Limit theorems for additive functionals of the fractional Brownian motion

Joint work with Nualart, Peccati, Nourdin

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$$\begin{aligned}\{\int_0^{nt} f(X_s) ds \ ; \ t \geq 0\} &\overset{Law}{=} \{\int_0^{nt} f(n^H X_{\frac{s}{n}}) ds \ ; \ t \geq 0\} \\ &= \{n \int_0^{t} f(n^H X_s) ds \ ; \ t \geq 0\},\end{aligned}$$

where f is a test function.

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This statistic is a particular instance of the family of processes

$$\{G_t^{(n)}\ ;\ t \geq 0\} := \{b_n \int_0^t f(n^H(X_s - \lambda))ds,\ ;\ t \geq 0\}, \tag{1}$$

where $b_n > 0$, $f : \mathbb{R} \to \mathbb{R}$ is a test function and $\lambda \in \mathbb{R}$.

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$$\text{Fluctuations} \longleftrightarrow \begin{cases} \text{Derivatives of the local time of } X \\ \text{Mixed Gaussian limits.} \end{cases}$$

Preliminaries: local times

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$$L_t(\lambda) := \lim_{\varepsilon \to 0} \int_0^t \phi_\varepsilon(X_s - \lambda)ds,$$

where the convergence holds in $L^2(\Omega)$ and ϕ_ε and

$$\phi_\varepsilon(x) := (2\pi\varepsilon)^{-\frac{1}{2}} \exp\{-\frac{1}{2\varepsilon}|x|^2\}$$

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Observation: Relation (2) implies a trivial condition when $\int_{\mathbb{R}} f(x)dx = 0$.

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$$n^{\frac{H+1}{2}} \int_0^t f(n^H(X_s - \lambda))ds \xrightarrow{Law} \sqrt{b}\, W_{L_t(\lambda)}, \tag{3}$$

where $b > 0$, W is a fractional Brownian motion independent of X and the convergence holds in the uniform topology over compact sets.

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- ¿What happens in the case $H \leq \frac{1}{3}$?
- ¿Can we say something about the non-zero energy case ($\int_{\mathbb{R}} f(y)dy \neq 0$)?

Handling the "rough case", when H is small.

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Lemma

Suppose $0 < H < \frac{1}{3}$. *Then, for every* $t \geq 0$ *and* $\lambda \in \mathbb{R}$, *the variables*

$$L^{(\prime)}_{t,\varepsilon}(\lambda) = \int_0^t \delta_0'(X_s - \lambda)ds := \int_0^t \phi_\varepsilon'(X_s - \lambda)ds, \quad \varepsilon > 0, \tag{4}$$

converge in $L^2(\Omega)$ *towards a limit* $L_t'(\lambda)$, *when* $\varepsilon \to 0$.

Main result I

Theorem (Jaramillo, Nourdin, Nualart, Peccati)
Suppose that $H < 1/3$ *and* $g : \mathbb{R} \to \mathbb{R}$ *satisfies* $\int_{\mathbb{R}} |f(y)|(1 + |y|^{1+v})dy$ *for some* $v > 0$.

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$$n^H \left(\int_0^t f(n^H(X_s - \lambda))ds - n^{-H} L_t(\lambda) \int_{\mathbb{R}} f(y)dy \right) \overset{L^2(\Omega)}{\to} - \left(\int_{\mathbb{R}} yf(y)dy \right) L'_t(\lambda).$$

Justification of the main result I

Recall that $L_t(\lambda) = \int_0^t \delta_0(X_s - \lambda)ds$ y $L'_t(\lambda) = \int_0^t \delta'_0(X_s - \lambda)ds$.

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$$2\pi L_t(\lambda) = \int_{\mathbb{R}} \int_0^t e^{\mathbf{i}\xi(X_s - \lambda)} ds d\xi,$$
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On the other hand,

$$n^{2H} \int_0^t f(n^H(X_s - \lambda))ds = \frac{n^H}{2\pi} \int_{\mathbb{R}^2} \int_0^t e^{\mathbf{i}\xi(X_s - \lambda - \frac{y}{n^H})} f(y) ds dy d\xi.$$

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From here it follows that

$$n^{2H}\int_0^t f(n^H(X_s-\lambda))ds - n^H\int_{\mathbb{R}} f(y)dyL_t(\lambda)$$
$$= -\frac{1}{2\pi}\int_{\mathbb{R}^2}\int_0^t \mathbf{i}\xi e^{\mathbf{i}\xi(X_s-\lambda)} yf(y)dsdyd\xi$$
$$+\frac{n^H}{2\pi}\int_{\mathbb{R}^2}\int_0^t e^{\mathbf{i}\xi(X_s-\lambda)}(\mathbf{i}\frac{y}{n^H}+e^{-\mathbf{i}\frac{y\xi}{n^H}}-1)f(y)dsdyd\xi. \tag{5}$$

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n^{2H}\int_0^t f(n^H(X_s-\lambda))ds - n^H\int_{\mathbb{R}} f(y)dyL_t(\lambda) \\
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The second term goes to zero and the first one is $-\left(\int_{\mathbb{R}} yf(y)dy\right) L_t'(\lambda)$.

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Theorem (Jaramillo, Nourdin, Nualart, Peccati)
Define $\ell_{n,H} := \mathbb{1}_{\{\{H>\frac{1}{3}\}\}} + \log(n)^{-\frac{1}{2}} \mathbb{1}_{\{\{H=\frac{1}{3}\}\}}$. *Then, for* $H \geq \frac{1}{3}$ *and* $\boldsymbol{f} : \mathbb{R} \to \mathbb{R}^d$ *of the form* $\boldsymbol{f} = (f_1 \ldots, f_d)$ *with* $f_i : \mathbb{R} \to \mathbb{R}$ *'nice',*

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$$\{n^{\frac{H+1}{2}}\ell_{n,H}\left(\int_0^t \boldsymbol{f}(n^H(X_s-\lambda))ds - n^{-H}L_t(\lambda)\int_{\mathbb{R}}\boldsymbol{f}(x)dx\right) \ ; \ t \geq 0\}$$
$$\overset{f.d.d}{\to} \{\mathcal{C}_H[\boldsymbol{f}]\tilde{\boldsymbol{W}}_{L_t(\lambda)} \ ; \ t \geq 0\}, \quad (6)$$

where $\tilde{\boldsymbol{W}} = \{\tilde{\boldsymbol{W}}_{\boldsymbol{t}} \ ; \ t \geq 0\}$ *is a d-dimensional Brownian motion independent of* X.

Justification of main result II, Step 1

Inclusion in the Wiener space: We will use a representation of X as a function of a Brownian motion $W = \{W_t \ ; \ t \geq 0\}$ via

$$X_t = \int_0^t K_H(s,t) dW_s,$$

where $K_H : \mathbb{R} \times \mathbb{R}$ is an explicit kernel.

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Itô representation for the fluctuations: Malliavin calculus allows us to express centered variables as stochastic integrals via the Clark-Ocone formula.

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$$n^{\frac{H+1}{2}}\ell_{n,H}\left(\int_0^t g(n^H(X_s-\lambda))ds - n^{-H}L_t(\lambda)\int_{\mathbb{R}} g(x)dx\right)$$
$$= \int_0^t F_t^{(n)}(s)dW_s + \mu_t,$$

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For analyzing $\int_0^t F_t^{(n)}(s)dW_s$, we compute the quadratic variation of

$$\{M_u \ ; \ u \leq t\} = \{\int_0^u F_t^{(n)}(s)dW_s \ ; \ u \leq t\}$$

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(C2) For all $u \in [0, T]$,

$$\langle M^{(n)}, W \rangle_u \xrightarrow{\mathbb{P}} 0.$$

Justification of main result II, main ingredients

For proving **(C1)** and **(C2)** we need

- Fourier inversion ofrmula and representation of local times.
- Clark-Ocone formula.
- Local non-determinism for the fractional Brownian motion (namely, estimations of $\mathbb{V}\mathrm{ar}[X_r \mid X_{r_1}, \ldots, X_{r_k}]$).

¡Gracias!

Arigato gozaimasu!

References

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