Exercise I:

Prove that any continuous map from a connected space X to a space Y, endowed with the discrete topology is constant.

Solution:

Take $f: X \to Y$ continuous. Suppose that f is not constant. Then $\exists$ $a, b \in Y$ s.t. $a \neq b$ and $\mathrm{Im}(f) \supset \{a, b\}$. (i.e. $\exists$ $x_a, x_b \in X$ s.t. $f(x_a) = a$ and $f(x_b) = b$).

- $\{a\}$ is an open set $\overset{\text{continuity of } f}{\Rightarrow}$ $f^{-1}(\{a\})$ is open
- $Y \setminus \{a\}$ is open $\overset{\text{continuity of } f}{\Rightarrow}$ $f^{-1}(Y \setminus \{a\})$ is open

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Now X connected implies that if $\exists$ $V, W \in \tau_X$ s.t. $V \cap W = \emptyset$, and $X = V \cup W$, then either $V = \emptyset$ or $W = \emptyset$. (*)

Take $\begin{cases} V = f^{-1}(\{a\}) \\ W = f^{-1}(Y \setminus \{a\}) \end{cases}$ Notice that $\begin{cases} x_a \in f^{-1}(\{a\}) \\ x_b \in f^{-1}(\{b\}) \end{cases} \Rightarrow f^{-1}(\{a\}), f^{-1}(\{b\}) \neq \emptyset$.

But by connectedness of X, since

$$X = f^{-1}(\{a\}) \cup f^{-1}(Y \setminus \{a\}) = V \cup W$$

and $V, W \in \tau_X$, then either $\begin{cases} V = \emptyset \\ W = \emptyset \end{cases}$ which contradicts (*)

Therefore, f must be constant.

Exercise II

Prove that (X, τ_X) is connected iff every continuous function $f: X \to \{0,1\}$ is constant

Note: $\{0,1\}$ is endowed with the discrete topology.

Solution:

$\Rightarrow$) Done in the previous exercise

$\Leftarrow$) Assume that every continuous function $f: X \to \{0,1\}$ is constant. (#)

We want to prove that if $V, W \in \tau_X$ are s.t. $V \cap W = \emptyset$ & $V \cup W = X$, then either $V = \emptyset$ or $W = \emptyset$.

Back to Jean-Paul's idea:

Facts: (**)

Take any function $f: X \to \{0,1\}$ that is continuous.

Then

$$X = \underbrace{\left(f^{-1}(\{0\})\right)}_{} \cup \underbrace{\left(f^{-1}(\{1\})\right)}_{}$$

open since f continuous and $\{0,1\}$ has the discrete topology.

The hypothesis (#) implies that f is constant.

$$\Rightarrow \text{either} \begin{cases} f(x) = 1 \quad \forall\ x \in X \Rightarrow f^{-1}(\{0\}) = \emptyset \\ f(x) = 0 \quad \forall\ x \in X \Rightarrow f^{-1}(\{1\}) = \emptyset \end{cases}$$

In particular, either $f^{-1}(\{0\}) = \emptyset$ or $f^{-1}(\{1\}) = \emptyset$.

Next step:

Take $V, W \in \tau_X$ as before (i.e. $V \cap W = \emptyset$ & $V \cup W = X$).

Consider f to be given by $f: X \to \{0,1\}$

$$f(x) = \begin{cases} 1 & \text{if } x \in V \\ 0 & \text{if } x \in W \end{cases}$$

(Note: f is well defined because $V \cap W = \emptyset$)

Observation: $f^{-1}(\{1\}) = V$ and $f^{-1}(\{0\}) = W$.

Claim: f is continuous. Reason:

- $f^{-1}(\{0\}) = W$ open ✓
- $f^{-1}(\{1\}) = V$ open ✓
- $f^{-1}(\emptyset) = \emptyset$ open ✓
- $f^{-1}(\{0,1\}) = X$ open ✓

so f is continuous. By the fact (**),

$f^{-1}(\{0\}) = \emptyset$ or $f^{-1}(\{1\}) = \emptyset \Rightarrow W = \emptyset$ or $V = \emptyset$.

Thus, X is connected, as required.

Exercise III:

Let (X, τ_X) and (Y, τ_Y) be two topological spaces. Consider $A \subset X$ be a subspace of X and $B \subset Y$ be a subspace of Y.
Prove that the product topology on $A \times B$ coincides with the subspace topology inherited from $X \times Y$.

Solution:

- τ_1 = product topology on $A \times B$
- τ_2 = subspace topology inherited from $X \times Y$.

We need to show

$$\begin{cases} \tau_1 \subset \tau_2 \\ \tau_2 \subset \tau_1 \end{cases} \xleftarrow{\text{shortcut}} \begin{cases} \text{Basis of } \tau_1 \subset \tau_2 \\ \text{Basis of } \tau_2 \subset \tau_1 \end{cases}$$

Proving that basis of $\tau_1 \subset \tau_2$:

- Take an element z in the basis of τ_1:
  then $z = V \times W$ with $V \in$ Basis of A $\subset$ Open sets of A and $W \in$ Basis of B $\subset$ Opens of B.
- Since $V \in \tau_A$:= subspace topology in $A \subset X$, then
  $V = \tilde{V} \cap A$, with $\tilde{V}$ an open set of X.
- Similarly, $W = \tilde{W} \cap B$, with $\tilde{W}$ an open set of Y.

} facts.

We want "z is an open set of $A \times B$ regarded as a subspace of $X \times Y$"
or equivalently,
we want "$z = \square \cap (A \times B)$, where $\square$ is an open set in $X \times Y$" (##)

By the facts above,

$z = V \times W = (\tilde{V} \cap A) \times (\tilde{W} \cap B)$, (recall that $\begin{cases} \tilde{V} \text{ is open in } X \\ \tilde{W} \text{ is open in } Y \end{cases}$)

$\Rightarrow z \overset{\text{check}}{=} (\tilde{V} \times \tilde{W}) \cap (A \times B)$, which gives (##), since $\tilde{V} \times \tilde{W}$ is an open of $X \times Y$.

This finishes the proof of $\tau_1 \subset \tau_2$.

Now let's prove $\tau_2 \subset \tau_1$. $\xleftarrow{\text{shortcut}}$ Basis of $(\tau_2) \subset \tau_1$

(recall τ_2 = subspace topology inherited from $X \times Y$).

If $z \in$ Basis of τ_1, then $\exists$ $\tilde{z} \in$ Basis of $X \times Y$ s.t. $z = \tilde{z} \cap (A \times B)$. $\tilde{z}$ must be of the form $\tilde{z} = \tilde{V} \times \tilde{W}$, where $\tilde{V}$ is open in X and $\hat{W}$ is open in Y.

$\Rightarrow z = (\hat{V} \times \hat{W}) \cap (A \times B) = (\tilde{V} \cap A) \times (\tilde{W} \cap B)$

(Recall that: τ_1 = product topology on $A \times B$)

Define $V := \hat{V} \cap A$ and $W = \tilde{W} \cap B$, so that

- V is open in A & W is open in B.
  and
- $z = V \times W$

$\Rightarrow z$ is in the product topology of $A \times B = \tau_1$.

Exercise IV:
Show that the product of two Hausdorff spaces is Hausdorff.

proof:
Check proposition 168 in the notes.