EXERCISE SESSION II

Recalling the notion of "neighborhood": Let X be a set endowed with a topology τ. We say that a set $A \subset X$ is a neighborhood of a given point $x \in X$ if there exists an open set $U_x \in \tau$ such that $x \in U_x \subset A$.

Exercise I
Prove *Lemma 24* in your lecture notes, which states the following

Lemma 0.1. Let X be a set endowed with a topology τ. Prove that $\Omega \subset X$ is an open set if and only if Ω is a neighborhood of x, for each $x \in \Omega$.

Exercise II
Prove *Proposition 28* in your lecture notes, which states the following

Proposition 0.2. Let X be a set and $\tau \subset 2^X$ a collection of subsets of X. Denote the associated set of closed sets by Γ. Then,

(a) $\emptyset, X \in \Gamma$
(b) For every $A_1, \dots, A_n \in \Gamma$, we have that $\bigcup_{k=1}^n A_k \in \Gamma$
(c) If $A_k \in \Gamma$ for every $k \in I$, we have that $\bigcap_{k \in I} A_k \in \Gamma$.

Reciprocally, if $\Gamma \subset 2^X$ is an arbitrary collection of subsets of X satisfying conditions $(a), (b)$ and (c), then the set τ consisting of the complements of elements in Γ is a topology.

Exercise III
Let (X, τ_X) and (Y, τ_Y) be topological spaces. Describe all the continuous functions $f : X \to Y$ in the following instances

(1) τ_X is the discrete topology
(2) τ_Y is the trivial topology
(3) τ_X is the trivial topology and τ_Y is the discrete topology.

Exercise IV
Consider the case where $X := \mathbb{R}^d$, $d \geq 1$ is endowed with the Euclidean topology (once you see the phrase "Euclidean topology", you are allowed to use all of your intuition from Calc I, which is great news!). Respond the following

(1) For the case $X = \mathbb{R}$, is it true that $[0, 2]$ is a neighborhood of 1, according to our definition? Is $[1, 2]$ a neighborhood of 1? Justify your answer.

Exercise V
Let $d : \mathbb{R}^2 \to [0, \infty)$ be the map defined by

$$d((x_1, y_1), (x_2, y_2)) := \max\{|x_2 - x_1|, |y_2 - y_1|\}.$$

(1) Prove that d is a distance on $\mathbb{R}^2$. The distance d is known as the Chebyshev distance.
(2) Give a geometric characterization of the d-ball centered at (x_0, y_0) with radius $r > 0$.
(3) Prove that the topology associated with d is the euclidean topology on $\mathbb{R}^2$.

Exercise VI
Let $d : X \times X \to \mathbb{R}$ be a distance on X. We consider the map $\overline{d} : X \times X \to \mathbb{R} : (x, y) \mapsto \min\{d(x, y), 1\}$.

(1) Prove that $\overline{d}$ is a distance on X.
(2) Prove that the topologies associated with $\overline{d}$ and d coincide.