Convergence of the empirical spectral distribution of Gaussian matrix processes

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Basic definitions

Consider a family of independent centered Gaussian processes $\{X_{i,j}\}_{i,j\in\mathbb{N}}$ with covariance function

$$R(s,t)=\mathbb{E}\left[X_{i,j}(s)X_{i,j}(t)\right],$$

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defined in a probability space $(\Omega,\mathcal{F},\mathbb{P})$. Consider as well the renormalized symmetric Gaussian matrix process $\{Y^{(n)}\}_{n\geq 1}$, defined by $Y^{(n)}(t):=\{Y_{i,j}^{(n)}(t)\}_{1\leq i,j\leq n}$, with

$$Y_{i,j}^{(n)}(t):=\begin{cases}\frac{1}{\sqrt{n}}X_{i,j}(t) & \text{if} \quad i<j\\ \frac{\sqrt{2}}{\sqrt{n}}X_{i,i}(t) & \text{if} \quad i=j.\end{cases}$$

Goal

Let $A^{(n)} := \{A_{i,j}^{(n)}\}_{1\leq i,j\leq n}$ be a random symmetric matrix independent of $Y^{(n)}$, and $\lambda_1^{(n)}(t) \geq \cdots \geq \lambda_n^{(n)}(t)$ denote the *n*-dimensional process of eigenvalues of

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Denote by $Pr(\mathbb{R})$ the space of random probability distributions, endowed with the weak convergence of probability measures. We are interested in the asymptotic behavior of the $Pr(\mathbb{R})$-valued process of empirical distributions $\{\mu^{(n)}\}_{n\geq 1}$, defined by $\mu^{(n)} := \{\mu_t^{(n)}\}_{t\geq 0}$, with

$$\mu_t^{(n)} := \frac{1}{n}\sum_{j=1}^{n} \delta_{\lambda_j^{(n)}(t)}, \qquad t \geq 0.$$

Framework

Let $d \geq 1$ and $T > 0$ be fixed. Consider a d-dimensional continuous centered Gaussian process $V = \{(V_t^1, \ldots, V_t^d)\}_{t \in [0,T]}$, defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

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Denote by $\mathscr{E}$ the space of step functions on $[0, T]$. We define in $\mathscr{E}$ the scalar product

$$\left\langle \mathbb{1}_{[0,s]}, \mathbb{1}_{[0,t]} \right\rangle_{\mathfrak{H}} := \mathbb{E}\left[V_s^1 V_t^1\right], \quad \text{for} \quad s,t \in [0, T].$$

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Let $\mathfrak{H}$ be the Hilbert space obtained by taking the completion of $\mathscr{E}$ with respect to this product.

Smooth random variables

For every $1 \leq i \leq n$, the mapping $\mathbb{1}_{[0,t]} \mapsto V^i(\mathbb{1}_{[0,t]}) := V^i_t$ can be extended to linear isometry, which we denote by $V^i(h)$, for $h \in \mathfrak{H}$.

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$$V(f) := \sum_{i=1}^{d} V^i(f_i).$$

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Let $\mathscr{S}$ denote the set of random variables of the form

$$F = g(V(h_1), \ldots, V(h_m)),$$

where $g : \mathbb{R}^m \to \mathbb{R}$ is $\mathcal{C}^\infty(\mathbb{R})$, and $h_j \in \mathscr{E}^d$.

The Malliavin derivative

The Malliavin derivative of F with respect to V, is the element of $L^2(\Omega; \mathfrak{H}^d)$, defined by

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For $p \geq 1$, the set $\mathbb{D}^{1,p}$ denotes the closure of $\mathscr{S}$ with respect to the norm $\|\cdot\|_{\mathbb{D}^{1,p}}$, defined by

$$\|F\|_{\mathbb{D}^{1,p}} := \left(\mathbb{E}\left[|F|^p\right] + \mathbb{E}\left[\|DF\|_{\mathfrak{H}^d}^p\right]\right)^{\frac{1}{p}}.$$

The operator D can be consistently extended to the set $\mathbb{D}^{1,p}$.

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$$\left|\mathbb{E}\left[\langle DF, u\rangle_{\mathfrak{H}^d}\right]\right| \leq C_u \mathbb{E}\left[F^2\right]^{\frac{1}{2}}, \quad \text{for every } F \in \mathbb{D}^{1,2},$$

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We will make use of the notation

$$\sum_{i=1}^{d} \int_0^t u_s^i \delta V_s^i := \delta(u \mathbb{1}_{[0,t]}).$$

Some technical issues associated to δ

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2. Can we interpret $\sum_{i=1}^{d} \int_0^t u_s^i \delta V_s^i$ as the stochastic integral of u?
3. In the case where V is a fractional Brownian motion with Hurst parameter $0 < H < \frac{1}{4}$, with covariance function

$$R(s,t) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}),$$

the trajectories of V do not belong to the space $\mathfrak{H}$, and hence, do not belong to the domain of δ.

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(H1) There exists $\alpha > 1$, such that for all $t \in [0, T]$, $s \mapsto R(s, t)$ is absolutely continuous on $[0, T]$, and

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Let β be the conjugate of α, defined by $\beta := \frac{\alpha}{\alpha - 1}$. For any pair of functions $\varphi \in L^{\beta}[0, T]$ and $\psi \in \mathscr{E}$ of the form $\psi = \sum_{j=1}^{m} c_j \mathbb{1}_{[0,t_j]}$, we define

$$\langle \psi, \varphi \rangle_{\mathfrak{H}} := \sum_{j=1}^{m} c_j \int_0^T \varphi(s) \frac{\partial R}{\partial s}(s, t_j) ds. \tag{1}$$

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for any smooth random variable $F \in \mathscr{S}$, where C_u is some constant depending on u. In this case, $\delta(u)$ is defined by the duality relationship

$$\mathbb{E}\left[F\delta(u)\right] = \mathbb{E}\left[\langle DF, u\rangle_{\mathfrak{H}^d}\right].$$

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1. *F is twice continuously differentiable.*
2. *There exist constants $C > 0$ and $M > 0$, such that*

$$|F(x)| + \left|\frac{\partial F}{\partial x_i}(x)\right| \leq C(1 + |x|^M), \qquad \text{for} \;\; i = 1, \dots, d. \tag{2}$$

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3. *There exists $0 < \delta < 1$, such that for every $p \geq 1$, and $s > 0$,*

$$\mathbb{E}\left[\left|\frac{\partial^2 F}{\partial x_i^2}(V_s)\right|^p\right] \leq C(1 + R(s,s)^{-p(1-\delta)}), \qquad \text{for } \quad i = 1, \dots, d. \tag{3}$$

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Then, the process $u_s = (u_s^1, \dots, u_s^d)$, *defined by* $u_s^l := \frac{\partial F}{\partial x_l}(V_s)\mathbb{1}_{[0,t]}(s)$, *belongs to* $\text{Dom}^*\delta$, *and*

$$F(V_t) = F(0) + \sum_{i=1}^{d} \int_0^t \frac{\partial F}{\partial x_i}(V_s)\delta V_s^i + \frac{1}{2}\sum_{i=1}^{d}\int_0^t \frac{\partial^2 F}{\partial x_i^2}(V_s)\frac{dR(s,s)}{ds}ds, \tag{4}$$

for every $t \in [0, T]$.

Back to our problem

Recall that

$$X^{(n)}(t) := A^{(n)} + Y^{(n)}(t),$$

and

$$\mu_t^{(n)} := \frac{1}{n}\sum_{j=1}^{n} \delta_{\lambda_j^{(n)}(t)}, \qquad t \geq 0.$$

For an element f in the space of continuously differentiable functions in $\mathbb{R}$, denoted by $\mathcal{C}_b^2(\mathbb{R})$, define

$$\langle \mu, f \rangle := \int f(x)\mu(dx).$$

Eigenvalues of a matrix

Lemma

For every $i = 1, \ldots, n$*, there exists a function* $\Phi_i^n : \mathbb{R}^{\frac{n(n+1)}{2}} \to \mathbb{R}$*, which is infinitely differentiable in an open subset* $G \subset \mathbb{R}^{\frac{n(n+1)}{2}}$*, with* $|G^c| = 0$*, such that* $\lambda_i^{(n)}(t) = \Phi_i^n(Z^{(n)}(t))$*.*

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Moreover, every element $X \in G$, viewed as an $n \times n$ matrix, has a factorization of the form $X = UDU^*$, where D is a diagonal matrix with entries $D_{i,i} = \lambda_i^n$ such that $\lambda_1^n > \cdots > \lambda_n^n$, U^n is an orthogonal matrix with $U_{i,i}^n > 0$ for all i, $U_{i,j}^n \neq 0$, all the minors of U^n have non zero determinants.

Some properties of Φ_i

Lemma

If U and Φ_i are as before, then for any $k \leq h$, we have

$$\frac{\partial \Phi_i^n}{\partial y_{k,h}} = 2U_{i,k}^n U_{i,h}^n \mathbb{1}_{\{k \neq h\}} + \sqrt{2}(U_{i,k}^n)^2 \mathbb{1}_{\{k=h\}}, \tag{5}$$

$$\frac{\partial^2 \Phi_i^n}{\partial y_{k,h}^2}, = 2\sum_{j \neq i} \frac{\left| U_{i,k}^n U_{j,h}^n + U_{i,h}^n U_{j,k}^n \right|^2}{\lambda_i^n - \lambda_j^n} \mathbb{1}_{\{k \neq h\}} + 4\sum_{j \neq i} \frac{\left| U_{i,k}^n U_{j,k}^n \right|^2}{\lambda_i^n - \lambda_j^n} \mathbb{1}_{\{k=h\}}. \tag{6}$$

Stochastic evolution of $\mu^{(n)}$

Lemma

Assume that $t \mapsto R(t,t)$ is continuously differentiable. Then we have the following evolution in time for $\left\langle \mu_t^{(n)}, f \right\rangle$, for $f \in \mathcal{C}_b^2(\mathbb{R})$ and $t \geq 0$

$$\begin{aligned}\left\langle \mu_t^{(n)}, f \right\rangle &= \left\langle \mu_0^{(n)}, f \right\rangle \\ &+ \frac{1}{n^{\frac{3}{2}}} \sum_{i=1}^{n} \sum_{k \leq h} \int_0^t f'(\Phi_i^n(X^{(n)}(s))) \frac{\partial \Phi_i^n}{\partial y_{k,l}}(X^{(n)}(s)) \delta X_{k,h}(s) \\ &+ \frac{1}{2} \int_0^t \int_{\mathbb{R}^2} \frac{f'(x) - f'(y)}{x - y} \frac{d}{ds} R(s,s) \mu_s^{(n)}(dx) \mu_s^{(n)}(dy) ds \\ &+ \frac{1}{2n^2} \sum_{i=1}^{n} \int_0^t f''(\Phi_i^n(X^{(n)}(s))) \frac{d}{ds} R(s,s) ds.\end{aligned} \tag{7}$$

Tightness of $\mu^{(n)}$

Proposition

Assume that $R(s,t)$ satisfies the condition
(H2) *There exist constants $\kappa, \gamma > 0$, such that for every $s, t > 0$,*

$$R(s,s) - 2R(s,t) + R(t,t) \leq \kappa \left| t - s \right|^{\gamma}.$$

Furthermore, assume that with probability one, the sequence of measures $\mu_0^{(n)}$ converges weakly to some measure ν. Then, almost surely, the family of measures $\{\mu^{(n)}\}_{n \geq 1}$ is tight in the space $C(\mathbb{R}_+, Pr(\mathbb{R}))$.

Tightness of $\mu^{(n)}$

Proof.

It suffices to show that for every $f \in \mathcal{C}^1(\mathbb{R})$, the process $\{\langle \mu_t^{(n)}, f\rangle \mid n \geq 1, t \geq 0\}$ is tight. Since $\mu_0^{(n)}$ converges weakly, by the Billingsley criterion, it suffices to show that there exist constants $C, p, q > 0$, independent of n, such that for every $0 \leq t_1 \leq t_2$,

$$\mathbb{E}\left[\left|\langle \mu_{t_1}^{(n)}, f\rangle - \langle \mu_{t_2}^{(n)}, f\rangle\right|^p\right] \leq C\left|t_2 - t_1\right|^q. \tag{8}$$

The left hand side can be estimated by using the inequality

$$\begin{aligned}\left(\frac{1}{n}\sum_{i=1}^{n}\left|\lambda_i(t) - \lambda_j(s)\right|\right)^2 &\leq \frac{1}{n}\sum_{i=1}^{n}\left(\lambda_i(t) - \lambda_j(s)\right)^2 \\ &\leq \frac{1}{n} tr\left[\left(\frac{1}{\sqrt{n}}(B_{i,j}(t) - B_{i,j}(s))\right)^2\right]\end{aligned}$$

Weak convergence of $\mu^{(n)}$

By the previous result $\mu^{(n)}$ has a subsequence $\mu^{(n_r)}$ converging in law to some random process μ. In addition, we have

Proposition

For every $t > 0$ fixed, the random variable

$$G_r := \frac{1}{n_r^{\frac{3}{2}}} \sum_{i=1}^{n_r} \sum_{k \leq h} \int_0^t f'(\Phi_i^{n_r}(X^{(n_r)}(s))) \frac{\partial \Phi_i^{n_r}}{\partial y_{k,l}}(X^{(n_r)}(s)) \delta X_{k,h}(s), \tag{9}$$

converges to zero in $L^2(\Omega)$.

Weak convergence of $\mu^{(n)}$

Proof.

From the evolution equation for $\mu^{(n)}$, we have that

$$\begin{aligned} G_r &= \left\langle \mu_t^{(n_r)}, f \right\rangle - \left\langle \mu_0^{(n_r)}, f \right\rangle \\ &- \frac{1}{2}\int_0^t \int_{\mathbb{R}^2} \frac{f'(x) - f'(y)}{x - y} \frac{d}{ds} R(s,s) \mu_s^{(n_r)}(dx) \mu_s^{(n_r)}(dy) ds \\ &- \frac{1}{2n_r^2} \sum_{i=1}^{n_r} \int_0^t f''(\Phi_i^n(X^{(n_r)}(s))) \frac{d}{ds} R(s,s) ds, \end{aligned} \tag{10}$$

So we can make estimations of the variance of G_r by using the duality relation of δ and the fact that

$$\mathbb{E}\left[G_r^2\right] = \mathbb{E}\left[\delta(\text{equation 9}) \times (\text{equation 10})\right]$$

Weak convergence of $\mu^{(n)}$

Using once more the evolution equation for μ, as well as the convergence of $\mu^{(n_r)}$ and the previous proposition, we get

$$\langle \mu_t, f\rangle = \langle \nu, f\rangle + \frac{1}{2}\int_0^t \int_{\mathbb{R}^2} \frac{f'(x)-f'(y)}{x-y}\frac{d}{ds}R(s,s)\mu_s(dx)\mu_s(dy)ds. \quad (11)$$

Using $f_z(x) = \frac{1}{x-z}, \quad z \in \mathbb{Q}^2 \cap \mathbb{C}^+$, and a continuity argument, we get that the Cauchy-Stieltjes transforms $G_t(z) := \int \frac{1}{x-z}\mu_t(dx)$, satisfy

$$\begin{aligned} G_t(z) &= \int_{\mathbb{R}} \frac{1}{x-z}\mu_0(dz) + \int_0^t \int_{\mathbb{R}^2} \frac{1}{(x-z)(y-z)^2}\frac{d}{ds}R(s,s)\mu_s(dx)\mu_s(dy) \\ &= G_0(z) + \int_0^t \frac{d}{ds}R(s,s)G_s(z)\frac{\partial}{\partial z}G_s(z)ds. \end{aligned}$$

Main result

Theorem

$\mu^{(n)}$ converges weakly in $C(\mathbb{R}_+, Pr(\mathbb{R}))$ to μ, where μ satisfies

$$\langle \mu_t, f \rangle = \langle \nu, f \rangle + \frac{1}{2} \int_0^t \int_{\mathbb{R}^2} \frac{f'(x) - f'(y)}{x - y} \frac{d}{ds} R(s,s) \mu_s(dx) \mu_s(dy) ds, \quad (12)$$

for each $t \geq 0$ and $f \in \mathcal{C}_b^2$. Moreover, the Cauchy transforms $G_t(z) := \int_{\mathbb{R}} \frac{1}{x-z} \mu_t(dz)$, satisfy $G_t(z) = F_{R(t,t)}(z)$, where $F_\tau(z)$ is the unique solution to the Burguers equation

$$\frac{\partial}{\partial \tau} F_\tau(z) = F_\tau(z) \frac{\partial}{\partial z} F_\tau(z),$$
$$F_0(z) = \int_{\mathbb{R}} \frac{1}{x - z} \nu(dx). \quad (13)$$

Thanks!