Free Berry-Esseen theorem via Stein's method

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Joint work with a great mathematician, great fan of Checo Pérez, and most of all: a great friend

Objective

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Moment of honesty...

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Lemma (Stein's lemma)
Consider the Stein identity

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where D denotes derivative and ι the identity function. If (1) holds for many choices of f, then μ is standard Gaussian.

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Why Stein's identity would be natural?

How to cook up Stein's identity? (personal favorite recipie)

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Then, for a given probability μ, and a test function h,

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This operator 'interpolates' towards Gaussian.

How to cook up Stein's identity?

Deriving and integrating, we get

$$\langle \gamma, h \rangle - \langle \mu, h \rangle = \int_0^\infty \frac{d}{d\theta} \langle \mu, P_\theta[h] \rangle d\theta = \langle \mu, \int_0^\infty \mathcal{L}[P_\theta[h]] d\theta \rangle.$$

where

$$\mathcal{L}[g](x) := xDg(x) - D^2 g(x) = (\iota \cdot Dg - D^2 g)(x).$$

What have we achieved so far?

If h is regular enough,

$$|\langle \gamma, h\rangle - \langle \mu, h\rangle| = |\langle \mu, \mathcal{L}[\mathcal{S}[h]]\rangle|,$$

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New bottleneck

Bound uniformly $|\langle \mu, \mathcal{L}[g]\rangle|$, with $g = \mathcal{S}[h]$.

Dealing with the bottleneck for getting CLT

Consider the case

$$\nu_n := \mu_{1,n} * \cdots * \mu_{n,n},$$

with $\mu_{j,n}$ centered, such that

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If $\xi_{1,n}, \ldots, \xi_{n,n}$ are independent with $\xi_{k,n} \sim \mu_{k,n}$, then

$$\langle \nu_n, \mathcal{L}[g] \rangle = \mathbb{E}[S_n Dg(S_n)] - \mathbb{E}[D^2 g(S_n)]$$

with $S_n := \xi_{1,1} + \cdots + \xi_{n,n}$.

A Lindeberg-type trick

Define $S_n^{(k)}$ as the part of S_n that does not involve $\xi_{k,n}$.

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Then,

$$\begin{aligned} \mathbb{E}[S_n Dg(S_n)] &\approx \sum_{k=1}^{n} \mathbb{E}[|\xi_{k,n}|^2 D^2 g(S_n^k))] = \sum_{k=1}^{n} \mathbb{E}[|\xi_{k,n}|^2]\mathbb{E}[D^2 g(S_n^k))] \\ &\approx \sum_{k=1}^{n} \mathbb{E}[|\xi_{k,n}|^2]\mathbb{E}[D^2 g(S_n))] = \mathbb{E}[D^2 g(S_n)] \end{aligned}$$

Conclusion: $\langle \nu_n, \mathcal{L}[g] \rangle \approx 0$

Elements of free probability

Let $\mathcal{A}$ be a unital C^*-algebra and $\tau : \mathcal{A} \to \mathbb{C}$ a positive unital linear functional. We then say that the pair $(\mathcal{A}, \tau)$ is a *C^*-probability space*.

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Definition (Freeness)
Let $\{A_n\}_{n\geq 1}$ be a sequence of subalgebras of $\mathcal{A}$. For $a \in \mathcal{A}$, denote the centering of a by $\bar{a} := a - \tau[a]$. We say that $\{A_n\}_{n\geq 1}$ are freely independent, or free, if

$$\tau[\bar{a}_1 \bar{a}_2 \cdots \bar{a}_k] = 0, \tag{2}$$

for $a_1, \ldots, a_k$ alternating algebras.

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Definition (Free convolution)
Sums of free random variables yields the free convolution $\boxplus$.

Recipie for cooking up a free Stein identity

Our building block before was

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Let $\{P^*_\theta\}_{\theta \geq 0}$ be operators **over measures**, defined by

$$P^*_\theta[\mu] := Law(e^{-\theta}X + \sqrt{1 - e^{-2\theta}}Y),$$

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This operator 'interpolates' from μ standardized to $\mathbf{s}$, as

$$\langle \mathbf{s}, h \rangle - \langle \mu, h \rangle = \langle P_\infty^*[\mu], h \rangle - \langle P_0^*[\mu], h \rangle$$

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Lemma (Diaz-Jaramillo)

$$\frac{d}{d\theta}\langle P^*_\theta[\mu], h\rangle = \langle P^*_\theta[\mu] \otimes P^*_\theta[\mu], \mathcal{L}_\boxplus[h]\rangle,$$

where $\mathcal{L}_\boxplus[h]$ is the real function in $\mathbb{R}^2$:

$$\mathcal{L}_\boxplus[h](x, y) := xDh(x) - \partial Dh,$$

where g is the non-commutative derivative

$$\partial g(x, y) := (g(x) - g(y))/(x - y).$$

What have we achieved in the free case?

For h regular enough,

$$\langle \boldsymbol{s}, h\rangle - \langle \mu, h\rangle = \int_0^{\infty} \langle P_\theta^*[\mu] \otimes P_\theta^*[\mu], \mathcal{L}_{\boxplus}[h]\rangle d\theta.$$

Lemma (Non-commutative Stein's lemma)
A law ν is semicircular if and only if

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for h smooth with second bounded derivative.

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As a consequence,

$$\langle \boldsymbol{s}, h\rangle - \langle \mu, h\rangle = \int_0^\infty \langle P_\theta^*[\mu] \otimes P_\theta^*[\mu] - P_\infty^*[\mu] \otimes P_\infty^*[\mu], \mathcal{L}_\boxplus[h]\rangle d\theta.$$

What have we achieved in the free case? Pt II

Writing what we have differently,

$$|\langle \mathbf{s}, h\rangle - \langle \mu, h\rangle| = |\langle \mathcal{S}^*_{\boxplus}[\mu], \mathcal{L}_{\boxplus}[h]\rangle|,$$

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$$\mathcal{S}^*_{\boxplus}[\mu] := \int_0^\infty (P^*_\theta[\mu] \otimes P^*_\theta[\mu] - P^*_\infty[\mu] \otimes P^*_\infty[\mu])d\theta.$$

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Compare this with the classical case, stating

$$|\langle \gamma, h\rangle - \langle \mu, h\rangle| = |\langle \mu, \mathcal{L}[\mathcal{S}[h]]\rangle|,$$

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Dealing with the bottleneck when $\mu = \nu_n$

Consider

$$\nu_n := \mu_{1,n} \boxplus \cdots \boxplus \mu_{n,n}.$$

Assume $Var[\nu_n] = 1$ and that $\mu_{k,n}$ have **small supports**.

Dealing with the bottleneck when $\mu = \nu_n$

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$$\nu_n := \mu_{1,n} \boxplus \cdots \boxplus \mu_{n,n}.$$

Assume $Var[\nu_n] = 1$ and that $\mu_{k,n}$ have **small supports**. If $\xi_{1,n}, \ldots, \xi_{n,n}$ are free with law equal to $\int_{\mathbb{R}_+} (P_\theta^* - P_\infty^*)[\mu_{k,n}] d\theta$, then

$$\langle \mathcal{S}_\boxplus^*[\nu_n], \mathcal{L}_\boxplus[h] \rangle = \tau[S_n Dh(S_n)] - \langle \mathcal{S}_\boxplus^*[\nu_n], \partial Dh \rangle,$$

with $S_n := \xi_{1,1} + \cdots + \xi_{n,n}$.

An important technicallity

Corollary (Superconvergence by Bercovici and Voiculescu)
For n large, Supp$(\nu_n) \subset [-3,3]$ *and Supp*$(\mathcal{S}^*_{\boxplus}[\nu_n]) \subset [-5,5]$.

The Cauchy formula yields

$$\langle \mathcal{S}^*_{\boxplus}[\nu_n], \mathcal{L}_{\boxplus}[h] \rangle = \frac{1}{2\pi i} \int_{\mathcal{R}} h(z)(\tau[S_n g_z(S_n)] - \langle \mathcal{S}^*_{\boxplus}[\nu_n], \partial g_z \rangle) dz,$$

where $\mathcal{R}$ strictly containing $[-5,5]$ and

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For total variation, h is bounded by one!

Moral of the story

For h very smooth and bounded by one, there is a constant only depending on $\mathcal{R}$, such that

$$|\langle \mathbf{s}, h\rangle - \langle \nu_n, h\rangle| \leq C \sup_{z\in\mathcal{R}} |\tau[S_n g_z(S_n)] - \langle \mathcal{S}^*_{\boxplus}[\nu_n], \partial g_z\rangle|.$$

By an approximation argument,

$$d_{TV}(\langle \mathbf{s}, h\rangle - \langle \nu_n, h\rangle) \leq C \sup_{z\in\mathcal{R}} |\tau[S_n g_z(S_n)] - \langle \mathcal{S}^*_{\boxplus}[\nu_n], \partial g_z\rangle|.$$

Non-commutative Lindeberg trick?

Define $S_n^{(k)}$ as the part of S_n that does not involve $\xi_{k,n}$. Observe that

$$\mathbb{E}[S_n g_z(S_n)] = \sum_{k=1}^{n} \mathbb{E}[\xi_{k,n} g_z(S_n)] = \sum_{k=1}^{n} \mathbb{E}[\xi_{k,n}(g_z(S_n) - g_z(S_n^k))]$$

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Do I need a commutative Taylor?

The only non-commutative calculus we need

Lemma

For non-commutative variables a, r*, define*

$$\Delta(a,r) := 2\mathfrak{s}[(z-a)r] - r^2,$$

where $\mathfrak{s}$ *denotes the symmetrization operator. Then, for all* $q \geq 1$*,*

$$g(a+r) = g(a+r)\left(\Delta(a,r)\, g(a)\right)^q + \sum_{j=0}^{q-1} g(a)\left(\Delta(a,r)\, g(a)\right)^j,$$

Consequence

We have the following expansion for all $q \geq 1$

$$\tau[S_n g_z(S_n)] = \sum_{k=1}^{n} \tau[\xi_{k,n} g\left(S_n\right)\left(\Delta\left(S_n^k, \xi_{k,n}\right) g(S_n^k)\right)^q] + \sum_{j=1}^{q-1} \tau[g\left(S_n^k\right)\left(\Delta\left(S_n^k, \xi_{k,n}\right) g\left(S_n^k\right)\right)^j].$$

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For $q = 2$, the boundedness of the g then yields

$$\begin{aligned}\tau[S_n g_z(S_n)] &= O(\sum_{k=1}^{n} \tau[|\xi_{k,n}|^3]) + \sum_{k=1}^{n} \tau[g\left(S_n^k\right) \Delta\left(S_n^k, \xi_{k,n}\right) g\left(S_n^k\right)] \\ &= O(\sum_{k=1}^{n} \tau[|\xi_{k,n}|^3]) + 2\sum_{k=1}^{n} \tau[g\left(S_n^k\right) \mathfrak{s}\left[(z - S_n^k)\xi_{k,n}\right] g\left(S_n^k\right)].\end{aligned}$$

Consequence

Finally, using freeness,

$$\begin{aligned}\tau[S_n g_z(S_n)] &= O(\sum_{k=1}^{n} \tau[|\xi_{k,n}|^3]) + 2\sum_{k=1}^{n} \tau[|\xi_{k,n}|^2]\tau[g\left(S_n^k\right)[(z - S_n^k)]\tau[g\left(S_n^k\right)] \\ &= O(\sum_{k=1}^{n} \tau[|\xi_{k,n}|^3]) + 2\sum_{k=1}^{n} \tau[|\xi_{k,n}|^2]\tau[g\left(S_n\right)[(z - S_n)]\tau[g\left(S_n\right)].\end{aligned}$$

The unit variance condition then implies

$$\sup_{z\in\mathcal{R}} |\tau[S_n g_z(S_n)] - \langle \mathcal{S}_{\boxplus}^*[\nu_n], \partial g_z\rangle| \leq C\sum_{k=1}^{n} \tau[|\xi_{k,n}|^3].$$

Wrapping things up

Theorem (Diaz-Jaramillo)
Under the above considerations,

$$d_{TV}(\mathbf{s}, \nu_n) \leq C \sum_{k=1}^{n} \int_{\mathbb{R}} |x|^3 \mu_{k,n}(dx).$$

Some improvements:

1. Neighborhoods of dependency.
2. Uniform convergence of the density.
3. The Mauricio Salazar trick holds.

Some unsolved improvements:

1. Uniform convergence of the derivatives of the density.

Some questions that I thought could be interesting

- Free law of rare events
- For Boolean or monotone convolutions, can we still say something in Wasserstein distance?
- Can we change $\boxplus$ by $\boxplus_m$ and still say something?
- Extended Mauricio Salazar trick or Edgeworth expansions
- Implementations in large matrix problems
- Multidimensional versions
- Free stable limits

Thanks!

References

Diaz M., Jaramillo A. Non-commutative Stein's method: Applications to free probability and sums of non-commutative variables.

G. P. Chistyakov and F. Gotze. Limit theorems in free probability theory.