Quantitative Erdös-Kac theorem for additive functions

joint work with X. Yang y L. Chen

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Objectives

- Study the asymptotic law of $\omega(J_n)$, when n is large.
- Generalize to the case where ω is replaced by a general function $\psi : \mathbb{N} \to \mathbb{R}$ only satisfying $\psi(ab) = \psi(a) + \psi(b)$ for $a, b \in \mathbb{N}$ coprime.

Plan

1. Historical context

2. Main results

3. Ideas of the proofs

    Simplification of the model

    Stein's method

Historical context

Classical Erdös-Kac theorem (1940)

Starting point: Paul Erdös and Mark Kac proved that

$$Z_n := \frac{\omega(J_n) - \log\log(n)}{\sqrt{\log\log(n)}} \tag{1}$$

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Intuition: Define $\mathcal{P}_n := \mathcal{P} \cap [1, n]$. The convergence in (1) can be heuristically justified by the decomposition

$$\omega(J_n) = \sum_{p \in \mathcal{P}_n} \mathbb{1}_{\{p \text{ divide } J_n\}}. \tag{2}$$

Question

Can we estimate the approximating error of the Gaussian approximation with respect to a suitable probability metric? Such as that defined by

$$d_{\mathrm{K}}(X, Y) = \sup_{z \in \mathbb{R}} |\mathbb{P}[X \leq z] - \mathbb{P}[Y \leq z]|$$

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$$d_1(X, Y) = \sup_{h \in \mathrm{Lip}_1} |\mathbb{E}[h(X)] - \mathbb{E}[h(Y)]|,$$

where Lip_1 is the family of Lipschitz functions with Lipschitz constant less than or equal to one.

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where Lip_1 is the family of Lipschitz functions with Lipschitz constant less than or equal to one. We define additionally,

$$d_{TV}(X,Y) = \sup_{A\in\mathcal{B}(\mathbb{R})} |\mathbb{P}[X \in A] - \mathbb{P}[Y \in A]|.$$

LeVeque's conjecture (1949)

LeVeque showed that

$$d_{\mathrm{K}}(Z_n, \mathcal{N}) \leq C \frac{\log\log\log(n)}{\log\log(n)^{\frac{1}{4}}},$$

for a constant $C > 0$ independent of n.

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Main ingredients: Perron's formula, Dirichlet series and some estimations of the Riemann ζ function around the band $\{z \in \mathbb{C}\ ;\ \Re(z) = 1\}$.

A probabilistic approach

For $p \in \mathcal{P}$ given, we define $\alpha_p : \mathbb{N} \to \mathbb{N}_0$ as

$$k = \prod_{p \in \mathcal{P}} p^{\alpha_p(k)}.$$

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What is the behavior of $\alpha_p(J_n)$?

Approximations for $\alpha_p(J_n)$

Let $\{\xi_p\}_{p\in\mathcal{P}}$ be a family of independent geometric random variables with law

$$\mathbb{P}[\xi_p = k] = p^{-k}(1 - p^{-1}),$$

for $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

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for $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Our heuristic is based on the well-known approximation

$$(\alpha_{p_1}(J_n), \ldots, \alpha_{p_m}(J_n)) \overset{Ley}{\approx} (\xi_{p_1}, \ldots, \xi_{p_m}),$$

valid for $m \in \mathbb{N}$ and $p_1, \ldots, p_m$ different.

Main results

Central limit theorem for additive functions

Let $\psi : \mathbb{N} \to \mathbb{R}$ be such that $\psi(ab) = \psi(a) + \psi(b)$ for a, b co-prime. We will assume that

$$\sup_{p\in\mathcal{P}} |\psi(p)| < \infty \qquad \text{y} \qquad \sum_{p\in\mathcal{P}} \sum_{k\geq 2} \frac{\psi(p^k)^2}{p^k} < \infty.$$

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Define

$$\mu_n = \sum_{p\in\mathcal{P}_n} \psi(p) p^{-1}(1-p^{-1})^{-1}$$
$$\sigma_n^2 = \sum_{p\in\mathcal{P}_n} \psi(p)^2 p^{-1}(1-p^{-1})^{-2}.$$

Main result for the Kolmogorov distance

Theorem (Chen, Jaramillo, Yang)

Under the above conditions,

$$d_{\mathrm{K}}\left(\frac{\psi(J_n)-\mu_n}{\sigma_n},\mathcal{N}\right)\leq\frac{\kappa_1}{\sigma_n}+\frac{\kappa_2}{\sigma_n^2}+\frac{\kappa_3\log\log(n)}{\log(n)}$$

$$d_1\left(\frac{\psi(J_n)-\mu_n}{\sigma_n},\mathcal{N}\right)\leq\frac{\kappa_4}{\sigma_n}+\frac{\kappa_5}{\sigma_n^2}+\kappa_6\frac{\log\log(n)^{\frac{1}{2}}}{\log(n)^{\frac{1}{2}}},$$

where $\kappa_1,\ldots,\kappa_6$ *are explicit functions of* ψ*.*

Ideas of the proofs

Simplified model: the harmonic distribution H_n

Let H_n be a random variable with $\mathbb{P}[H_n = k] = \frac{1}{L_n k}$ for $k \leq n$, where $L_n := \sum_{k=1}^{n} \frac{1}{k}$.

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Proposition

Suppose that $n \geq 21$. We define the event

$$A_n := \Big\{ \prod_{p \in \mathcal{P}_n} p^{\xi_p} \leq n \Big\}. \tag{3}$$

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We have

$$\mathcal{L}(\psi(H_n)) = \mathcal{L}(\sum_{p \in \mathcal{P}_n} \psi(p^{\alpha_p(H_n)})) = \mathcal{L}(\sum_{p \in \mathcal{P}_n} \psi(p^{\xi_p}) | A_n). \tag{4}$$

Relation with the harmonic distribution

Let $\{Q(k)\}_{k\geq 1}$ be a sequence of independent random variables and independent of (J_n, H_n), where $Q(k)$ is uniformly distributed over the set

$$\mathcal{P}_k^* := \{1\} \cup \mathcal{P}_k.$$

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Lemma (Chen, Jaramillo y Yang)

For $n \geq 21$*,*

$$d_{\mathrm{TV}}(J_n, H_n Q(n/H_n)) \leq 61 \frac{\log \log n}{\log n}$$

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$$\mathbb{P}[Q(n/H_n) \textit{ divides } H_n] \leq 6.4 \frac{\log \log n}{\log n}.$$

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Recall that conditionally over $A_n := \left\{ \prod_{p \in \mathcal{P}_n} p^{\xi_p} \leq n \right\}$,

$$\psi(H_n) \overset{Law}{=} \sum_{p \in \mathcal{P}_n} \psi(p^{\xi_p}).$$

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The problem reduces to estimate

$$d_1\left(Law\left(\frac{\sum_{p\in\mathcal{P}_n}\psi(p)\xi_p-\mu_n}{\sigma_n}\mid A_n\right),\mathcal{N}\right).$$

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We will use Stein' s method.

Stein's method

Lemma (Stein's lemma)
For every smooth $f : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}[f'(\mathcal{N})] = \mathbb{E}[\mathcal{N} f(\mathcal{N})]$$

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Lemma (Stein's lemma)
For every smooth $f : \mathbb{R} \to \mathbb{R}$,

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Stein's heuristic: If X is an $\mathbb{R}$-valued random variable such that

$$\mathbb{E}[f'(X)] \approx \mathbb{E}[X f(X)],$$

for a sufficiently large class of functions f, then Z approximates $\mathcal{N}$.

Lemma
Let $h : \mathbb{R} \to \mathbb{R}$ *be* 1-*Lipchitz. Then the equation*

$$f'(x) - xf(x) = h(x) - \mathbb{E}[h(\mathcal{N})]$$

has a unique solution $f = f_h$,

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has a unique solution $f = f_h$, which satisfies

$$\sup_{w \in \mathbb{R}} |f_h(w)| \leq 2 \qquad \sup_{w \in \mathbb{R}} |f_h'(w)| \leq \sqrt{2/\pi} \qquad \sup_{w \in \mathbb{R}} |f_h'(w)| \leq 2. \tag{5}$$

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Thereofore, if X is a random variable,

$$d_K(X, \mathcal{N}) \leq \sup_f |\mathbb{E}[f'(X) - Xf(X)]|$$

where f belongs to the family of functions satisfying (5).

Poisson space representation

Define

$$W_n := \frac{\sum_{p \in \mathcal{P}_n} \psi(p)\xi_p - \mu_n}{\sigma_n}$$

y $I_n := \mathbb{1}_{A_n}$.

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$$|\mathbb{E}[Z_n f(W_n) - f'(W_n)|A_n]| = \mathbb{P}[A_n]^{-1}|\mathbb{E}[f(W_n)W_n I_n] - \mathbb{E}[f'(W_n)I_n]|$$

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$$\begin{aligned} |\mathbb{E}[Z_n f(W_n) - f'(W_n)|A_n]| &= \mathbb{P}[A_n]^{-1} |\mathbb{E}[f(W_n)W_n I_n] - \mathbb{E}[f'(W_n)I_n]| \\ &\leq 2|\mathbb{E}[f(W_n){\color{red}W_n}I_n] - \mathbb{E}[f'(W_n)I_n]|. \end{aligned}$$

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Let η be a Poisson point process over $\mathbb{X}$, with intensity $\lambda : \mathbb{X} \to \mathbb{R}_+$ given by

$$\lambda(p,k) = \frac{1}{kp^k}, \quad \textit{para } p \in \mathcal{P}, k \in \mathbb{N}.$$

Stein's method

Using characteristic functions, one can show that

$$W_n \stackrel{Law}{=} \tilde{\eta}(\rho_n), \tag{6}$$

where $\tilde{\eta} = \eta(p, k) - \mathbb{E}[\eta(p, k)]$ is the compensation of $\eta(p, k)$ and

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$$\mathbb{E}[\tilde{\eta}(\rho_n) G_n(\eta)] = \int_{\mathbb{X}} \rho_n(x) \mathbb{E}[D_x G_n(\eta)] \lambda(dx), \tag{8}$$

where $D_x G_n(\eta) := G_n(\eta + \delta_x) - G_n(\eta)$.

Stein' s formula

For the case where $G_n = f(W_n)I_n$, by the previous formula,

$$\mathbb{E}[W_n f(W_n)] = \int_{\mathbb{X}} \rho_n(x)\mathbb{E}[D_x G_n(\eta)]\lambda(dx).$$

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$$\mathbb{E}[W_n f(W_n)] = \int_{\mathbb{X}} \rho_n(x)\mathbb{E}[D_x G_n(\eta)]\lambda(dx).$$

One can verify the approximation $D_x(f(W_n)I_n) \approx f'(W_n)\rho_n(x)I_n$, so that

$$\mathbb{E}[W_n f(W_n) I_n] \approx \int_{\mathbb{X}} \rho_n(x)^2 \mathbb{E}[f'(W_n)I_n]\lambda(dx) = \mathbb{E}[f'(W_n)I_n].$$

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From the above analysis we get

$$d_1(Z_n, \mathcal{N})$$

The result follows from a suitable measurement of the error of the approximations.

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Poisson case

Theorem (Chen, Jaramillo y Yang)

Suppose that

$$\lambda_n := \sum_{p \in \mathcal{P}_n} \frac{\psi(p)}{p-1} > 0, \quad n \in \mathbb{N}. \tag{9}$$

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Theorem (Chen, Jaramillo y Yang)

Suppose that

$$\lambda_n := \sum_{p \in \mathcal{P}_n} \frac{\psi(p)}{p-1} > 0, \quad n \in \mathbb{N}. \tag{9}$$

Let M_n be a Poisson random variable with intensity λ_n. Then,

$$d_{TV}(\psi(J_n), M_n) \leq \frac{\gamma_1}{\sqrt{\lambda_n}} + \frac{\gamma_2}{\lambda_n} + \frac{2\gamma_3}{\lambda_n} \sum_{p \in \mathcal{P}_n} \frac{|\psi(p) - 1|}{p}. \tag{10}$$

for $\gamma_1, \ldots, \gamma_3$ explicit in terms of ψ.

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