EXERCISE SESSION V

Exercise I
Let $\mathcal{B}$ be the family of subsets of $\mathbb{R}$ defined as

$$\mathcal{B} := \{[a, \infty) \ ; \ a \in \mathbb{R}\}.$$

Prove that $\mathcal{B}$ is a basis of topology. We call by *lower topology* the topology generated by $\mathcal{B}$, and we denote it by τ_{low} . Compare τ_{low} with the Euclidean topology.

Exercise II
Let $\mathcal{B}$ be the family of subsets of $\mathbb{R}$ defined as

$$\mathcal{B} := \{[a, b) \ ; \ a, b \in \mathbb{R} \ \text{ with } a < b\}.$$

Prove that $\mathcal{B}$ is a basis of topology. We call by *lower limit topology* the topology generated by $\mathcal{B}$, and we denote it by τ_{lowlim} . Compare τ_{lowlim} with the Euclidean topology.

Exercise III
Compare the topologies τ_{low} and τ_{lowmin}.

Exercise IV
Let τ_0 be the family of subsets of $\mathbb{R}$ defined as

$$\tau_0 := \{\emptyset, \mathbb{R}\} \cup \{(a, \infty) \ ; \ a \in \mathbb{R}\}.$$

Recall from previous sessions we proved that τ_0 is a topology. Answer and justify the following questions:

- Is τ_0 finer than τ_{low}? Is τ_{low} finer than τ_0?
- Is τ_0 finer than τ_E? Is τ_E finer than τ_0?
- Is τ_0 finer than τ_{lowlim}? Is τ_{lowmin} finer than τ_0?

Comments:
With the notion of basis, you can redo the following exercises from Session II easier.
Exercise V
Let $d : \mathbb{R}^2 \to [0, \infty)$ be distance

$$d((x_1, y_1), (x_2, y_2)) := \max\{|x_2 - x_1|, |y_2 - y_1|\}.$$

- Prove that the topology associated with d is the Euclidean topology on $\mathbb{R}^2$.

Exercise VI
Let $d : X \times X \to \mathbb{R}$ be a distance on X. We consider the distance $\overline{d} : X \times X \to \mathbb{R} : (x, y) \mapsto \min\{d(x, y), 1\}$.

- Prove that the topologies associated with $\overline{d}$ and d coincide.