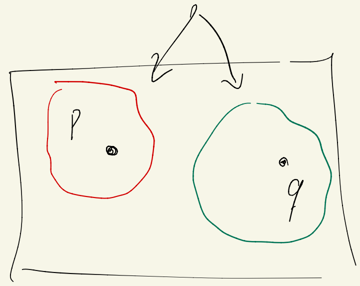
Exercise I:

$f, g: X \to Y$ Assume

i) Y is Hausdorff: $\longleftrightarrow$ open & disjoint. if $p \neq q$.



ii) $f(x) = g(x) \quad \forall \quad x \in D$,

where D is dense in X.

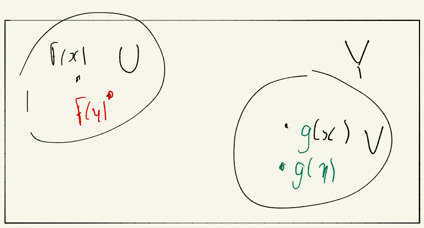
iii) f and g are continuous.

proof:

If $f(x) \neq g(x)$, then $\exists \quad U, V \in \tau_Y$ s.t. $f(x) \in U$, $g(x) \in V$ (*)

& $U \cap V = \emptyset$. Then denote by $A = f^{-1}(U) \cap g^{-1}(V)$. Since f, g are continuous, then A is open.

Note: $\forall \quad y \in A$ is such that $f(y) \in U$ and $g(y) \in V$, but by (*)



$\Rightarrow \quad f(y) \neq g(y). \quad \forall \quad y \in A \leftarrow$ open.

this contradicts the fact that they coincide in a dense set.

To justify this claim:

case I): if X only has one point, then $X = D$, and there is nothing to prove

case I): If X has at least 2 points, then A meets D (since D is dense & A is open), then $\exists \quad p \in A \cap D$, but then

i) $f(p) = g(p)$ since f and g coincide in D ii) $f(p) \neq g(p)$ since $p \in A$. ∇

Exercise 2:)

Let (X, τ_X) be a topological space.

$\Delta_X = \{(x,x) \,;\, x \in X\}$.

We want to show: X is Hausdorff iff. Δ_X is a closed subset of $X \times X$.

proof:

Let's start with $\Leftarrow$): (Δ_X closed $\Rightarrow$ Hausdorff).

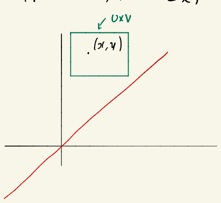
If Δ_X is closed in $X \times X$, then $(\Delta_X)^c$ is open.

Then

What we have: #

if $(x,y) \in (\Delta_X)^c$, we can find an element of the basis of $X \times X$ (recall that $\{U \times V \,;\, U, V \in \tau_X\}$ form a basis of $X \times X$), of the form $U \times V$, with $U, V \in \tau_X$ and

$$(x,y) \in U \times V \iff \begin{cases} x \in U \\ y \in V \end{cases}$$



and such that $(U \times V) \cap \Delta_X = \emptyset$. (see picture!

What we need:

Take $x, y \in X$ s.t. $x \neq y$. We want to find $A, B \in \tau_X$, s.t. $x \in A$, $y \in B$ and $A \cap B = \emptyset$.

Since $x \neq y$, then $p = (x, y) \in (\Delta_X)^c$, then, the statement # holds, namely, we can find $U, V \in \tau_X$ s.t.

$$p \in U \times V \iff \begin{cases} x \in U \\ y \in V \end{cases} \quad \text{and} \quad (U \times V) \cap \Delta_X = \emptyset.$$

Claim: $U \cap V = \emptyset$.

Reason: if not, then $\exists\ a \in U \cap V \Rightarrow p = (a, a) \in \Delta_X$

but $a \in U \cap V \Rightarrow (a, a) \in U \times V$

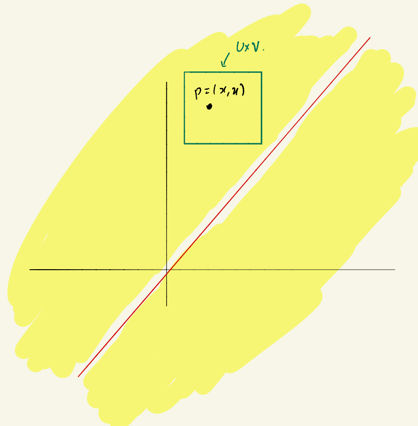
$$\Rightarrow \begin{cases} (a,a) \in \Delta_X \\ (a,a) \in U \times V \end{cases}$$ contradicting the fact that $(U \times V) \cap \Delta_X = \emptyset$.

Let's now prove $\Rightarrow$) (X is Hausdorff $\Rightarrow$ Δ_X is closed).

What we have:

$\forall\ x, y \in X$ s.t. $x \neq y$, then $\exists\ U, V \in \tau_X$ s.t. $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

What we need:



Δ_X closed $\Leftrightarrow$ $(\Delta_X)^c$ is open:

It suffices to show that $\forall\ p=(x,y) \in (\Delta_X)^c$ $\exists$ an element of the basis $U \times V \subset X \times X$ (with $U, V \in \tau_X$) s.t. $p=(x,y) \in U \times V$ and $U \times V \subset (\Delta_X)^c \Leftrightarrow (U \times V) \cap \Delta_X = \emptyset$ (**),

Take $p=(x,y) \in (\Delta_X)^c \Leftrightarrow x \neq y$

Hausdorff property $\Rightarrow$ $\exists\ U, V \in \tau_X$ s.t. $x \in U$, $y \in V$ & $U \cap V = \emptyset$. (*)

Claim: these U, V are such that

i) $p \in U \times V$ (since $x \in U$ & $y \in V$)

ii) $(U \times V) \cap \Delta_X = \emptyset$ Reason: if not, then $\exists\ (a,a) \in \Delta_X$ s.t. $(a,a) \in U \times V \Rightarrow a \in U \cap V$, which contradicts (*)

By i) & ii), and the discussion at (**) $(\Delta_X)^c$ is open.

Comment: From Jean-Paul:

Using exercise II to prove I:

idea:

define $\Psi : X \longrightarrow Y \times Y$

$x \longmapsto (f(x), g(x))$.

$\Psi(D) \subset \Delta_X$ when $f = g$ in D.

By continuity, $\Psi(X) = \Psi(\bar{D}) \subset \bar{\Delta}_X$ ⊕

Since X is Hausdorff, by Ex II, Δ_X is closed $\Rightarrow \bar{\Delta}_X = \Delta_X$

Thus, by ⊕, $\Psi(X) \subset \Delta_X \Rightarrow f(x) = g(x) \;\; \forall \;\; x \in X$.

Exercise III

Prove that if $f : S^1 \to \mathbb{R}$ (unit circle) is continuous, then $\exists \; p \in S^1$ s.t.

$f(p) = f(-p)$

proof:

(hint: if $f(p) \neq f(-p) \;\; \forall \; p \in S^1$,

we consider the function $g : S^1 \to \mathbb{R}$.

$$p \longmapsto \frac{f(p) - f(-p)}{|f(p) - f(-p)|} = \text{sign}(f(p) - f(-p)) \,)$$

If $f(p) \neq f(-p) \;\; \forall \; p \in S^1$, then

the mapping g is well defined and furthermore,

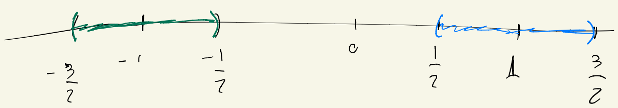
since $\alpha \mapsto \frac{\alpha}{|\alpha|} = \text{sign}(\alpha)$ is continuous in the domain $\mathbb{R} \setminus \{0\}$.

So g is continuous.

Then, in particular,

i) $g^{-1}((\tfrac{1}{2}, \tfrac{3}{2})) = g^{-1}(\{1\}) = U$ is open. in S^1

ii) $g^{-1}((-\tfrac{3}{2}, -\tfrac{1}{2})) = g^{-1}(\{-1\}) = V$ is open in S^1



Furthermore, $U \cup V = S^1$ and $U \cap V = \emptyset$

Since S^1 is connected, then, either $U = \emptyset$ or $V = \emptyset$.

$\Rightarrow g$ is constant.

From here we conclude that either

i) $f(p) > f(-p)\ \forall\ p \in S'$ or ii) $f(p) < f(-p)\ \forall\ p \in S'$.

If $f(p) > f(-p)\ \forall\ p \in S'$, we have that by choosing $p = 1$, $f(1) > f(-1)$, but by choosing $p = -1$, we get $f(-1) > f(1)$, which is a contradiction.

A similar conclusion holds when $f(p) < f(-p)\ \forall\ p \in S'$.

Therefore, the assumption $f(p) \neq f(-p)\ \forall\ p$ doesn't hold, as required