Integrated noise scale estimation for stochastic integrals

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Collaborators

Joint work with Jorge Cazares

Key works by Vladimir Fomichov, Jorge Gonzalez Cazares, Jevgenijs Ivanovs

Objective

Consider a process of the form

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- V is Gaussian with $Var[V_{s+h} - V_s]$ behaving like h^{2H}.
- $\eta > 0$.
- $\circledast$ is a notion of integration.
- L has low activity:

$$\|L\|_p^p := \sup_{n \geq 1} \left(|\Delta_{1,n} L|^p + \cdots + |\Delta_{n,n} L|^p \right) < \infty.$$

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Canonical example

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- $I[b]$ is a finite-variation drift term.
- σ is a square integrable continuous adapted process.
- J an integral of a Levy, with finite $(2-\varepsilon)$-variation.

We observe X on a regular grid, and want to recover $I[\sigma^2]$

Thresholding strategies

At the level of increments,

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Heuristic considerations

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Discretizing and rescaling by n^H:

$$X_{n,k} = \sigma_{n,k} Z_{n,k} + Y^o_{n,k},$$

where $Z_{n,k} := n^H \Delta_{k,n} V$, and

- $X_{n,k} := n^H \Delta_{n,k} \xi$
- $\sigma_{n,k} Z_{n,k} =$ Riemann sum approximation
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Precise definitions for later

Big blocks-small blocks methodology

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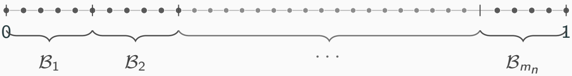

In each block $\mathcal{B}_\ell$, we use the local approximation

$$X_{n,k} = \sigma Z_{n,k} + Y_{n,k},$$

for k in $\mathcal{B}_\ell$.

- σ is a **constant** noise scale inside the block.
- $Y_{n,k}$ absorbs both the term $Y^o_{n,k}$ and the approximation $\sigma_{n,k}$ with the blockwise constant.

A change of coordinates?

Apply a transformation (change of coordinates) $\mathcal{T}$ to the block

$$\boldsymbol{X}_n^{[\ell]} = (X_{n,k};\, k \in \mathcal{B}_\ell).$$

We want two design $\mathcal{T}$ with the properties:

- The vector $\mathcal{T}(\boldsymbol{X}_n^{[\ell]})$ is more predictable
- The equation $\boldsymbol{X}_n^{[\ell]} = \sigma \boldsymbol{Z}_n^{[\ell]} + \boldsymbol{Y}_n^{[\ell]}$, induces a tractable equation

$$\mathfrak{T}(\mathcal{T}(\boldsymbol{X}_n^{[\ell]}), \boldsymbol{Z}_n^{[\ell]}, \boldsymbol{Y}_n^{[\ell]}) = 0 \tag{1}$$

with $\boldsymbol{Y}_n^{[\ell]}$ not being of big influence in (1).

Detour: scale noise estimate in time series

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Lemma
Let $p \geq 1$ and let $A, B, C \in \mathbb{R}^n$ satisfy $A_k = B_k + C_k$. Then

$$\|A^{\uparrow} - B^{\uparrow}\|_p \leq \|C\|_p,$$

where $\|\cdot\|_p$ denotes p-norm.

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We then obtain the key identity **(Speaker note)**

$$X^{\uparrow}_{k,n} = \sigma Z^{\uparrow}_{k,n} + \widetilde{Y}_{k,n},$$

with

$$\|\widetilde{\boldsymbol{Y}}^{[\ell]}_n\|_p \leq \|\boldsymbol{Y}^{[\ell]}_n\|_p.$$

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where

$$\psi_{n,k} := \Psi^{-1}\big(k/(n+1)\big),$$

for Ψ denoting the standard Gaussian law. The approximation $X^{\uparrow}_{k,n} \approx \sigma\psi_{n,k}$ suggest to choose σ so that minimizes the loss **(Speaker note)**

$$\theta(\varsigma) = \sum_{k=1}^{n} \left| X^{\uparrow}_{k,n} - \varsigma\psi_{n,k} \right|.$$

We define our estimator for σ within a single block as

$$\Sigma^{r}(\boldsymbol{X}_n^{[\ell]}) := \arg\min_{\varsigma>0} \theta(\varsigma).$$

Detour for constant volatility estimate

For concreteness: $Z_{n,k}$ are the increments of a fBm (V is a fractional Brownian motion).

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Theorem (González Cázares and Jaramillo)
If η is constant equal to $\sigma > 0$ and we have one single block, then,
(Speaker note)

$$\mathbb{E}\left[\left|\frac{\Sigma(\boldsymbol{X}_n)}{\sigma} - 1\right|\right] \leq C\left(n^{-1/p}\mathbb{E}\left[\frac{\|\boldsymbol{Y}_n\|_p}{\sigma}\right] + n^{-((1-H)\wedge 1/2)}\right).$$

Interest connections with robustness

Ingredients of the proof

First step

$$|\Sigma(\boldsymbol{X}_n) - \sigma| \, \|\psi_n\|_1 = \|\Sigma(\boldsymbol{X}_n)\psi_n - \sigma\psi_n\|_1$$

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Second step

$$\|Z_n^{\uparrow} - \psi_n\| = n d_{W^1}(\iota[Z_n], \iota[\psi_n]),$$

where ι denotes taking the empirical distribution. **(speaker note)**

Details on the choices of the time series

The model:

$$\xi_t = \int_0^t \eta_s \circledast dV_s + L_t.$$

Define

$$s_{k,n}^2 := \mathrm{Var}(\Delta_{k,n} V), \qquad X_{k,n} \quad := s_{k,n}^{-1} \Delta_{k,n} \xi, \qquad Z_{k,n} := s_{k,n}^{-1} \Delta_{k,n} V.$$

Then

$$X_{k,n} = \eta_{t_{k-1}} Z_{k,n} + Y_{k,n},$$

where

$$Y_{k,n} = s_{k,n}^{-1} \Delta_{k,n} L + s_{k,n}^{-1} \left(\int_{t_{k-1}}^{t_k} \eta_s \circledast dV_s - \eta_{t_{k-1}} \Delta_{k,n} V \right)$$

$$\sigma_{k,n} = \eta_{t_{k-1}}.$$

Details on the choices of the blocks

We divide $\{1, \ldots, n\}$ into $M_n = \lfloor n/m_n \rfloor$ consecutive blocks of size m_n:

$$\mathcal{B}_\ell = \{(\ell - 1)m_n + 1, \ldots, \ell m_n\}.$$

For each block, recall the notation

$$\boldsymbol{X}_{m_n}^{[\ell]} = (X_{k,n} : k \in \mathcal{B}_\ell)$$

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Integrated scale estimator

$$\mathcal{IV}^{(m_n)}(\boldsymbol{X}_n) := \frac{T}{M_n} \sum_{\ell=1}^{M_n} \Sigma(\boldsymbol{X}_{m_n}^{[\ell]}).$$

Assumptions for convergence

We assume low activity

$$\sup_{n\to\infty}\sum_{k=1}^{n}\mathbb{E}[|L_{t_k}-L_{t_{k-1}}|^p]<\infty.$$

The dependence of the noise is measured by

$$\kappa_n(r) := \sup_{a\geq 0}\sum_{i,j=1}^{r}\left|\mathrm{Cov}\left(Z_{a+i,n}, Z_{a+j,n}\right)\right|.$$

We assume that the integral satisfies the local approximation

$$\left\|\int_s^t U_u \circledast dV_u - U_s(V_t - V_s)\right\|_{L^2(\Omega)} \leq C|t-s|^{H+\gamma-\varepsilon}.$$

for parameters $\gamma, \varepsilon > 0$.

Theoretical guarantees

Theorem (González Cázares-Jaramillo)
Under the previous assumptions,

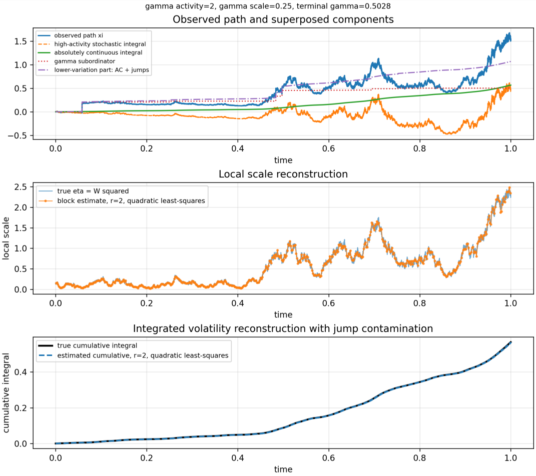
$$\left\| \mathcal{IV}^{(m_n)}(\boldsymbol{X}_n) - \int_0^T \eta_s \, ds \right\|_{L^1(\Omega)} \leq C \left(n^{H - \frac{\gamma(1-p(H-\varepsilon))}{1+\gamma p}} + n^{-\frac{H-\varepsilon+\gamma}{1+\gamma p}} \kappa_n(m_n)^{1/(2p)} \right).$$

where

$$m_n := n^{\frac{H-\varepsilon+\gamma}{\gamma+1/p}}.$$

Numerical illustration

$$\xi_t = \int_0^t W_s^2 \, dW_s + \int_0^t W_s^2 \, ds + J_t,$$



Open problems and conjectures

- Extend the method to noises with α-stable tails.
- Formulate multidimensional versions of the estimator.
- Study random fields, not only stochastic processes.
- To what degree, can the scheme play similar roles as the quadratic variation?

Acknowledgements

Thank you!

Questions and comments are welcome.

References and related work

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Work in progress.

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