Fluctuations of the spectrum of matrix-valued Gaussian processes.

Arturo Jaramillo

National University of Singapore
Université du Luxembourg

April 2019

Goal

If $Y^{(n)} = (Y^{(n)}(t);\ t \geq 0)$ are centered Gaussian processes with values in the set of real symmetric matrices of dimension n

Goal

If $Y^{(n)} = (Y^{(n)}(t);\ t \geq 0)$ are centered Gaussian processes with values in the set of real symmetric matrices of dimension n and $(\mu_t^{(n)}\ ;\ n \geq 1)$ is the measure that assigns mass $\frac{1}{n}$ to each eigenvalue of $Y^{(n)}(t)$.

Goal

If $Y^{(n)} = (Y^{(n)}(t);\ t \geq 0)$ are centered Gaussian processes with values in the set of real symmetric matrices of dimension n and $(\mu_t^{(n)}\ ;\ n \geq 1)$ is the measure that assigns mass $\frac{1}{n}$ to each eigenvalue of $Y^{(n)}(t)$.

Question

For $r \in \mathbb{N}$ fixed and a given $F : \mathbb{R} \to \mathbb{R}^r$, what can we say about

$$\left(\int_{\mathbb{R}} F(x) \mu_t^{(n)}(dx) - \mathbb{E}\left[\int_{\mathbb{R}} F(x) \mu_t^{(n)}(dx) \right]\ ;\ t \geq 0 \right)?$$

Notation

Denote by $\mathbb{R}^{n\times n}$ the set of square matrices of dimension n. Let $Y^{(n)} = (Y^{(n)}(t);\ t \geq 0)$ be a sequence of $\mathbb{R}^{n\times n}$-valued processes, defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Notation

Denote by $\mathbb{R}^{n\times n}$ the set of square matrices of dimension n. Let $Y^{(n)} = (Y^{(n)}(t);\ t \geq 0)$ be a sequence of $\mathbb{R}^{n\times n}$-valued processes, defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assume that $Y^{(n)}(t) = [Y_{i,j}^{(n)}(t)]_{1\leq i,j\leq n}$ is real and symmetric, with

$$Y_{i,j}^{(n)}(t) = \begin{cases} \frac{1}{\sqrt{n}} X_{i,j}(t) & \text{if} \quad i < j, \\ \frac{\sqrt{2}}{\sqrt{n}} X_{i,i}(t) & \text{if} \quad i = j, \end{cases} \tag{1}$$

where $X_{i,j} := (X_{i,j}(t);\ t \geq 0)$ are i.i.d. centered Gaussian processes with covariance

$$R(s,t) := \mathbb{E}[X_{1,1}(s) X_{1,1}(t)].$$

Notation

We will use the notation

$$\sigma_s := \sqrt{R(s,s)} \qquad \text{y} \qquad \rho_{s,t} := \frac{R(s,t)}{\sigma_s \sigma_t},$$

Notation

We will use the notation

$$\sigma_s := \sqrt{R(s,s)} \qquad \text{y} \qquad \rho_{s,t} := \frac{R(s,t)}{\sigma_s \sigma_t},$$

and assume that $\sigma_1 = 1$,

(H1) There exists $\alpha > 1$, such that for all $T > 0$ and $t \in [0, T]$, the mapping $s \mapsto R(s,t)$ is absolutely continuous in $[0, T]$ and

$$\sup_{0 \le t \le T} \int_0^T \left| \frac{\partial R}{\partial s}(s,t) \right|^{\alpha} \mathrm{d}s < \infty.$$

Notation

We will use the notation

$$\sigma_s := \sqrt{R(s,s)} \qquad \text{y} \qquad \rho_{s,t} := \frac{R(s,t)}{\sigma_s \sigma_t},$$

and assume that $\sigma_1 = 1$,

(H1) There exists $\alpha > 1$, such that for all $T > 0$ and $t \in [0, T]$, the mapping $s \mapsto R(s,t)$ is absolutely continuous in $[0, T]$ and

$$\sup_{0 \leq t \leq T} \int_0^T \left| \frac{\partial R}{\partial s}(s,t) \right|^{\alpha} \mathrm{d}s < \infty.$$

(H2) The mapping $s \mapsto \sigma_s^2$ is continuously differentiable in $(0, \infty)$ and continuous at zero. Moreover, we have that $\frac{d}{ds}\sigma_s^2 \in L^1[0, T]$ for all $T > 0$.

Notation

Examples:

- Brownian motion.

Notation

Examples:

- Brownian motion.
- Fractional Brownian motion with Hurst parameter $H \in (0, 1)$.

Notation

Examples:

- Brownian motion.
- Fractional Brownian motion with Hurst parameter $H \in (0, 1)$.
- Ornestein-Uhlenbeck process.

Notation

We will denote by $\lambda_1^{(n)}(t) \geq \cdots \geq \lambda_n^{(n)}(t)$ the ordered eigenvalues of $Y^{(n)}(t)$ and by $\mu_t^{(n)}$ the spectral empirical distribution

$$\mu_t^{(n)}(\mathrm{d}x) = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i^{(n)}(t)}(\mathrm{d}x).$$

Wigner theorem

Wigner theorem establishes that for all $\varepsilon > 0$ and all function f belonging to the set $\mathcal{C}_b(\mathbb{R})$ of continuous and bounded functions,

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\int_{\mathbb{R}} f(x)\mu_1^{(n)}(\mathrm{d}x) - \int_{\mathbb{R}} f(x)\mu_1^{sc}(\mathrm{d}x)\right| > \epsilon\right) = 0, \tag{2}$$

where μ_σ^{sc}, for $\sigma > 0$, denotes the rescaled semicircle distribution

$$\mu_\sigma^{sc}(\mathrm{d}x) := \frac{\mathbb{1}_{[-2\sigma,2\sigma]}(x)}{2\pi\sigma^2}\sqrt{4\sigma^2 - x^2}\mathrm{d}x.$$

Functional Wigner theorem

In a paper by Jaramillo, Pardo and Pérez (based on previous works by Rogers, Shi, Cépa, Lepingale and Pérez-Abreu), it was proved that

Theorem

Denote by $\mathcal{C}(\mathbb{R}_+, \mathtt{Pr}(\mathbb{R}))$ the set of continuous functions defined in $\mathbb{R}_+$, with values in the set of probability measures. If $\mu_0^{(n)}$ converges in law to ν, then $\{(\mu_t^{(n)}(dx);\ t \geq 0) : n \geq 1\}$ converges weakly to a function $(\mu_t; t \geq 0)$, such that

$$\int f(x)\mu_t(dx) = \int f(x)\nu(dx) + \frac{1}{2}\int_0^t \int_{\mathbb{R}^2} \frac{f'(x) - f'(y)}{x - y} \frac{d}{ds}(R(s,s))\mu_s(dx)\mu_s(dy)ds,$$

for all $t \geq 0$ and $f : \mathbb{R} \to \mathbb{R}$ three times differentiable, with derivatives of polynomial growth.

Fluctuations of Wigner's theorem

In a paper by Lytova y Pastur, it was proved (in a much more general context than the one described before), that

Theorem

for all $f \in \mathcal{C}_b(\mathbb{R})$,

$$n \int_{\mathbb{R}} f(x) \mu_1^{(n)}(\mathrm{d}x) - n\mathbb{E}\left[\int_{\mathbb{R}} f(x) \mu_1^{(n)}(\mathrm{d}x) \right] \xrightarrow{d} \mathcal{N}(0, \sigma_f^2), \tag{3}$$

where $\mathcal{N}(0, \sigma_f^2)$ *is a Gaussian random variable with variance*

$$\sigma_f^2 := \frac{1}{4} \int_{\mathbb{R}^2} \left(\frac{f(x) - f(y)}{x - y} \right)^2 \frac{4 - xy}{(4 - x^2)(4 - y^2)} \mu_1^{sc}(\mathrm{d}x) \mu_1^{sc}(\mathrm{d}y).$$

Functional fluctuations of Wigner's theorem

There are some results on the functional fluctuations of Wigner's theorem in the following particular cases:

Functional fluctuations of Wigner's theorem

There are some results on the functional fluctuations of Wigner's theorem in the following particular cases:

- The entries $X_{i,j}$ are Ornstein-Uhlenbeck processes. This problem was studied by Israelson, Bender and Unterberger. We know that the limit is Gaussian and the limiting covariance function can be explicitly described.

Functional fluctuations of Wigner's theorem

There are some results on the functional fluctuations of Wigner's theorem in the following particular cases:

- The entries $X_{i,j}$ are Ornstein-Uhlenbeck processes. This problem was studied by Israelson, Bender and Unterberger. We know that the limit is Gaussian and the limiting covariance function can be explicitly described.
- The entries $X_{i,j}$ are **complex** Brownian motions and $f : \mathbb{R} \to \mathbb{R}$ is a polynomial. This problem has been studied by Pérez-Abreu and Tudor. It is known that the limit is Gaussian, but the covariance of the limit hasn't been described in an explicit way.

Main results (notation)

Consider the set of test functions

$$\mathcal{P} := \{f \in \mathcal{C}^3(\mathbb{R};\mathbb{R}) \mid f''' \text{ has polynomial growth}\}.$$

Main results (notation)

Consider the set of test functions

$$\mathcal{P} := \{f \in \mathcal{C}^3(\mathbb{R};\mathbb{R}) \mid f''' \text{ has polynomial growth}\}.$$

For $f \in \mathcal{P}$, $F = (f_1, \dots, f_r) \in \mathcal{P}^r$ and $z \in (0,1)$, define the processes

$$Z_f^{(n)}(t) := n\int_{\mathbb{R}} f(x)\mu_t^{(n)}(dx) - n\mathbb{E}\left[\int_{\mathbb{R}} f(x)\mu_t^{(n)}(dx)\right]$$
$$Z_F^{(n)}(t) := n\int_{\mathbb{R}} F(x)\mu_t^{(n)}(dx) - n\mathbb{E}\left[\int_{\mathbb{R}} F(x)\mu_t^{(n)}(dx)\right],$$

Main results (notation)

Consider the set of test functions

$$\mathcal{P} := \{f \in \mathcal{C}^3(\mathbb{R};\mathbb{R}) \mid f''' \text{ has polynomial growth}\}.$$

For $f \in \mathcal{P}$, $F = (f_1, \dots, f_r) \in \mathcal{P}^r$ and $z \in (0,1)$, define the processes

$$Z_f^{(n)}(t) := n \int_{\mathbb{R}} f(x)\mu_t^{(n)}(dx) - n\mathbb{E}\left[\int_{\mathbb{R}} f(x)\mu_t^{(n)}(dx)\right]$$
$$Z_F^{(n)}(t) := n \int_{\mathbb{R}} F(x)\mu_t^{(n)}(dx) - n\mathbb{E}\left[\int_{\mathbb{R}} F(x)\mu_t^{(n)}(dx)\right],$$

and the kernel

$$K_z(x,y) := \frac{1-z^2}{z^2(x-y)^2 - xyz(1-z)^2 + (1-z^2)^2}.$$

Main results

Theorem (Díaz, Jaramillo, Pardo, Pérez)

For all $f, g \in \mathcal{P}$,

$$\lim_{n\to\infty} \mathrm{Cov}\left[Z_f^{(n)}(s), Z_g^{(n)}(t)\right] = 2\int_{\mathbb{R}^2} f'(x)g'(y)\nu_{\sigma_s,\sigma_t}^{\rho_{s,t}}(\mathrm{d}x, \mathrm{d}y),$$

where

$$\nu_{\sigma_s,\sigma_t}^{\rho_{s,t}}(A, B) = \int_0^1 \int_{A\times B} K_{z\rho_{s,t}}(x/\sigma_s, y/\sigma_t)\mu_{\sigma_s}^{sc}(\mathrm{d}x)\mu_{\sigma_t}^{sc}(\mathrm{d}y)\mathrm{d}z.$$

Main results

Theorem (Díaz, Jaramillo, Pardo, Pérez)

There exists a centered Gaussian process with values in $\mathbb{R}^r$*, denoted by* $\Lambda_F = ((\Lambda_{f_1}(t), \ldots, \Lambda_{f_r}(t)); t \geq 0)$*, independent of* $\{X_{i,j}; j \geq i \geq 1\}$*, defined in an extended probability space* $(\Omega, \mathcal{G}, \mathbb{P})$*, such that*

$$(Z_F^{(n)}(t) \ ; \ t \geq 0) \stackrel{Stably}{\longrightarrow} \Lambda_F,$$

in the topology of uniform convergence over compact sets.

Main results

Theorem (Díaz, Jaramillo, Pardo, Pérez)

There exists a centered Gaussian process with values in $\mathbb{R}^r$*, denoted by* $\Lambda_F = ((\Lambda_{f_1}(t), \dots, \Lambda_{f_r}(t)); t \geq 0)$*, independent of* $\{X_{i,j}; j \geq i \geq 1\}$*, defined in an extended probability space* $(\Omega, \mathcal{G}, \mathbb{P})$*, such that*

$$(Z_F^{(n)}(t)\ ;\ t \geq 0) \stackrel{Stably}{\longrightarrow} \Lambda_F,$$

in the topology of uniform convergence over compact sets. The law of Λ_F *is characterized by*

$$\mathbb{E}\left[\Lambda_{f_i}(s)\Lambda_{f_j}(t)\right] = \int_{\mathbb{R}^2} f_i'(x) f_j'(y) \nu_{\sigma_s,\sigma_t}^{\rho_{s,t}}(\mathrm{d}x, \mathrm{d}y).$$

In addition, we have that $d_{TV}(Z_f^{(n)}(t), \Lambda_f(t)) \leq \frac{C}{\sqrt{n}}$

Basic definitions

Let $T > 0$ be fixed and define $d := \frac{n(n+1)}{2}$, we can identify the process $(X_{i,j}(t)\ ;\ 1 \leq i \leq j \leq n,\ t \geq 0)$ with a $\mathbb{R}^d$-valued process $V = (V_t^1, \ldots, V_t^d\ ;\ t \geq 0)$ with i.i.d. entries

Basic definitions

Let $T > 0$ be fixed and define $d := \frac{n(n+1)}{2}$, we can identify the process $(X_{i,j}(t)\ ;\ 1 \leq i \leq j \leq n,\ \ t \geq 0)$ with a $\mathbb{R}^d$-valued process $V = (V_t^1, \dots, V_t^d\ ;\ t \geq 0)$ with i.i.d. entries

We will denote by $\mathscr{E}$ the space of step functions over $[0, T]$. Consider the inner product

$$\left\langle \mathbb{1}_{[0,s]}, \mathbb{1}_{[0,t]} \right\rangle_{\mathfrak{H}} := \mathbb{E}\left[V_s^1 V_t^1 \right], \qquad s, t \in [0, T],$$

defined in $\mathscr{E}$.

Basic definitions

Let $T > 0$ be fixed and define $d := \frac{n(n+1)}{2}$, we can identify the process $(X_{i,j}(t)\ ;\ 1 \leq i \leq j \leq n,\ t \geq 0)$ with a $\mathbb{R}^d$-valued process $V = (V_t^1, \dots, V_t^d\ ;\ t \geq 0)$ with i.i.d. entries

We will denote by $\mathscr{E}$ the space of step functions over $[0, T]$. Consider the inner product

$$\left\langle \mathbb{1}_{[0,s]}, \mathbb{1}_{[0,t]} \right\rangle_{\mathfrak{H}} := \mathbb{E}\left[V_s^1 V_t^1 \right], \qquad s, t \in [0, T],$$

defined in $\mathscr{E}$. Let $\mathfrak{H}$ obtained as the completion of $\mathscr{E}$ with respect to the inner product above.

Basic definitions

Example: If $X_{1,1}$ is a Brownian motion, then $\mathfrak{H} = L^2[0, T]$.

Basic definitions

Example: If $X_{1,1}$ is a Brownian motion, then $\mathfrak{H} = L^2[0, T]$.

For all $1 \leq i \leq n$, the mapping $\mathbb{1}_{[0,t]} \mapsto V^i(\mathbb{1}_{[0,t]}) := V_t^i$ can be extended into a linear isometry, which we will denote by $V^i(h)$, for $h \in \mathfrak{H}$.

Basic definitions

Example: If $X_{1,1}$ is a Brownian motion, then $\mathfrak{H} = L^2[0, T]$.

For all $1 \leq i \leq n$, the mapping $\mathbb{1}_{[0,t]} \mapsto V^i(\mathbb{1}_{[0,t]}) := V_t^i$ can be extended into a linear isometry, which we will denote by $V^i(h)$, for $h \in \mathfrak{H}$. If $f \in \mathfrak{H}^d$ is of the form $f = (f_1, \dots, f_d)$, we define

$$V(f) := \sum_{i=1}^{d} V^i(f_i).$$

Basic definitions

Example: If $X_{1,1}$ is a Brownian motion, then $\mathfrak{H} = L^2[0, T]$.

For all $1 \leq i \leq n$, the mapping $\mathbb{1}_{[0,t]} \mapsto V^i(\mathbb{1}_{[0,t]}) := V_t^i$ can be extended into a linear isometry, which we will denote by $V^i(h)$, for $h \in \mathfrak{H}$. If $f \in \mathfrak{H}^d$ is of the form $f = (f_1, \ldots, f_d)$, we define

$$V(f) := \sum_{i=1}^{d} V^i(f_i).$$

Example: If $X_{1,1}$ is a Brownian motion, then

$$V(f) = \sum_{i=1}^{d} \int_0^T f_i(t) dV_t^i.$$

Chaos decomposition

For $q \in \mathbb{N}$ fixed, define the q-th Wiener chaos, as the subspace

$$\mathcal{H}_q = \overline{\text{span}\{H_q(V(h)) \mid \|h\|_{\mathfrak{H}^d} = 1\}} \subset L^2(\Omega),$$

where H_q denotes the q-th Hermite polynomial, defined by $H_0 = 1$ and $H_{q+1}(x) = xH_q(x) - qH_{q-1}(x)$.

Chaos decomposition

For $q \in \mathbb{N}$ fixed, define the q-th Wiener chaos, as the subspace

$$\mathcal{H}_q = \overline{\mathrm{span}\{H_q(V(h)) \mid \|h\|_{\mathfrak{H}^d} = 1\}} \subset L^2(\Omega),$$

where H_q denotes the q-th Hermite polynomial, defined by $H_0 = 1$ and $H_{q+1}(x) = xH_q(x) - qH_{q-1}(x)$.

Theorem (Chaos decomposition)

We have that

$$L^2(\Omega, \mathbb{P}) = \bigoplus_{q=0}^{\infty} \mathcal{H}_q.$$

Chaos decomposition

For $q \in \mathbb{N}$ fixed, define the q-th Wiener chaos, as the subspace

$$\mathcal{H}_q = \overline{\text{span}\{H_q(V(h)) \mid \|h\|_{\mathfrak{H}^d} = 1\}} \subset L^2(\Omega),$$

where H_q denotes the q-th Hermite polynomial, defined by $H_0 = 1$ and $H_{q+1}(x) = xH_q(x) - qH_{q-1}(x)$.

Theorem (Chaos decomposition)

We have that

$$L^2(\Omega, \mathbb{P}) = \bigoplus_{q=0}^{\infty} \mathcal{H}_q.$$

The projection of an element $Y \in L^2(\Omega)$ over the space $\mathcal{H}_q$, will be denoted by $J_q[Y]$.

Derivative and divergence operators

For $q \in \mathbb{N}$, denote by $(\mathfrak{H}^d)^{\otimes q}$ and $(\mathfrak{H}^d)^{\odot q}$ the q-th tensor product and q-th symmetrized tensor product of $\mathfrak{H}^d$.

Derivative and divergence operators

For $q \in \mathbb{N}$, denote by $(\mathfrak{H}^d)^{\otimes q}$ and $(\mathfrak{H}^d)^{\odot q}$ the q-th tensor product and q-th symmetrized tensor product of $\mathfrak{H}^d$.

Definition (Derivative operator)

For a random variable F of the form $F = f(V(h_1), ..., V(h_n))$, where $f \in C^\infty(\mathbb{R}^n; \mathbb{R})$, has derivatives with polynomial growth, define the Malliavin derivative of F as the $\mathfrak{H}^d$-valued random vector

$$DF = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(V(h_1), ..., V(h_n))h_k.$$

Derivative and divergence operators

For $q \in \mathbb{N}$, denote by $(\mathfrak{H}^d)^{\otimes q}$ and $(\mathfrak{H}^d)^{\odot q}$ the q-th tensor product and q-th symmetrized tensor product of $\mathfrak{H}^d$.

Definition (Derivative operator)

For a random variable F of the form $F = f(V(h_1), ..., V(h_n))$, where $f \in C^\infty(\mathbb{R}^n; \mathbb{R})$, has derivatives with polynomial growth, define the Malliavin derivative of F as the $\mathfrak{H}^d$-valued random vector

$$DF = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(V(h_1), ..., V(h_n))h_k.$$

For $p \geq 1$, the operator D can be extended to a subspace $\mathbb{D}^{1,p} \subset L^2(\Omega)$, closed with respect to the norm $\|F\|_{\mathbb{D}^{1,p}} := \left(\mathbb{E}\left[|F|^p\right] + \mathbb{E}\left[\|DF\|_{\mathfrak{H}}^p\right]\right)^{\frac{1}{p}}$.

Derivative and divergence operators

For $q \in \mathbb{N}$, denote by $(\mathfrak{H}^d)^{\otimes q}$ and $(\mathfrak{H}^d)^{\odot q}$ the q-th tensor product and q-th symmetrized tensor product of $\mathfrak{H}^d$.

Definition (Derivative operator)

For a random variable F of the form $F = f(V(h_1), ..., V(h_n))$, where $f \in C^\infty(\mathbb{R}^n; \mathbb{R})$, has derivatives with polynomial growth, define the Malliavin derivative of F as the $\mathfrak{H}^d$-valued random vector

$$DF = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(V(h_1), ..., V(h_n))h_k.$$

For $p \geq 1$, the operator D can be extended to a subspace $\mathbb{D}^{1,p} \subset L^2(\Omega)$, closed with respect to the norm $\|F\|_{\mathbb{D}^{1,p}} := \left(\mathbb{E}\left[|F|^p\right] + \mathbb{E}\left[\|DF\|_{\mathfrak{H}}^p\right]\right)^{\frac{1}{p}}$. Define D^r as the r-th iteration of D.

Derivative and divergence operators

Definition (Divergence operator)

Denote the adjoint of D by δ. Namely,

- δ is defined in a domain $Dom(\delta) \subset L^2(\Omega; \mathfrak{H}^d)$, characterized by the property that $u \in Dom(\delta)$ if there exists a constant $c > 0$, only depending on u, such that for all $F \in \mathbb{D}^{1,2}$,

$$|\mathbb{E}\left[\langle DF, u\rangle_{\mathfrak{H}^d}\right]| \leq c\|F\|_{L^2(\Omega)}.$$

- If $u \in Dom(\delta)$, then $\delta(u)$ is characterized by

$$\mathbb{E}\left[F\delta(u)\right] = \mathbb{E}\left[\langle DF, u\rangle_{\mathfrak{H}^d}\right].$$

Derivative and divergence operators

Definition (Divergence operator)

Denote the adjoint of D by δ. Namely,

- δ is defined in a domain $Dom(\delta) \subset L^2(\Omega; \mathfrak{H}^d)$, characterized by the property that $u \in Dom(\delta)$ if there exists a constant $c > 0$, only depending on u, such that for all $F \in \mathbb{D}^{1,2}$,

$$|\mathbb{E}\left[\langle DF, u\rangle_{\mathfrak{H}^d}\right]| \leq c\|F\|_{L^2(\Omega)}.$$

- If $u \in Dom(\delta)$, then $\delta(u)$ is characterized by

$$\mathbb{E}\left[F\delta(u)\right] = \mathbb{E}\left[\langle DF, u\rangle_{\mathfrak{H}^d}\right].$$

analogously, we define δ^r as the adjoint of D^r.

The Ornstein-Uhlenbeck semigroup

Definition

The Ornstein-Uhlenbeck semigroup $\{P_t\}_{t\geq 0}$ is defined by $P_tF := \sum_{q=0}^{\infty} e^{-qt} J_q(F) \in L^2(\Omega)$,

The Ornstein-Uhlenbeck semigroup

Definition

The Ornstein-Uhlenbeck semigroup $\{P_t\}_{t\geq 0}$ is defined by $P_t F := \sum_{q=0}^{\infty} e^{-qt} J_q(F) \in L^2(\Omega)$, and the generator of the Ornstein-Uhlenbeck L, is defined by

$$LF = -\sum_{q=1}^{\infty} q J_q[F].$$

Its domain is formed by the random variables F such that $\sum_{q=1}^{\infty} q^2 \mathbb{E}\left[J_q[F]^2\right] < \infty$.

Relations between D, δ y L

Mehler's formula stablishes that $F \in L^2(\Omega)$ and Ψ_F is a measurable mapping from $\mathbb{R}^{\mathfrak{H}^d}$ to $\mathbb{R}$, such that $F = \Psi_F(V)$, then

$$P_\theta F = \widetilde{\mathbb{E}}\left[\Psi_F(e^{-\theta} V + \sqrt{1 - e^{-2\theta}}\widetilde{V})\right],$$

where $\widetilde{V}$ is an independent copy of V and $\widetilde{\mathbb{E}}$ is the expectation with respect to $\widetilde{V}$.

Relations between D, δ y L

Mehler's formula stablishes that $F \in L^2(\Omega)$ and Ψ_F is a measurable mapping from $\mathbb{R}^{\mathfrak{H}^d}$ to $\mathbb{R}$, such that $F = \Psi_F(V)$, then

$$P_\theta F = \widetilde{\mathbb{E}}\left[\Psi_F(e^{-\theta}V + \sqrt{1-e^{-2\theta}}\widetilde{V})\right],$$

where $\widetilde{V}$ is an independent copy of V and $\widetilde{\mathbb{E}}$ is the expectation with respect to $\widetilde{V}$. Additionally, we have that $F \in Dom(L)$ if and only if $F \in \mathbb{D}^{1,2}$ and $DF \in Dom(\delta)$, in which case

$$LF = -\delta(DF).$$

Furthermore, if $F \in L^2(\Omega)$, then

$$-L^{-1}F = \int_{\mathbb{R}_+} P_\theta F \mathrm{d}\theta.$$

Contractions

Let $\{b_j\}_{j\in\mathbb{N}} \subset \mathfrak{H}^d$ be an orthonormal basis of $\mathfrak{H}^d$. Given $f \in (\mathfrak{H}^d)^{\odot p}$, $g \in (\mathfrak{H}^d)^{\odot q}$ and $r \in \{1, \dots, p \wedge q\}$, the r-th contraction of f and g is the element $f \otimes_r g \in (\mathfrak{H}^d)^{\otimes(p+q-2r)}$ given by

$$f \otimes_r g = \sum_{i_1,\dots,i_r=1}^{\infty} \langle f, b_{i_1}, \dots, b_{i_r} \rangle_{(\mathfrak{H}^d)^{\otimes r}} \otimes \langle g, b_{i_1}, \dots, b_{i_r} \rangle_{(\mathfrak{H}^d)^{\otimes r}}.$$

CLT via Malliavin calculus

Theorem (Nourdin, Peccati and Réveillac)

Suppose that $r \geq 1$ is fixed. Consider random vectors $Z_n = (Z_{1,n}, \ldots, Z_{r,n})$, $n \geq 1$, with $\mathbb{E}[Z_{i,n}] = 0$ and $Z_{i,n} \in \mathbb{D}^{2,4}$. Let C be a non-negative definite, symmetric matrix of dimensioin r, and let $N = (N_1, \ldots, N_r) \sim \mathcal{N}_r(0, C)$.

CLT via Malliavin calculus

Theorem (Nourdin, Peccati and Réveillac)

Suppose that $r \geq 1$ *is fixed. Consider random vectors* $Z_n = (Z_{1,n}, \ldots, Z_{r,n})$, $n \geq 1$, *with* $\mathbb{E}[Z_{i,n}] = 0$ *and* $Z_{i,n} \in \mathbb{D}^{2,4}$. *Let* C *be a non-negative definite, symmetric matrix of dimensioin* r, *and let* $N = (N_1, \ldots, N_r) \sim \mathcal{N}_r(0, C)$. *Suppose that:*

(i) *For all* $i, j = 1, \ldots, r$, $\mathbb{E}[Z_{i,n} Z_{j,n}] \to C(i,j)$ *when* $n \to \infty$;

CLT via Malliavin calculus

Theorem (Nourdin, Peccati and Réveillac)

Suppose that $r \geq 1$ *is fixed. Consider random vectors* $Z_n = (Z_{1,n}, \dots, Z_{r,n})$, $n \geq 1$, *with* $\mathbb{E}[Z_{i,n}] = 0$ *and* $Z_{i,n} \in \mathbb{D}^{2,4}$. *Let* C *be a non-negative definite, symmetric matrix of dimensioin* r, *and let* $N = (N_1, \dots, N_r) \sim \mathcal{N}_r(0, C)$. *Suppose that:*

(i) *For all* $i, j = 1, \dots, r$, $\mathbb{E}[Z_{i,n} Z_{j,n}] \to C(i,j)$ *when* $n \to \infty$;

(ii) *For all* $i = 1, \dots, r$, $\sup_{n \geq 1} \mathbb{E}\left[\|DZ_{i,n}\|_{\mathfrak{H}}^4\right] < \infty$;

CLT via Malliavin calculus

Theorem (Nourdin, Peccati and Réveillac)

Suppose that $r \geq 1$ is fixed. Consider random vectors $Z_n = (Z_{1,n}, \ldots, Z_{r,n})$, $n \geq 1$, with $\mathbb{E}[Z_{i,n}] = 0$ and $Z_{i,n} \in \mathbb{D}^{2,4}$. Let C be a non-negative definite, symmetric matrix of dimensioin r, and let $N = (N_1, \ldots, N_r) \sim \mathcal{N}_r(0, C)$. Suppose that:

(i) *For all $i, j = 1, \ldots, r$, $\mathbb{E}[Z_{i,n} Z_{j,n}] \to C(i,j)$ when $n \to \infty$;*

(ii) *For all $i = 1, \ldots, r$, $\sup_{n \geq 1} \mathbb{E}\left[\|DZ_{i,n}\|_{\mathfrak{H}}^4\right] < \infty$;*

(iii) *For all $i = 1, \ldots, r$, $\mathbb{E}\left[\|D^2 Z_{i,n} \otimes_1 D^2 Z_{i,n}\|_{(\mathfrak{H}^d)^{\otimes 2}}^2\right] \to 0$ when $n \to \infty$.*

CLT via Malliavin calculus

Theorem (Nourdin, Peccati and Réveillac)

Suppose that $r \geq 1$ is fixed. Consider random vectors $Z_n = (Z_{1,n}, \ldots, Z_{r,n})$, $n \geq 1$, with $\mathbb{E}[Z_{i,n}] = 0$ and $Z_{i,n} \in \mathbb{D}^{2,4}$. Let C be a non-negative definite, symmetric matrix of dimensioin r, and let $N = (N_1, \ldots, N_r) \sim \mathcal{N}_r(0, C)$. Suppose that:

(i) *For all $i, j = 1, \ldots, r$, $\mathbb{E}[Z_{i,n} Z_{j,n}] \to C(i,j)$ when $n \to \infty$;*

(ii) *For all $i = 1, \ldots, r$, $\sup_{n \geq 1} \mathbb{E}\left[\|DZ_{i,n}\|_{\mathfrak{H}}^4\right] < \infty$;*

(iii) *For all $i = 1, \ldots, r$, $\mathbb{E}\left[\|D^2 Z_{i,n} \otimes_1 D^2 Z_{i,n}\|_{(\mathfrak{H}^d)^{\otimes 2}}^2\right] \to 0$ when $n \to \infty$.*

Then $Z_n \stackrel{Law}{\to} N$ when $n \to \infty$.

CLT via Malliavin calculus

Theorem (Nourdin, Peccati and Réveillac)

Suppose that $r \geq 1$ is fixed. Consider random vectors $Z_n = (Z_{1,n}, \dots, Z_{r,n})$, $n \geq 1$, with $\mathbb{E}[Z_{i,n}] = 0$ and $Z_{i,n} \in \mathbb{D}^{2,4}$. Let C be a non-negative definite, symmetric matrix of dimensioin r, and let $N = (N_1, \dots, N_r) \sim \mathcal{N}_r(0, C)$. Suppose that:

(i) *For all $i, j = 1, \dots, r$, $\mathbb{E}[Z_{i,n} Z_{j,n}] \to C(i,j)$ when $n \to \infty$;*

(ii) *For all $i = 1, \dots, r$, $\sup_{n \geq 1} \mathbb{E}\left[\|DZ_{i,n}\|_{\mathfrak{H}}^4\right] < \infty$;*

(iii) *For all $i = 1, \dots, r$, $\mathbb{E}\left[\|D^2 Z_{i,n} \otimes_1 D^2 Z_{i,n}\|_{(\mathfrak{H}^d)^{\otimes 2}}^2\right] \to 0$ when $n \to \infty$.*

Then $Z_n \overset{Law}{\to} N$ when $n \to \infty$. Moreover,

$$d_{TV}(Z_{1,n}, N_1)) \leq \mathbb{E}\left[\left\|D^2 Z_{i,n} \otimes_1 D^2 Z_{i,n}\right\|_{(\mathfrak{H}^d)^{\otimes 2}}^2\right]^{\frac{1}{4}}$$

Bibliography

- Díaz M., Jaramillo A., Pardo J.C., y Pérez J.L. (2018). Functional Central Limit theorem for Matrix-valued Gaussian processes.
- Perez-Abreu V. y Tudor C. (2007). Functional Limit Theorem for Trace processes in a Dyson Brownian motion. *Communications on Stochastic Analysis*. **3** 415-428.
- Jaramillo, A., Pardo, J. y Pérez, J. (2018). Convergence of the empirical spectral distribution of a Gaussian matrix process. *Electronic Journal of Probability*.
- Israelson S. (2001). Asymptotic fluctuations of a particle system with singular interaction. *Stochastic Process and their Applications*. **93** 25-56.

Proving tightness

The main observation is that the random variable $\int f(x)\mu_t^{(n)}(dx)$ satisfies the following stochastic equation

$$
\begin{aligned}
&\int f(x)\mu_t^{(n)}(dx) \\
&\quad = f(0) + \frac{1}{n^{\frac{3}{2}}} \sum_{i=1}^{n} \sum_{k \leq h} \int_0^t f'(\Phi_i(Y^{(n)}(s))) \frac{\partial \Phi_i}{\partial y_{k,l}}(Y^{(n)}(s)) \delta X_{k,h}(s) \\
&\quad + \frac{1}{2} \int_0^t \int_{\mathbb{R}^2} \mathbb{1}_{\{x \neq y\}} \frac{f'(x) - f'(y)}{x - y} \mu_s^{(n)}(dx) \mu_s^{n}(dy) v_s' ds \\
&\quad + \frac{1}{2n^2} \sum_{i=1}^{n} \int_0^t f''(\Phi_i(Y^{(n)}(s))) v_s' ds,
\end{aligned}
$$

where $v_s := \sigma_s^2$.