Exercise I:

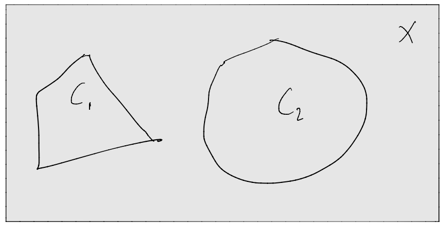
Let $f : X \to Y$ be an $\overset{\text{surjective}}{\text{onto}}$ continuous function. Prove that

"if X is normal, then Y is normal".

Recall:

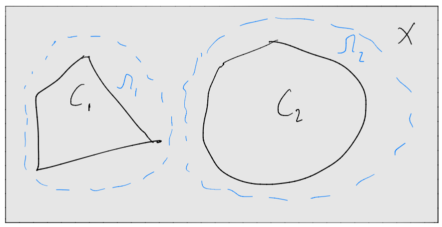
(Z, τ) ~ topological space. We say that Z is normal if.

every time we have $C_1, C_2 \subset Z$ closed sets which are disjoint. $\longleftrightarrow$



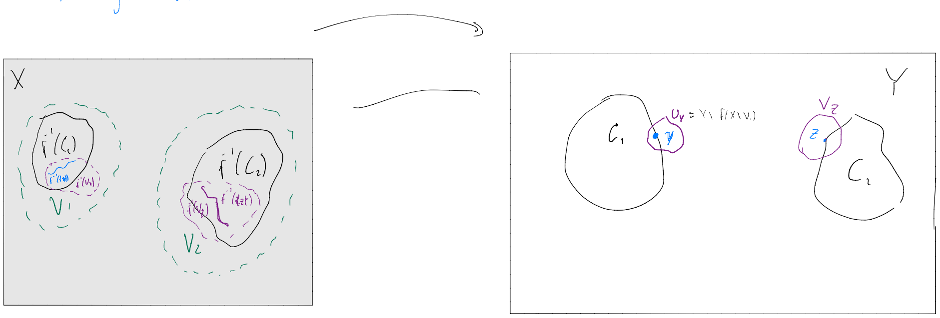
there exist open sets $\Omega_1, \Omega_2 \in \tau$ s.t.

- $\Omega_1 \cap \Omega_2 = \emptyset$
- $\begin{cases} C_1 \subset \Omega_1 \\ C_2 \subset \Omega_2 \end{cases}$ $\longleftrightarrow$



proof:

Exploratory guesses: f continuous, closed and onto.



We want: that for every closed sets C_1 & C_2, $\exists\ W_1, W_2 \in \tau_Y$

s.t. $\begin{cases} W_1 \cap W_2 = \emptyset \\ C_1 \subset W_1 \quad \& \quad C_2 \subset W_2 \end{cases}$

Consider $K_1 := f^{-1}(C_1)$ $\quad K_2 := f^{-1}(C_2)$.

Notice that K_1, K_2 are
- closed sets of X.
- $K_1 \cap K_2 = f^{-1}(C_1 \cap C_2) = \emptyset$

Since X is normal, there are $V_1, V_2 \in \tau_X$ s.t. $V_1 \cap V_2 = \emptyset$ and $K_1 \subset V_1$ & $K_2 \subset V_2$

Take $y \in C_1$ and consider $f^{-1}(\{y\})$, which is closed by continuity of f. and satisfies

$f^{-1}(\{y\}) \subset V_1 \implies X \setminus V_1 \subset X \setminus f^{-1}(\{y\})$

$\implies f(X \setminus V_1) \subset f(X \setminus f^{-1}(\{y\})) \overset{\text{claim}}{\subset} Y \setminus \{y\}. \implies \{y\} \subset Y \setminus f(X \setminus V_1)$

$\implies y \in Y \setminus f(X \setminus V_1)$

Reason:
$z \in f(X \setminus f^{-1}(\{y\}))$, then $z = f(a)$, with $a \in X \setminus f^{-1}(\{y\})$
$\implies f(a) \notin \{y\} \implies f(a) \neq y \implies z \neq y \implies z \in Y \setminus \{y\}$

We have proved that

- $y \in Y \setminus f(X \setminus V_1)$

Claim II: $Y \setminus f(X \setminus V_1)$ is open:

Reason:

$V_1 \in \mathcal{T}_X \Rightarrow X \setminus V_1$ is closed $\xRightarrow{f \text{ closed map}} f(X \setminus V_1)$ is closed.

$\Rightarrow Y \setminus f(X \setminus V_1)$ is open

Therefore:

- $y \in Y \setminus f(X \setminus V_1)$
- $Y \setminus f(X \setminus V_1)$ is open:

$\Rightarrow \quad C_1 \subset Y \setminus f(X \setminus V_1)$

By the same reason,

$$C_2 \subset Y \setminus f(X \setminus V_2).$$

Define

$W_1 := Y \setminus f(X \setminus V_1)$

$W_2 := Y \setminus f(X \setminus V_2)$

open.

Claim: $W_1 \cap W_2 = \phi$:

If $w \in W_1 \cap W_2 = f(X \setminus V_1)^c \cap f(X \setminus V_2)^c$

$\Rightarrow \begin{cases} w \neq f(a) \quad \forall \quad a \in X \setminus V_1 \\ w \neq f(b) \quad \forall \quad b \in X \setminus V_2 \end{cases}$ (#).

but the surjectivity $\Rightarrow$ $w = f(c)$ for some $c \in X$.

Since $V_1 \cap V_2 = \phi$, then $(X \setminus V_1) \cup (X \setminus V_2) = X$

$\Rightarrow \begin{cases} c \in X \setminus V_1 \\ \text{or} \\ c \in X \setminus V_2 \end{cases}$ which contradicts (#)

This proves that $W_1 \cap W_2 = \phi$, as required.

Note: We used the fact that $\{y\}$ is closed, so we need Y to be Hausdorff.

Exercise II:

Let $f, g : X \to Y$ two continuous maps, with

. Y Hausdorff.

Prove that $C = \{ x \in X \,;\, f(x) = g(x) \}$ forms a closed subspace of X.

proof:

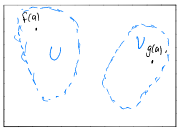
Lets show that $X \setminus C$ is open:

take $a \in X \setminus C$. $\Leftrightarrow$ $f(a) \neq g(a)$.

by the Hausdorff property, $\exists$ $U, V \in \tau_Y$ s.t.

. $U \cap V = \emptyset$

. $f(a) \in U$ & $f(a) \in V$



We want to find $W_a \in \tau_X$

s.t. $\begin{cases} a \in W_a \\ W_a \subset X \setminus C \end{cases}$ (##)

Consider $W_a = f^{-1}(U) \cap f^{-1}(V)$ $\leftarrow$ open since U, V are open & f is continuous,

By construction, . $f(a) \in U \Rightarrow a \in f^{-1}(U)$ $\Rightarrow$ $a \in f^{-1}(U) \cap g^{-1}(V)$

. $g(a) \in V \Rightarrow a \in g^{-1}(V)$

In addition, $\forall$ $w \in W_a = f^{-1}(U) \cap g^{-1}(V)$, we have

$\begin{cases} f(w) \in U \\ g(w) \in V \end{cases} \xRightarrow[\text{since}]{U \cap V = \emptyset} f(w) \neq g(w)$

Since (##) holds $\forall$ $a \in X \setminus C$, we conclude that $X \setminus C$ is open

$\Leftrightarrow$ C is closed.

Exercise III:

Let $X = Y = \mathbb{N}$ with the cofinite topology.

i) Prove that Y is not Hausdorff.

proof:

If Y was Hausdorff, then $\forall$ $x \neq y$ $x, y \in Y$ $\exists$
$U_x \ni x$ and $V_y \ni y$, $U_x, U_y \in \tau_Y$ s.t $U_x \cap U_y = \emptyset$

$\Updownarrow$

. $Y \setminus U_x$ is finite
. $Y \setminus U_y$ is finite

$$\underbrace{(Y \setminus U_x)}_{\text{finite}} \cup \underbrace{(Y \setminus U_y)}_{\text{finite}} = Y = \underbrace{\mathbb{N}}_{\text{infinite}} \quad \nabla$$

Therefore, Y is not Hausdorff.

Part ii) of the exercise:

ii) Define $f, g : X \to Y$ as $f(x) = x$, and $g(x) = \max\{x, 5\}$.

Prove that

I) f, g are continuous.

II) $C = \{x \in X ;\ f(x) = g(x)\}$ is not closed.

proof:

. f = identity $\Rightarrow$ f is continuous.

. Proving that g is continuous:

$\Leftrightarrow$ $g^{-1}(U)$ is open $\forall$ $U \in \tau_Y$

$\Leftrightarrow$ $g^{-1}(K)$ is closed $\forall$ closed set K.

$\Leftrightarrow$ $g^{-1}(K)$ is closed $\forall$ finite set $K \subset \mathbb{N}$.

Observe that $K = \{a_1, \dots, a_n\}$ for some $a_1, \dots, a_n \in \mathbb{N}$.

$g^{-1}(K) = \{x \in X = \mathbb{N} ;\ g(x) \in \{a_1, \dots, a_n\}\}$.

$= \{x \in \mathbb{N} ;\ \max\{x, 5\} \in \{a_1, \dots, a_n\}\}$.

$\subset \{x \in \mathbb{N} ;\ x \leq \max\{a_1, \dots, a_n\}\}$.

$\Rightarrow$ $g^{-1}(K)$ is finite

$\Leftrightarrow$ $g^{-1}(K)$ is closed.

$\Rightarrow$ g is continuous.

Now let's prove

$C = \{x \in X ;\ f(x) = g(x)\}$ is not closed.

$\Leftrightarrow$

$C = \{x \in X ;\ f(x) = g(x)\}$ is infinite ⊛

Notice that

$C = \{x \in Y ;\ x \overset{\oplus}{=} \max\{x, 5\}\}$.

Notice that if $x \in \mathbb{N} \cap [6, \infty)$, then $\max\{x, 5\} = x$

$\Rightarrow$ $\oplus$ holds $\Rightarrow$ $C \supset \mathbb{N} \cap [6, \infty)$ ← infinite.

$\therefore$ C is infinite, which proves ⊛

So C is not closed.