High frequency statistics and local times for the fractional Brownian motion

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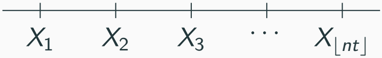

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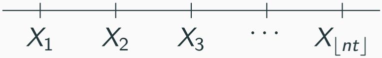

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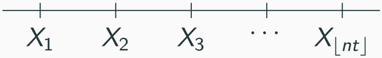

where $t > 0$. Our starting point is the study of a suitable normalization of the variables

$$\sum_{i=1}^{\lfloor nt \rfloor} f(X_i) \overset{Law}{=} \sum_{i=1}^{\lfloor nt \rfloor} f(n^H X_{\frac{i}{n}})$$

where f is some test function, with properties to be specified later.

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These type of variables are a special instance of the statistics

$$G^{(n)}_{a,t} := b_n \sum_{i=1}^{\lfloor nt \rfloor} f(n^a(X_{\frac{i}{n}} - \lambda)), \tag{1}$$

where $b_n > 0$ is a normalizing constant, $f : \mathbb{R} \to \mathbb{R}$ is a test function, $\lambda \in \mathbb{R}$ and $a > 0$.

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What will come later: The variables $G_{a,t}^{(n)}$ are closely related to the local time of X at the level λ. The fluctuations are going to be closely related to the formal derivatives of the local time of X!

Preliminaries: local times

The local time of X at $\lambda \in \mathbb{R}$, is heuristically defined by $\int_0^t \delta_0(X_s - \lambda)ds$.

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$$L_t(\lambda) := \lim_{\varepsilon \to 0} \int_0^t \phi_\varepsilon(X_s - \lambda) ds,$$

where the convergence is taken in the L^2-sense and ϕ_ε is the heat kernel with variance $\varepsilon > 0$, defined by $\phi_\varepsilon(x) := (2\pi\varepsilon)^{-\frac{1}{2}} \exp\{-\frac{1}{2\varepsilon}|x|^2\}$.

Historical context (first order approximation)

The first order approximation of our problem was solved by Jeganathan (2004), who showed that for all $H \in (0,1)$, if $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$,

$$n^{H-1} \sum_{i=1}^{\lfloor nt \rfloor} f(n^H(X_{\frac{i-1}{n}} - \lambda)) \xrightarrow{L^2(\Omega)} L_t(\lambda) \int_{\mathbb{R}} f(x)dx. \tag{2}$$

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Remark: one sees immediately that (2) implies a trivial conclusion in the case $\int_{\mathbb{R}} f(x)dx = 0$.

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A refinement of the previous result was first obtained by Jeganathan, who showed that if $1/3 < H < 1$, $\int_{\mathbb{R}} (|f(x)|^p + |xf(x)|)dx < \infty$ for $p = 1, 2, 3, 4$, and $\int_{\mathbb{R}} f(x)dx = 0$, then,

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$$n^{\frac{H-1}{2}} \sum_{i=1}^{\lfloor nt \rfloor} f(n^H(X_{\frac{i-1}{n}} - \lambda)) \xrightarrow{f.d.d.} \sqrt{b}\, W_{L_t(\lambda)}, \tag{3}$$

where $\xrightarrow{f.d.d.}$ indicates convergence in the sense of finite-dimensional distributions.

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Questions

- What happens when $H \leq \frac{1}{3}$?
- Can something be said in the non-zero energy case?

Handling the “rough case”, where H is small.

The main ingredient for handling the remaining cases of the theorems previously studied by Jeganathan and Nualart et.al. is the **spatial derivative of the local time.**

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Lemma

Let $\ell \in \mathbb{Z}$ be such that $0 < H < \frac{1}{2\ell+1}$. Then, for every $t \geq 0$ and $\lambda \in \mathbb{R}$, the random variables

$$L_{t,\varepsilon}^{(\ell)}(\lambda) = \int_0^t \delta_0^{(\ell)}(X_s - \lambda) ds := \int_0^t \phi_\varepsilon^{(\ell)}(X_s - \lambda) ds, \quad \varepsilon > 0, \qquad (4)$$

converge in L^2 to a limit $L_t^{(\ell)}(\lambda)$, as $\varepsilon \to 0$.

A more general model

One can easily prove that all functions f of the form $f = g'$ with $g, g' \in L^1(\mathbb{R})$ satisfy the property $\int_{\mathbb{R}} f(x)dx = 0$.

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For $\ell \in \mathbb{N}$ and a function g in the set $W^{\ell,1}$ of real functions with weak derivatives of order ℓ satisfying

$$\sum_{i=1}^{\ell} \int_{\mathbb{R}} |g^{(i)}(x)| dx < \infty,$$

we define

$$G_{t,\lambda,a}^{(n,\ell)}[g] := \sum_{i=2}^{\lfloor nt \rfloor} g^{(\ell)}(n^a(X_{\frac{i-1}{n}} - \lambda)).$$

Main results

Theorem (Jaramillo, Nourdin, Peccati (2019))
Assume that $H(2\ell+1) < 1$ *and* $g \in W^{1,\ell}$ *satisfies* $g^{(\ell)} \in L^2(\mathbb{R})$ *and* $(1+|x|^{1+\varepsilon})|g(x)| \in L^1(\mathbb{R})$. *Then, for all* $0 < a \leq H$,

$$n^{a(\ell+1)-1} G_{t,\lambda,a}^{(n,\ell)}[g] \xrightarrow{L^2} L_t^{(\ell)}(\lambda) \int_{\mathbb{R}} g(x)dx.$$

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Moreover, for $T_1, T_2 > 0$,

$$\sup_{t \in [T_1, T_2]} |n^{a(\ell+1)-1} G_{t,\lambda,a}^{(n,\ell)}[g] - L_t^{(\ell)}(\lambda) \int_{\mathbb{R}} g(x)dx| \xrightarrow{\mathbb{P}} 0.$$

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If $\ell \geq 1$, $(2\ell+2)H<1$ *and* $T>0$, *then*

$$\sup_{t\in[0,T]} |n^{a(\ell+1)-1} G_{t,\lambda,a}^{(n,\ell)}[g] - L_t^{(\ell)}(\lambda) \int_{\mathbb{R}} g(x)dx| \xrightarrow{\mathbb{P}} 0.$$

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$$n^a\left(n^{a(\ell+1)-1} G_{t,\lambda,a}^{(n,\ell)}[g] - L_t^{(\ell)}(\lambda)\int_{\mathbb{R}} g(x)dx\right) \xrightarrow{L^2} -L_t^{(\ell+1)}(\lambda)\int_{\mathbb{R}} xg(x)dx.$$

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If $T_1, T_2>0$, *then*

$$\sup_{t\in[T_1,T_2]}\Big|n^a\Big(n^{a(\ell+1)-1}G_{t,\lambda,a}^{(n,\ell)}[g]-L_t^{(\ell)}(\lambda)\int_{\mathbb{R}}g(x)dx\Big)+L_t^{(\ell+1)}(\lambda)\int_{\mathbb{R}}xg(x)dx\Big|\xrightarrow{\mathbb{P}}0.$$

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Moreover, if $H(2\ell+4)<1$ *and* $T>0$*, then*

$$\sup_{t\in[0,T]}\left|n^a\left(n^{a(\ell+1)-1}G_{t,\lambda,a}^{(n,\ell)}[g]-L_t^{(\ell)}(\lambda)\int_{\mathbb{R}}g(x)dx\right)+L_t^{(\ell+1)}(\lambda)\int_{\mathbb{R}}xg(x)dx\right|\xrightarrow{\mathbb{P}}0.$$

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