Free Breuer-Major theorem

Arturo Jaramillo Gil (work in progress with Sefika Kuzgun)

Centro de Investigación en Matemáticas (CIMAT)

Motivation

Let Y_k be standardized i.i.d. The central limit theorem establishes that the normalized sum

$$Z_n := \frac{1}{\sqrt{n}} \sum_{k=1}^{n} Y_k$$

satisfies $Z_n \overset{Law}{\approx} \mathcal{N}$, where $\mathcal{N}$ is a standard Gaussian variable.

Motivation

Let Y_k be standardized i.i.d. The central limit theorem establishes that the normalized sum

$$Z_n := \frac{1}{\sqrt{n}} \sum_{k=1}^{n} Y_k$$

satisfies $Z_n \overset{Law}{\approx} \mathcal{N}$, where $\mathcal{N}$ is a standard Gaussian variable.

Theme of the talk

Motivation

Let Y_k be standardized i.i.d. The central limit theorem establishes that the normalized sum

$$Z_n := \frac{1}{\sqrt{n}} \sum_{k=1}^{n} Y_k$$

satisfies $Z_n \overset{Law}{\approx} \mathcal{N}$, where $\mathcal{N}$ is a standard Gaussian variable.

Theme of the talk

Can we allow the variables Y_k to be dependent?

Motivation

Let Y_k be standardized i.i.d. The central limit theorem establishes that the normalized sum

$$Z_n := \frac{1}{\sqrt{n}} \sum_{k=1}^{n} Y_k$$

satisfies $Z_n \overset{Law}{\approx} \mathcal{N}$, where $\mathcal{N}$ is a standard Gaussian variable.

Theme of the talk

Can we allow the variables Y_k to be dependent?

How fast is this convergence?

Motivation

Let Y_k be standardized i.i.d. The central limit theorem establishes that the normalized sum

$$Z_n := \frac{1}{\sqrt{n}} \sum_{k=1}^{n} Y_k$$

satisfies $Z_n \overset{Law}{\approx} \mathcal{N}$, where $\mathcal{N}$ is a standard Gaussian variable.

Theme of the talk

Can we allow the variables Y_k to be dependent?

How fast is this convergence?

Is there an analogue in free probability?

Classical Breuer-Major theorem

Let $\{\xi_k\}_{k\geq 1}$ be a stationary standardized Gaussian process.

Classical Breuer-Major theorem

Let $\{\xi_k\}_{k\geq 1}$ be a stationary standardized Gaussian process. Take $Y_k := g(\xi_k)$, where g is a real test function.

Classical Breuer-Major theorem

Let $\{\xi_k\}_{k\geq 1}$ be a stationary standardized Gaussian process. Take $Y_k := g(\xi_k)$, where g is a real test function. Let H_q denote the q-th Hermite polynomial.

Classical Breuer-Major theorem

Let $\{\xi_k\}_{k\geq 1}$ be a stationary standardized Gaussian process. Take $Y_k := g(\xi_k)$, where g is a real test function. Let H_q denote the q-th Hermite polynomial.

Theorem (Breuer-Major (1983))
Assumptions

- *Expansion* $g = \sum_{q\geq Q} a_q H_q$.

Classical Breuer-Major theorem

Let $\{\xi_k\}_{k\geq 1}$ be a stationary standardized Gaussian process. Take $Y_k := g(\xi_k)$, where g is a real test function. Let H_q denote the q-th Hermite polynomial.

Theorem (Breuer-Major (1983))
Assumptions

- *Expansion* $g = \sum_{q\geq Q} a_q H_q$. *$Q$ is called Hermite rank.*

Classical Breuer-Major theorem

Let $\{\xi_k\}_{k\geq 1}$ be a stationary standardized Gaussian process. Take $Y_k := g(\xi_k)$, where g is a real test function. Let H_q denote the q-th Hermite polynomial.

Theorem (Breuer-Major (1983))
Assumptions

- *Expansion* $g = \sum_{q\geq Q} a_q H_q$. *$Q$ is called Hermite rank.*
- *The function* $\rho(k) := \mathbb{E}[X_1 X_{k+1}]$ *satisfies* $\sum_k |\rho(k)|^Q < \infty$.

Classical Breuer-Major theorem

Let $\{\xi_k\}_{k\geq 1}$ be a stationary standardized Gaussian process. Take $Y_k := g(\xi_k)$, where g is a real test function. Let H_q denote the q-th Hermite polynomial.

Theorem (Breuer-Major (1983))
Assumptions

- *Expansion* $g = \sum_{q\geq Q} a_q H_q$. *$Q$ is called Hermite rank.*
- *The function* $\rho(k) := \mathbb{E}[X_1 X_{k+1}]$ *satisfies* $\sum_k |\rho(k)|^Q < \infty$.

Classical Breuer-Major theorem

Let $\{\xi_k\}_{k\geq 1}$ be a stationary standardized Gaussian process. Take $Y_k := g(\xi_k)$, where g is a real test function. Let H_q denote the q-th Hermite polynomial.

Theorem (Breuer-Major (1983))
Assumptions

- *Expansion* $g = \sum_{q\geq Q} a_q H_q$. *$Q$ is called Hermite rank.*
- *The function* $\rho(k) := \mathbb{E}[X_1 X_{k+1}]$ *satisfies* $\sum_k |\rho(k)|^Q < \infty$.

Then $S_n \overset{Law}{\to} \mathcal{N}(0, \sigma^2)$, *for some* $\sigma > 0$.

Proof by two different approaches

Classical methodology

Direct computation of $\mathbb{E}[H_{q_1}(\xi_{q_1}) \cdots H_{q_m}(\xi_{q_m})]$ (diagram formula).

Proof by two different approaches

Classical methodology

Direct computation of $\mathbb{E}[H_{q_1}(\xi_{q_1})\cdots H_{q_m}(\xi_{q_m})]$ (diagram formula).

The Malliavin-Stein approach

Understand very well $\mathbb{E}[Z_n f(Z_n)]$ and use Stein's method.

Proof by two different approaches

Classical methodology

Direct computation of $\mathbb{E}[H_{q_1}(\xi_{q_1})\cdots H_{q_m}(\xi_{q_m})]$ (diagram formula).

The Malliavin-Stein approach

Understand very well $\mathbb{E}[Z_n f(Z_n)]$ and use Stein's method. Nourdin, Peccati and Podolskij construct $\Gamma_1[Z_n]$ satisfying

$$\mathbb{E}[\Gamma_1[Z_n]f'(Z_n) - Z_n f(Z_n)] = 0.$$

Proof by two different approaches

Classical methodology

Direct computation of $\mathbb{E}[H_{q_1}(\xi_{q_1})\cdots H_{q_m}(\xi_{q_m})]$ (diagram formula).

The Malliavin-Stein approach

Understand very well $\mathbb{E}[Z_n f(Z_n)]$ and use Stein's method. Nourdin, Peccati and Podolskij construct $\Gamma_1[Z_n]$ satisfying

$$\mathbb{E}[\Gamma_1[Z_n]f'(Z_n) - Z_n f(Z_n)] = 0.$$

This allows to obtain bounds for

$$\mathbb{E}[f'(Z_n) - Z_n f(Z_n)],$$

in terms of $\mathbb{E}[|\Gamma_1[Z_n] - 1|]$,

Proof by two different approaches

Classical methodology

Direct computation of $\mathbb{E}[H_{q_1}(\xi_{q_1})\cdots H_{q_m}(\xi_{q_m})]$ (diagram formula).

The Malliavin-Stein approach

Understand very well $\mathbb{E}[Z_n f(Z_n)]$ and use Stein's method. Nourdin, Peccati and Podolskij construct $\Gamma_1[Z_n]$ satisfying

$$\mathbb{E}[\Gamma_1[Z_n]f'(Z_n) - Z_n f(Z_n)] = 0.$$

This allows to obtain bounds for

$$\mathbb{E}[f'(Z_n) - Z_n f(Z_n)],$$

in terms of $\mathbb{E}[|\Gamma_1[Z_n] - 1|]$, which is combined with Stein's method.

Constructing Γ_1 in a simple case

Simplified problem

Suppose $N \sim \mathcal{N}(0,1)$ and we want to understand $\mathbb{E}[Yf(Y)]$ for Y measurable with respect to N.

Constructing Γ_1 in a simple case

Simplified problem

Suppose $N \sim \mathcal{N}(0,1)$ and we want to understand $\mathbb{E}[Yf(Y)]$ for Y measurable with respect to N.

Further simplifications

- If $f(x) = x$, we need to deal with $\mathbb{E}[Y^2]$.

Constructing Γ_1 in a simple case

Simplified problem

Suppose $N \sim \mathcal{N}(0,1)$ and we want to understand $\mathbb{E}[Yf(Y)]$ for Y measurable with respect to N.

Further simplifications

- If $f(x) = x$, we need to deal with $\mathbb{E}[Y^2]$.
- Since $\mathbb{E}[Y^2] = \langle Y, Y \rangle_{L^2(\Omega)}$, we could try orthogonality arguments.

Constructing Γ_1 in a simple case

Simplified problem

Suppose $N \sim \mathcal{N}(0,1)$ and we want to understand $\mathbb{E}[Yf(Y)]$ for Y measurable with respect to N.

Further simplifications

- If $f(x) = x$, we need to deal with $\mathbb{E}[Y^2]$.
- Since $\mathbb{E}[Y^2] = \langle Y, Y\rangle_{L^2(\Omega)}$, we could try orthogonality arguments.
- Since $Y = \psi(N)$ for some ψ, we could try to expand ψ in Hermite.

Constructing Γ_1 in a simple case

Simplified problem

Suppose $N \sim \mathcal{N}(0,1)$ and we want to understand $\mathbb{E}[Yf(Y)]$ for Y measurable with respect to N.

Further simplifications

- If $f(x) = x$, we need to deal with $\mathbb{E}[Y^2]$.
- Since $\mathbb{E}[Y^2] = \langle Y, Y \rangle_{L^2(\Omega)}$, we could try orthogonality arguments.
- Since $Y = \psi(N)$ for some ψ, we could try to expand ψ in Hermite.
- The variable $f(Y)$ can also be expanded in a Hermite basis.

Constructing Γ_1 in a simple case

Theorem
If $F = \varphi(N)$, for very smooth and integrable φ, then

$$F = \sum_{q=0}^{\infty} a_q H_q(N), \qquad \textit{or} \qquad \varphi = \sum_{q=0}^{\infty} a_q H_q$$

where $a_q = \mathbb{E}[D^q \varphi(N)]/q!$.

Constructing Γ_1 in a simple case

Theorem
If $F = \varphi(N)$, for very smooth and integrable φ, then

$$F = \sum_{q=0}^{\infty} a_q H_q(N), \qquad \text{or} \qquad \varphi = \sum_{q=0}^{\infty} a_q H_q$$

where $a_q = \mathbb{E}[D^q \varphi(N)]/q!$.

Corollary
If $F = \varphi(N)$, for smooth and integrable enough φ,

$$\mathbb{E}[H_q(N)\varphi(N)] = \mathbb{E}[D^q \varphi(N)].$$

Constructing Γ_1 in a simple case

Theorem
If $Y = H_q(N)$*, and* $\Gamma_1[Y] = (\frac{1}{q} H_{q-1}(N)) \cdot (\frac{1}{q} H_{q-1}(N))$*, then*

$$\mathbb{E}[\Gamma_1[Y] f'(Y)] - \mathbb{E}[Y f(Y)] = 0.$$

Constructing Γ_1 in a simple case

Theorem
If $Y = H_q(N)$, *and* $\Gamma_1[Y] = (\frac{1}{q}H_{q-1}(N)) \cdot (\frac{1}{q}H_{q-1}(N))$, *then*

$$\mathbb{E}[\Gamma_1[Y]f'(Y)] - \mathbb{E}[Yf(Y)] = 0.$$

Proof.

Constructing Γ_1 in a simple case

Theorem
If $Y = H_q(N)$, and $\Gamma_1[Y] = (\frac{1}{q} H_{q-1}(N)) \cdot (\frac{1}{q} H_{q-1}(N))$, then

$$\mathbb{E}[\Gamma_1[Y] f'(Y)] - \mathbb{E}[Y f(Y)] = 0.$$

Proof.

Keep in mind that for adequate φ, we have
$\mathbb{E}[H_q(N)\varphi(N)] = \mathbb{E}[D^q \varphi(N)]$.

- Show that $\mathbb{E}[H_q(N)\varphi(N)] = \mathbb{E}[H_{q-1}(N)\varphi'(N)]$

$\square$

Constructing Γ_1 in a simple case

Theorem
If $Y = H_q(N)$*, and* $\Gamma_1[Y] = (\frac{1}{q}H_{q-1}(N)) \cdot (\frac{1}{q}H_{q-1}(N))$*, then*

$$\mathbb{E}[\Gamma_1[Y]f'(Y)] - \mathbb{E}[Yf(Y)] = 0.$$

Proof.
Keep in mind that for adequate φ, we have $\mathbb{E}[H_q(N)\varphi(N)] = \mathbb{E}[D^q\varphi(N)]$.

- Show that $\mathbb{E}[H_q(N)\varphi(N)] = \mathbb{E}[H_{q-1}(N)\varphi'(N)]$
- Take $\varphi = f \circ H_q$ and use the fact that $H_q' = qH_{q-1}$.

□

Constructing Γ_1 in a simple case

Theorem
If $Y = H_q(N)$*, and* $\Gamma_1[Y] = (\frac{1}{q}H_{q-1}(N)) \cdot (\frac{1}{q}H_{q-1}(N))$*, then*

$$\mathbb{E}[\Gamma_1[Y]f'(Y)] - \mathbb{E}[Yf(Y)] = 0.$$

Proof.
Keep in mind that for adequate φ, we have $\mathbb{E}[H_q(N)\varphi(N)] = \mathbb{E}[D^q\varphi(N)]$.

- Show that $\mathbb{E}[H_q(N)\varphi(N)] = \mathbb{E}[H_{q-1}(N)\varphi'(N)]$
- Take $\varphi = f \circ H_q$ and use the fact that $H_q' = qH_{q-1}$.

□

Constructing Γ_1 in a simple case

Theorem

If $Y = H_q(N)$*, and* $\Gamma_1[Y] = (\frac{1}{q} H_{q-1}(N)) \cdot (\frac{1}{q} H_{q-1}(N))$*, then*

$$\mathbb{E}[\Gamma_1[Y] f'(Y)] - \mathbb{E}[Y f(Y)] = 0.$$

Proof.

Keep in mind that for adequate φ, we have $\mathbb{E}[H_q(N)\varphi(N)] = \mathbb{E}[D^q \varphi(N)]$.

- Show that $\mathbb{E}[H_q(N)\varphi(N)] = \mathbb{E}[H_{q-1}(N)\varphi'(N)]$
- Take $\varphi = f \circ H_q$ and use the fact that $H_q' = qH_{q-1}$.

□

Same ideas can be carried to more general functions and settings.

The Malliavin-Stein approach

If we can construct $\Gamma_1[Z_n]$ such that

$$\mathbb{E}[\Gamma_1[Z_n]f'(Z_n) - Z_n f(Z_n)] = 0,$$

Recalling Malliavin-Stein method in BM

The Malliavin-Stein approach

If we can construct $\Gamma_1[Z_n]$ such that

$$\mathbb{E}[\Gamma_1[Z_n]f'(Z_n) - Z_n f(Z_n)] = 0,$$

we can obtain bounds for

$$\mathbb{E}[f'(Z_n) - Z_n f(Z_n)],$$

in terms of $\mathbb{E}[|\Gamma_1[Z_n] - 1|]$

Rates of convergence

Theorem (Sefika Kuzgun, David Nualart (2019))
Define $\sigma_n^2 := Var[Z_n]$. Suppose g is twice Malliavin differentiable and integrable enough and $Q \geq 3$. Then

$$d_{TV}(Z_n/\sigma_n, \mathcal{N}(0,1)) \leq Cn^{-1/2} \sum_{|k|\leq n} |\rho(k)|^{Q-1} \left(\sum_{|k|\leq n} |\rho(k)|^2 \right)^{1/2} + Cn^{-1/2} \left(\sum_{|k|\leq n} |\rho(k)|^2 \right)^{1/2} \left(\sum_{|k|\leq n} |\rho(k)| \right)^{1/2}$$

Results for $Q = 2$ are also available.

Rates of convergence

Theorem (Sefika Kuzgun, David Nualart (2019))
Define $\sigma_n^2 := Var[Z_n]$. Suppose g is twice Malliavin differentiable and integrable enough and $Q \geq 3$. Then

$$d_{TV}(Z_n/\sigma_n, \mathcal{N}(0,1)) \leq Cn^{-1/2} \sum_{|k|\leq n} |\rho(k)|^{Q-1} \left(\sum_{|k|\leq n} |\rho(k)|^2 \right)^{1/2} + Cn^{-1/2} \left(\sum_{|k|\leq n} |\rho(k)|^2 \right)^{1/2} \left(\sum_{|k|\leq n} |\rho(k)| \right)^{1/2}$$

Results for $Q = 2$ are also available.

Test generality taking $\rho(k) = O(k^{-\alpha})$.

Rates of convergence

Theorem (Sefika Kuzgun, David Nualart (2019))
Define $\sigma_n^2 := Var[Z_n]$. Suppose g is twice Malliavin differentiable and integrable enough and $Q \geq 3$. Then

$$d_{TV}(Z_n/\sigma_n, \mathcal{N}(0,1)) \leq Cn^{-1/2} \sum_{|k|\leq n} |\rho(k)|^{Q-1} \left(\sum_{|k|\leq n} |\rho(k)|^2 \right)^{1/2} + Cn^{-1/2} \left(\sum_{|k|\leq n} |\rho(k)|^2 \right)^{1/2} \left(\sum_{|k|\leq n} |\rho(k)| \right)^{1/2}$$

Results for $Q = 2$ are also available.

Test generality taking $\rho(k) = O(k^{-\alpha})$. Worst case: $\alpha \approx -1/Q$.

Rates of convergence pt. II

Many related results are available!

- David Nualart and Hongjuan Zhou (2018). Optimal bounds, but only for the cases $Q = 1$ and $Q = 2$.

Rates of convergence pt. II

Many related results are available!

- David Nualart and Hongjuan Zhou (2018). Optimal bounds, but only for the cases $Q = 1$ and $Q = 2$.
- Ivan Nourdin, Giovanni Peccati, Xiaochuan Yang (2019). Optimal bounds, minimal smoothness in g, but only $Q = 1$.

Rates of convergence pt. II

Many related results are available!

- David Nualart and Hongjuan Zhou (2018). Optimal bounds, but only for the cases $Q = 1$ and $Q = 2$.
- Ivan Nourdin, Giovanni Peccati, Xiaochuan Yang (2019). Optimal bounds, minimal smoothness in g, but only $Q = 1$.
- Nourdin, Nualart, Peccati (2019). Optimal bounds, minimal smoothness on g, but only $Q = 2$.

Rates of convergence pt. II

Many related results are available!

- David Nualart and Hongjuan Zhou (2018). Optimal bounds, but only for the cases $Q = 1$ and $Q = 2$.
- Ivan Nourdin, Giovanni Peccati, Xiaochuan Yang (2019). Optimal bounds, minimal smoothness in g, but only $Q = 1$.
- Nourdin, Nualart, Peccati (2019). Optimal bounds, minimal smoothness on g, but only $Q = 2$.
- Guillaume Poly (2023). Optimal bounds, minimal smoothnes and minimal integrability on g, but only $Q = 2$.

Rates of convergence pt. II

Many related results are available!

- David Nualart and Hongjuan Zhou (2018). Optimal bounds, but only for the cases $Q = 1$ and $Q = 2$.
- Ivan Nourdin, Giovanni Peccati, Xiaochuan Yang (2019). Optimal bounds, minimal smoothness in g, but only $Q = 1$.
- Nourdin, Nualart, Peccati (2019). Optimal bounds, minimal smoothness on g, but only $Q = 2$.
- Guillaume Poly (2023). Optimal bounds, minimal smoothnes and minimal integrability on g, but only $Q = 2$.
- A general sharp statement for $Q \geq 3$ seems to not be stated in detail.

Free Breuer Major theorem

Let U_q denote the q-th Chebyshev polynomial of the second kind.

Free Breuer Major theorem

Let U_q denote the q-th Chebyshev polynomial of the second kind.

Theorem (Kemp, Nourdin, Peccati, Speicher (2012))
Let $\{\xi_k\}_{k\geq 1}$ be a stationary standardized semicircular process. Suppose that $g = U_q$ and let Z_n be given by

$$Z_n := \frac{1}{\sqrt{n}} \sum_{k=0}^{n} g(\xi_k).$$

Then $Z_n \stackrel{Law}{\to} \mathcal{S}(0, \sigma^2)$, where $\mathcal{S}(\mu, \sigma)$ denotes the semicircle law and $\sigma > 0$.

Free Breuer Major theorem

Let U_q denote the q-th Chebyshev polynomial of the second kind.

Theorem (Kemp, Nourdin, Peccati, Speicher (2012))
Let $\{\xi_k\}_{k\geq 1}$ be a stationary standardized semicircular process. Suppose that $g = U_q$ and let Z_n be given by

$$Z_n := \frac{1}{\sqrt{n}} \sum_{k=0}^{n} g(\xi_k).$$

Then $Z_n \overset{Law}{\to} \mathcal{S}(0, \sigma^2)$, where $\mathcal{S}(\mu, \sigma)$ denotes the semicircle law and $\sigma > 0$.

A sharp quantitative version using Malliavin calculus was proved by Bourgin, Campese (2017).

Free Breuer Major theorem

Let U_q denote the q-th Chebyshev polynomial of the second kind.

Theorem (Kemp, Nourdin, Peccati, Speicher (2012))
Let $\{\xi_k\}_{k\geq 1}$ be a stationary standardized semicircular process. Suppose that $g = U_q$ and let Z_n be given by

$$Z_n := \frac{1}{\sqrt{n}} \sum_{k=0}^{n} g(\xi_k).$$

Then $Z_n \stackrel{Law}{\to} \mathcal{S}(0, \sigma^2)$, where $\mathcal{S}(\mu, \sigma)$ denotes the semicircle law and $\sigma > 0$.

A sharp quantitative version using Malliavin calculus was proved by Bourgin, Campese (2017).

Ongoing research: consider a general g.

The basic tools: Free Malliavin derivative

For convenience, we will assume that $\xi_k = \int_{\mathbb{R}_+} h_k(s) dW_s$, where W is a free Brownian motion and $h_k \in L^2(\mathbb{R}_+; \mathbb{R})$.

The basic tools: Free Malliavin derivative

For convenience, we will assume that $\xi_k = \int_{\mathbb{R}_+} h_k(s) dW_s$, where W is a free Brownian motion and $h_k \in L^2(\mathbb{R}_+; \mathbb{R})$.

Definition
The Malliavin derivative of an element $A \in \mathcal{A}$ is the process $\nabla = \{\nabla_t[A] \ ; \ t \geq 0\}$, taking values in $\mathcal{A} \otimes \mathcal{A}$, satisfying

- For all $h \in \mathfrak{H}$, $\nabla_t[\int_{\mathbb{R}_+} h_k(s) dW_s] = h(t) \cdot (1 \otimes 1)$.
- For all $a, b \in \mathcal{A}$, it holds that

$$\nabla[ab] = \nabla[a] \cdot b + a \cdot \nabla[b],$$

  where $\cdot$ denotes the action of $\mathcal{A}$ on $\mathcal{A} \otimes \mathcal{A}$ over the right leg when the multiplication is of the form $F \cdot a$, with $F \in \mathcal{A} \otimes \mathcal{A}$ and $a \in \mathcal{A}$.

The basic tools: Free Malliavin derivative

For convenience, we will assume that $\xi_k = \int_{\mathbb{R}_+} h_k(s)dW_s$, where W is a free Brownian motion and $h_k \in L^2(\mathbb{R}_+;\mathbb{R})$.

Definition
The Malliavin derivative of an element $A \in \mathcal{A}$ is the process $\nabla = \{\nabla_t[A]\ ;\ t \geq 0\}$, taking values in $\mathcal{A} \otimes \mathcal{A}$, satisfying

- For all $h \in \mathfrak{H}$, $\nabla_t[\int_{\mathbb{R}_+} h_k(s)dW_s] = h(t) \cdot (1 \otimes 1)$.
- For all $a, b \in \mathcal{A}$, it holds that

$$\nabla[ab] = \nabla[a] \cdot b + a \cdot \nabla[b],$$

  where $\cdot$ denotes the action of $\mathcal{A}$ on $\mathcal{A} \otimes \mathcal{A}$ over the right leg when the multiplication is of the form $F \cdot a$, with $F \in \mathcal{A} \otimes \mathcal{A}$ and $a \in \mathcal{A}$. The action of $\mathcal{A}$ on $\mathcal{A} \otimes \mathcal{A}$ over the left leg when the multiplication is of the form $a \cdot F$

Free Stein kernel

For $a \in \mathcal{A}$, define

$$\Gamma_1[a] := \int_{\mathbb{R}_+} (\iota \otimes \tau)[\nabla_t[a]] \cdot (\nabla_t[a])^* dt.$$

with $\iota \otimes \tau$ given as the functional $(\iota \otimes \tau)[a \otimes b] = \tau[b]a$.

Free Stein kernel

For $a \in \mathcal{A}$, define

$$\Gamma_1[a] := \int_{\mathbb{R}_+} (\iota \otimes \tau)[\nabla_t[a]] \cdot (\nabla_t[a])^* dt.$$

with $\iota \otimes \tau$ given as the functional $(\iota \otimes \tau)[a \otimes b] = \tau[b]a$.

Theorem (Cebron 2018)

If $a \in \mathcal{A}$ is in the domain of ∇ and f is a polynomial,

$$(\tau \otimes \tau)[\Gamma_1[a]\partial f(a)] - \tau[af(a)] = 0$$

where $\partial f : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ denotes the non-commutative derivative.

Free Stein kernel

For $a \in \mathcal{A}$, define

$$\Gamma_1[a] := \int_{\mathbb{R}_+} (\iota \otimes \tau)[\nabla_t[a]] \cdot (\nabla_t[a])^* dt.$$

with $\iota \otimes \tau$ given as the functional $(\iota \otimes \tau)[a \otimes b] = \tau[b]a$.

Theorem (Cebron 2018)

If $a \in \mathcal{A}$ is in the domain of ∇ and f is a polynomial,

$$(\tau \otimes \tau)[\Gamma_1[a]\partial f(a)] - \tau[af(a)] = 0$$

where $\partial f : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ denotes the non-commutative derivative. Recall that a is semicircular if

$$(\tau \otimes \tau)[\partial f(a)] - \tau[af(a)] = 0$$

Free Malliavin-Stein

Theorem (Cebron 2018)

If S is a semicircular random variable and $F \in \mathcal{A}$ is sufficiently smooth and integrable, the 2-Wasserstein distance d_{W_2} satisfies

$$d_{W_2}(F, S) \leq \|\Gamma_1[F] - 1_{\mathcal{A}} \otimes 1_{\mathcal{A}}\|_{L^2(\mathcal{A},\tau)}. \tag{1}$$

Free Malliavin-Stein

Theorem (Cebron 2018)

If S is a semicircular random variable and $F \in \mathcal{A}$ is sufficiently smooth and integrable, the 2-Wasserstein distance d_{W_2} satisfies

$$d_{W_2}(F, S) \leq \|\Gamma_1[F] - 1_{\mathcal{A}} \otimes 1_{\mathcal{A}}\|_{L^2(\mathcal{A},\tau)}. \tag{1}$$

Fundamental applications:

Free Malliavin-Stein

Theorem (Cebron 2018)

If S is a semicircular random variable and $F \in \mathcal{A}$ is sufficiently smooth and integrable, the 2-Wasserstein distance d_{W_2} satisfies

$$d_{W_2}(F, S) \leq \|\Gamma_1[F] - 1_{\mathcal{A}} \otimes 1_{\mathcal{A}}\|_{L^2(\mathcal{A},\tau)}. \tag{1}$$

Fundamental applications:

- The case where F is a Wigner integral.

Free Malliavin-Stein

Theorem (Cebron 2018)

If S is a semicircular random variable and $F \in \mathcal{A}$ is sufficiently smooth and integrable, the 2-Wasserstein distance d_{W_2} satisfies

$$d_{W_2}(F, S) \leq \|\Gamma_1[F] - 1_{\mathcal{A}} \otimes 1_{\mathcal{A}}\|_{L^2(\mathcal{A},\tau)}. \tag{1}$$

Fundamental applications:

- The case where F is a Wigner integral.
- The case where $F = Z_n$, where

$$Z_n := \frac{1}{\sqrt{n}} \sum_{k=0}^{n} g(\xi_k).$$

Free Malliavin-Stein applied to free Breuer-Major

Let Z_n be as before,

Free Malliavin-Stein applied to free Breuer-Major

Let Z_n be as before,

$$\Gamma_1[Z_n] := \int_{\mathbb{R}_+} (\iota \otimes \tau)[\nabla_t[Z_n]] \cdot (\nabla_t[Z_n])^* dt.$$

Free Malliavin-Stein applied to free Breuer-Major

Let Z_n be as before,

$$\Gamma_1[Z_n] := \int_{\mathbb{R}_+} (\iota \otimes \tau)[\nabla_t[Z_n]] \cdot (\nabla_t[Z_n])^* dt.$$

We have that

$$\nabla_t[Z_n] = \frac{1}{\sigma_n} \sum_{k=1}^{n} h_k(t) \partial g(\xi_k).$$

Free Malliavin-Stein applied to free Breuer-Major

Let Z_n be as before,

$$\Gamma_1[Z_n] := \int_{\mathbb{R}_+} (\iota \otimes \tau)[\nabla_t[Z_n]] \cdot (\nabla_t[Z_n])^* dt.$$

We have that

$$\nabla_t[Z_n] = \frac{1}{\sigma_n} \sum_{k=1}^{n} h_k(t) \partial g(\xi_k).$$

From here it follows that

$$\Gamma_1[Z_n] = \frac{1}{\sigma_n^2} \sum_{1 \leq k_1, k_2 \leq n} (\iota \otimes \tau)[\partial g(\xi_{k_1})] \cdot \partial g(\xi_{k_2}) \rho(k_2 - k_1),$$

Some calculations later...

If $g = \sum_{q \geq Q} a_q U_q$, then

$$
\begin{aligned}
&\|\Gamma_1[Z_n] - (\tau \otimes \tau)[\Gamma_1[Z_n]]\|^2_{L^2(\mathcal{A}\otimes\mathcal{A},\tau\otimes\tau)} \\
&= \frac{1}{n^2} \sum_{1 \leq k_1, k_2, \kappa_1, \kappa_2 \leq n} \sum_{q_1, q_2, p_1, p_2 \geq Q} \sum_{1 \leq \ell_1 \leq q_2} \sum_{1 \leq \ell_2 \leq p_2} \sum_{j_1=0}^{(q_1-1)\wedge(\ell_1-1)} \sum_{j_2=0}^{(p_1-1)\wedge(\ell_2-1)} \\
&\times \mathbb{1}_{\{\{q_1+\ell_1-2j_1=p_1+\ell_2-2j_2\}\}} \mathbb{1}_{\{\{q_2-\ell_1=p_2-\ell_2\}\}} \\
&\times \mathbb{1}_{\{\{(q_1,\ell_1,j_1+1)\neq(q_2,q_2,q_2)\}\}} \mathbb{1}_{\{\{(p_1,\ell_2,j_2+1)\neq(p_2,p_2,p_2)\}\}} \\
&\times a_{q_1} a_{q_2} a_{p_1} a_{p_2} \rho(k_2 - k_1)^{j_1+1} \rho(\kappa_2 - \kappa_1)^{j_2+1} \\
&\times \langle f_{k_1}^{\otimes(q_1-1-j_1)} \otimes f_{k_2}^{\otimes(\ell_1-1-j_1)}, f_{\kappa_1}^{\otimes(p_1-1-j_2)} \otimes f_{\kappa_2}^{\otimes(\ell_2-1-j_2)} \rangle_{\mathfrak{H}^{\otimes(q_1+\ell_1-2-2j_1)}} \\
&\times \rho(k_2 - \kappa_2)^{q_2-\ell_1}.
\end{aligned}
$$

Main result?

Theorem (Conjecture: Jaramillo, Kuzgun)
Define

$$\mathfrak{p}(r) := \sup_{|k|\geq|r|} |\rho(r)|.$$

Main result?

Theorem (Conjecture: Jaramillo, Kuzgun)
Define

$$\mathfrak{p}(r) := \sup_{|k| \geq |r|} |\rho(r)|.$$

Then we can guarantee the existence of a universal constant $C_Q > 0$ only depending on Q, such that if $Q \geq 2$,

$$d_{W_2}(Z_n, S) \leq C_Q \|g\|_{\mathbb{D}^{2,2}} n^{-1/2} \left(\sum_{r=1}^{n} \mathfrak{p}(r)^{Q-1} \right)^{1/2} \left(\sum_{r=1}^{n} \mathfrak{p}(r) \right)^{1/2}.$$

Main result?

Theorem (Conjecture: Jaramillo, Kuzgun)
Define

$$\mathfrak{p}(r) := \sup_{|k| \geq |r|} |\rho(r)|.$$

Then we can guarantee the existence of a universal constant $C_Q > 0$ only depending on Q, such that if $Q \geq 2$,

$$d_{W_2}(Z_n, S) \leq C_Q \|g\|_{\mathbb{D}^{2,2}} n^{-1/2} \left(\sum_{r=1}^{n} \mathfrak{p}(r)^{Q-1} \right)^{1/2} \left(\sum_{r=1}^{n} \mathfrak{p}(r) \right)^{1/2}.$$

and if $Q = 1$,

$$d_{W_2}(Z_n, S) \leq C_Q \|g\|_{\mathbb{D}^{2,2}} \frac{1}{\sqrt{n}}.$$

Related lines of research

- The result is not yet proved with this level of generality in the classical version.

Related lines of research

- The result is not yet proved with this level of generality in the classical version.
- Expansions (Edgeworth-type expansions, but with smoother distances) seem to be reasonable to study, both in classical and free versions.

Related lines of research

- The result is not yet proved with this level of generality in the classical version.
- Expansions (Edgeworth-type expansions, but with smoother distances) seem to be reasonable to study, both in classical and free versions.
- Uniform convergence of densities are available in the classical case. Is there an analog in the free version?

Related lines of research

- The result is not yet proved with this level of generality in the classical version.
- Expansions (Edgeworth-type expansions, but with smoother distances) seem to be reasonable to study, both in classical and free versions.
- Uniform convergence of densities are available in the classical case. Is there an analog in the free version?
- What happens if $\sum_k |\rho(k)|^Q = \infty$? Free version of Rosenblatt variable? functional versions of Breuer Major.

Gracias!

Contacto

Arturo Jaramillo
jagil@cimat.mx

References classical BM

- Kuzgun, Nualart (2019). Rate of convergence in the Breuer-Major theorem via chaos expansions. Stochastic Analysis and Applications.
- Nualart and H. Zhou. Total variation estimates in the Breuer-Major theorem (2018). Annals of the Institut Henri Poincare.
- Nourdnin, Peccati, Yang (2019). Berry-Esseen bounds in the Breuer-Major CLT and Gebelein's inequality. Electron. Commun. Probab.
- Nourdin, Nualart and Peccati (2021). The Breuer-Major theorem in total variation: improved rates of convergence under minimal regularity. Stochastic Processes and Their Applications.
- Angst, Dalmao, Poly (2023). A total variation version of Breuer-Major Central Limit Theorem under $\mathbb{D}^{1,2}$-assumption. Arxiv

References Free BM

- Cebron (2018). Quantitative fourth moment theorem in free probability. Advances in mathematics
- Biane, Speicher (1998). Stochastic calculus with respect to free Brownian motion and analysis on Wigner space. Probability Theory and Related Fields
- Bourguin, Campese (2018). Free quantitative fourth moment theorems on Wigner space. International Mathematics Research Notices.
- Kemp, Nourdin, Peccati, Speicher (2010). Wigner chaos and the fourth moment. Annals of Probability.