Prove that

\[ \|Z_s(u)\| \geq k \|E\|_1 \approx k \|x^*x\|_1 c^2 \]

with $k = k(u)$

uniform for every geodesic segment of length $1$ and $\forall E \in U$

$\Rightarrow \{d\phi^s | geU\}$ covers a neighbourhood of the original linearized Poincaré map $d\phi^0$ of size depending only on the $C^2$ norm of the perturbation.
Derivative of the Jacobi equation

\[ \dot{X}_s = A_s X_s \]

\[ Z = \frac{dX_s}{ds}, \quad A_s = A + sB, \quad B = \begin{bmatrix} 0 & 0 \\ I(t) & 0 \end{bmatrix} \]

\[ \dot{Z} = AZ + B \]

"variation of parameters": \[ Z = XY \]

\[ X \dot{Y} = BX \]

\[ Y(t) = \int_0^t X^{-1}BX \]

\[ Z(t) = X(t) \int_0^t X^{-1}B(t)X \, dt \]

Integrating by parts:

\[ \int_0^t X^{-1}B(t)X \, dt \approx \]

\[ \approx X_2^{-1} \left[ \begin{bmatrix} a & b \\ -b & -\rho \end{bmatrix} + \begin{bmatrix} -2c & -Kd-3dK \\ 0 & \rho \end{bmatrix} \right] X_2 \]

To solve

\[ b - (Kd + 3dK) = \rho \]

is equiv. to solve \[ Ke - eK = f \]

may not have solution unless \( K \) has no repeated eigenvectors.
A generic condition on the curvature

**Theorem**

\[ \exists \phi \in C^\infty(M) \text{ residual s.t. } \forall \phi \in \mathcal{E} \forall \theta \in M \exists \varepsilon \in [0, \varepsilon] \text{ s.t. the Jacobi matrix } \]

\[ K_{ij}(\theta_\varepsilon) = \langle R(\theta_\varepsilon, e_i) \theta_\varepsilon, e_j \rangle \]

has no repeated eigenvalue.

Why need this and not just a preliminary perturbation?

preliminary perturb.
to separate the eigenvalues

\[ g_0 \rightarrow \]

Franks lemma
depends on amount of separation of e.v.'s

\[ g_1 \]

\[ \partial \circ g_0 \]

\[ \partial \circ g_1 \]

Franks Lemma on \( g_1 \)
Strategy: Use a transversality argument.

Know: can perturb Jacobi matrix (curvature) at will.

\[ \Sigma = \{ A \in \text{Sym}(n \times n) \mid A \text{ has repeated } \text{eigenvalues} \} \]

it is an algebraic set with singularities

\[ A \in \Sigma \iff \det \ p_A(A) = \prod (\lambda_i - \lambda_j)^2 = 0. \]

Enough to show that

geodesic vector field "crosses \( \Sigma \) transversally."

Example:

- Flow in \( \mathbb{R}^2 \) without sing.
- \( \Sigma = S' \)
- Can ask that a chosen orbit segment is \( \not\subset S' \) but not all.
- Can ask that tangency is not of order 2.
If $\Sigma$ were a smooth manifold:

$$J^k \Sigma = k\text{-jets of curves inside } \Sigma.$$  
$$\dim J^k \Sigma = (k+1) \dim \Sigma$$  

coeffs Taylor series $t \mapsto a_0 + a_1 t + \ldots + a_k t^k$ in local chart $a_i \in \mathbb{R}^\sigma, \sigma = \dim \Sigma$  

$$\dim J^k S(n) = (k+1) \dim S(n)$$  
$$\operatorname{codim} S(n) \Sigma = r \geq 1$$  
$$\operatorname{codim} J^k S(n) \quad J^k \Sigma = (k+1)r \to \infty \quad \text{when } k \to \infty$$

$$F: \mathcal{C}^\infty(M) \times SM \times J_0, \Omega \to J^k S(n)$$  
$$(g, \theta, \varepsilon) \mapsto J^k \kappa(g, \theta)$$

If $F \in J^k \Sigma$  

$\Rightarrow \exists$ residual $g \in \mathcal{C}^\infty(M)$ s.t. 

$g \in \hat{Y} \Rightarrow F(g, \ldots, t) \in J^k \Sigma.$
\[ k \text{ large} \Rightarrow \text{codim } J^k \Sigma > \dim (SM \times [0,1]) \]
\[ \tilde{\psi} \Rightarrow \text{no intersection} \]
\[ + \text{ compactness argument} \Rightarrow \text{required bounds} \]
\[ \text{on eigenvalues} \]
\[ \text{use } \min_{\theta \in SM} \max_{t \in [0,1]} \prod |\lambda_i - \lambda_j|^2 > 0 \text{ when } \tilde{\psi} \]

**But \( \Sigma \) has singularities**

**Algebraic Jet space**
\[ \mathcal{J}_k(\Sigma) = \text{polynomials } a_0 + a_1 t + \ldots + a_k t^k = p(t) \]
\[ \text{s.t. } f_0 p(t) \equiv 0 \text{ (mod } t^{k+1}) \]

**Arc space**
\[ \mathcal{L}_\infty(\Sigma) = \text{formal power series } p(t) \]
\[ \text{s.t. } f_0 p \equiv 0 \]

\[ \Pi_k : \mathcal{L}_\infty(\Sigma) \rightarrow \mathcal{J}_k(\Sigma) \text{ truncation} \]
$Z_k(\Sigma)$ is an algebraic variety.

$\Pi_k(Z_{\infty}(\Sigma)) \subset Z_k(\Sigma)$ is a finite union of algebraic subsets. (it is "constructible")

$J^k\Sigma = k$-jets of $C^\infty$ curves in $\Sigma$

$\Rightarrow J^k\Sigma \subset \Pi_k(Z_{\infty}(\Sigma)) \subset Z_k(\Sigma)$.

Denef & Loeser:

$\dim \Pi_k(Z_{\infty}(\Sigma)) \leq (k+1) \dim \Sigma$.

(same bound as in smooth case).