



CIMAT

Centro de Investigación en Matemáticas, A.C.

Método de Stein no-conmutativo

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Un gusto haber trabajado en esto, un gusto aún mayor haberlo compartido contigo

Objective

Let $\{\mu_{k,n} ; k, n \geq 1\}$ be a sequence of centered probability measures, and define

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Goal

Bound $d_{TV}(\nu_n, \mathbf{s})$ in a probabilistic way.

Let $\mathcal{A}$ be a unital C^* -algebra and $\tau : \mathcal{A} \rightarrow \mathbb{C}$ a positive unital linear functional.

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Definition

Let $\{A_n\}_{n \geq 1}$ be subalgebras of $\mathcal{A}$. Define $\bar{a} := a - \tau[a]$. We say that $\{A_n\}_{n \geq 1}$ are freely independent if

$$\tau[\bar{a}_1 \bar{a}_2 \cdots \bar{a}_k] = 0, \tag{1}$$

for $a_1, \dots, a_k$ alternating. Sums of free random variables yields the free convolution $\boxplus$.

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A not so mathematical slide

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Trick

If distance is like walking, let's think of walking from ν_n to $\mathbf{s}$ through a trail.

A trail for walking, driven by Stein identity

Definition

Let $\{P_\theta^*\}_{\theta \geq 0}$ be operators **over measures**, defined by

$$P_\theta^*[\mu] := \text{Law}(e^{-\theta}X + \sqrt{1 - e^{-2\theta}}Y),$$

with $X \sim \mu$ and $Y \sim m_1[\mu] + \sqrt{\text{Var}[\mu]}s$.

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$$\langle \mathbf{s}, h \rangle - \langle \mu, h \rangle = \langle P_\infty^*[\mu], h \rangle - \langle P_0^*[\mu], h \rangle$$

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furthermore: P_θ^* is a semigroup.

Actual connection to Stein identity

Deriving and integrating, we get

$$\langle \mathbf{s}, h \rangle - \langle \mu, h \rangle = \int_0^\infty \frac{d}{d\theta} \langle P_\theta^*[\mu], h \rangle d\theta.$$

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Lemma

The red term equals

$$\langle P_\theta^*[\mu] \otimes P_\theta^*[\mu], \mathcal{L}_\boxplus[h] \rangle,$$

where

$$\mathcal{L}_\boxplus[h](x, y) := xDh(x) - \partial Dh,$$

for D denoting derivative and

$$\partial g(x, y) := (g(x) - g(y))/(x - y).$$

What have we achieved in the free case?

For h regular enough,

$$\langle \mathbf{s}, h \rangle - \langle \mu, h \rangle = \int_0^\infty \langle P_\theta^*[\mu] \otimes P_\theta^*[\mu], \mathcal{L}_\boxplus[h] \rangle d\theta.$$

Lemma (Non-commutative Stein's lemma)

A law ν is semicircular if and only if

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$$\langle \mathbf{s}, h \rangle - \langle \mu, h \rangle = \int_0^\infty \langle P_\theta^*[\mu] \otimes P_\theta^*[\mu] - P_\infty^*[\mu] \otimes P_\infty^*[\mu], \mathcal{L}_\boxplus[h] \rangle d\theta.$$

What have we achieved in the free case? Part II

Writing what we have differently,

$$|\langle \mathbf{s}, h \rangle - \langle \mu, h \rangle| = |\langle \mathcal{S}_{\boxplus}^*[\mu], \mathcal{L}_{\boxplus}[h] \rangle|,$$

where

$$\mathcal{S}_{\boxplus}^*[\mu] := \int_0^\infty (P_\theta^*[\mu] \otimes P_\theta^*[\mu] - P_\infty^*[\mu] \otimes P_\infty^*[\mu]) d\theta.$$

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Compare this with the classical case:

$$|\langle \gamma, h \rangle - \langle \mu, h \rangle| = |\langle \mu, \mathcal{L}[\mathcal{S}[h]] \rangle|,$$

where γ is the standard Gaussian and $\mathcal{S}$ is defined similarly.

New bottleneck

Dealing with the bottleneck when $\mu = \nu_n$

Let $\xi_{1,n}, \dots, \xi_{n,n}$ be free random variables with law

$$\int_{\mathbb{R}_+} (P_\theta^* - P_\infty^*)[\mu_{k,n}] d\theta.$$

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$$\langle \mathcal{S}_{\boxplus}^*[\nu_n], \mathcal{L}_{\boxplus}[h] \rangle = \tau[S_n Dh(S_n)] - \langle \mathcal{S}_{\boxplus}^*[\nu_n], \partial Dh \rangle,$$

where $S_n := \xi_{1,n} + \dots + \xi_{n,n}$.

Corollary (Bercovici and Voiculescu)

For n large, (ν_n) supported in $[-3, 3]$

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If h is holomorphic

$$\langle S_{\boxplus}^*[\nu_n], \mathcal{L}_{\boxplus}[h] \rangle = \frac{1}{2\pi i} \int_{\mathcal{R}} h(z) (\tau[S_n g_z(S_n)] - \langle S_{\boxplus}^*[\nu_n], \partial g_z \rangle) dz,$$

where $\mathcal{R}$ is a contour strictly containing $[-5, 5]$ and $g_z(x) := (z - x)^{-2}$.

For h holomorphic and bounded by one over $[-5, 5]$, there is a constant only depending on $\mathcal{R}$, such that

$$|\langle \mathbf{s}, h \rangle - \langle \nu_n, h \rangle| \leq C \sup_{z \in \mathcal{R}} |\tau[S_n g_z(S_n)] - \langle \mathcal{S}_{\boxplus}^*[\nu_n], \partial g_z \rangle|.$$

By an approximation argument,

$$d_{TV}(\mathbf{s}, \nu_n) \leq C \sup_{z \in \mathcal{R}} |\tau[S_n g_z(S_n)] - \langle \mathcal{S}_{\boxplus}^*[\nu_n], \partial g_z \rangle|.$$

Non-commutative Lindeberg trick?

Define $S_n^{(k)}$ as the part of S_n that does not involve $\xi_{k,n}$. Observe that

$$\tau[S_n g_z(S_n)] = \sum_{k=1}^n \tau[\xi_{k,n} g_z(S_n)] = \sum_{k=1}^n \tau[\xi_{k,n} (g_z(S_n) - g_z(S_n^{(k)}))]$$

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Idea: use a bit of non-commutative Taylor

If we (i) applying Taylor (ii) use the boundedness of the g_z , then

$$\begin{aligned}\tau[S_n g_z(S_n)] &= O\left(\sum_{k=1}^n \tau[|\xi_{k,n}|^3]\right) \\ &\quad + 2 \sum_{k=1}^n \tau[\xi_{k,n} g_z(S_n^{(k)}) \mathfrak{s} \left[(z - S_n^{(k)}) \xi_{k,n} \right] g_z(S_n^{(k)})].\end{aligned}$$

where $\mathfrak{s}$ means symmetrization.

Using freeness, and approximating $S_n^{(k)}$ by S_n ,

$$\tau[S_n g_z(S_n)] = O\left(\sum_{k=1}^n \tau[|\xi_{k,n}|^3]\right) + 2 \sum_{k=1}^n \tau[|\xi_{k,n}|^2] \tau[g_z(S_n)(z - S_n)g_z(S_n)] .$$

The unit variance condition then implies

$$\sup_{z \in \mathcal{R}} |\tau[S_n g_z(S_n)] - \langle \mathcal{S}_{\boxplus}^*[\nu_n], \partial g_z \rangle| \leq C \sum_{k=1}^n \tau[|\xi_{k,n}|^3].$$

Wrapping things up

Theorem (Diaz-Jaramillo)

Under the above considerations,

$$d_{TV}(\mathbf{s}, \nu_n) \leq C \sum_{k=1}^n \int_{\mathbb{R}} |x|^3 \mu_{k,n}(dx).$$

Some improvements:

- Neighborhoods of dependency.
- Uniform convergence of the density.
- Including sharper approximations under more conditions.

Some unsolved improvements:

- Uniform convergence of the derivatives of the density.

Some questions that might be interesting

- For Boolean or monotone convolutions, can we still say something (in Wasserstein distance)?
- Can we change $\boxplus$ by $\boxplus_m$ and still say something?
- Extended to Edgeworth expansions
- Implementations in large matrix problems
- Free stable limits



To the memory of the four Gaussians



Diaz M., Jaramillo A. Non-commutative Stein's method:
Applications to free probability and sums of non-commutative
variables.



G. P. Chistyakov and F. Gotze. Limit theorems in free probability
theory.